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VERMA MODULES OVER LIE ALGEBRAS DEFINED BY TORSION-FREE ABELIAN GROUPS

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We study a submodule structure of standard highest weight modules over Lie algebras defined by torsion-free abelian groups. All such modules for higher rank Virasoro algebras and for algebras defined by $(\mathbb{Q},+)$ are classified. We also prove Futorny theorem for the algebras defined by $(\mathbb{Q},+)$.

1. Introduction

A lot of different examples of simple infinite-dimensional Lie algebras appears in literature during last decade (see for example [3], [9], [6] and references therein). But the representation theory of such algebras is not developed satisfactorily till now. All known results are concerned with the Kac-Moody algebras, Heisenberg algebras, Witt algebra, Virasoro algebra and trigonometrical algebras ([4], [7], [1], [3]). From the other hand, there exists a lot of different generalizations of the classical Witt algebra and Virasoro algebra (see [7], [3], [8], [6]). It turns out that such algebras play an important role in mathematical physics.

In this paper we investigate the structure of standard highest weight representations of simple Lie (Virasoro-like) algebras defined by torsion-free abelian groups. We obtain criteria for such representations to be irreducible and investigate all submodules and quotients for the reducible representations. We also describe all possible types of standard Borel subsets for the Lie algebras defined by finitely-generated torsion-free abelian group or the additive group of rationales.

Let us briefly describe the structure of the paper. In section 2 we give all necessary notations and preliminaries. In section 5 we investigate the structure of Verma modules over algebras defined by torsion-free abelian group. In section 6 we prove the uniqueness theorem for the algebras associated with the additive group of rationales and describe all possible Borel subsets for such algebras. We also prove the Futorny-like theorem about a support of irreducible weight module.

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2. Notations and preliminary results

Let $\mathbb{C}$ denote the field of complex numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{Q}$ the field of all rationales and $\mathbb{Z}$ the ring of all integers. We also write $\mathbb{N}$ for the semigroup of all positive integers.

Let $(A, +)$ be certain torsion-free abelian group and $\varphi: A \to \mathbb{C}$ its additive character. We will denote by 0 the zero element of A.

Consider vector-space $\mathfrak{G}$ with the base e_a, $a \in A$ and Lie brackets defined by

$$[e_a, e_b] = \varphi(b-a)e_{a+b}. \tag{1}$$

Lemma 1.

1) *The space $\mathfrak{G}$, together with the operation defined by (1) is A-graded Lie algebra $\mathfrak{G} = \mathfrak{G}(\varphi)$.*
2) *$\mathfrak{G}$ is simple if and only if φ is a monomorphism.*
3) *If $\mathfrak{G}$ is simple, it possesses a unique universal central extension which we will also denote by $\mathfrak{G}$.*

Proof. The first part of the lemma can be obtained by direct calculation. To prove the second part we note that $\mathfrak{G}$ is simple if and only if the following condition holds:

$$[e_a, e_b] = 0 \iff a = b. \tag{2}$$

By (1) this is equivalent to the condition that φ is a monomorphism. The last part follows from [5].

We set $\mathfrak{H} = \langle e_0 \rangle$ to be the Cartan subalgebra of $\mathfrak{G}$ and will call the elements $a \in A$ the roots of $\mathfrak{G}$. We will also call the space $\langle e_a \rangle$ the root space of $\mathfrak{G}$ corresponding to the root a. For the rest of the paper we will consider only weight $\mathfrak{G}$-modules with respect to the Cartan subalgebra and keep all standard notations (see for example [6]). From now on we assume that the algebra $\mathfrak{G}$ is simple. Let $U(\mathfrak{G})$ be the universal enveloping algebra of $\mathfrak{G}$. We will also denote by $\mathfrak{W}$ the classical Virasoro algebra. Set $\mathrm{spec}(\varphi) = \{\varphi(a) \,:\, a \in A\}$.

We will call a subset $T \subset A$ the Borel subset if T is a subsemigroup, $-T \cap T = \varnothing$ and $-T \cup T \cup \{0\} = A$. With each Borel subset T we associate the following partition of the algebra $\mathfrak{G}$: $\mathfrak{G} = \mathfrak{G}_T \oplus \mathfrak{H} \oplus \mathfrak{G}_{-T}$, where $\mathfrak{G}_{\pm T} = \langle e_a \,|\, a \in \pm T \rangle$. Let T be a Borel subset. For $x \in \mathfrak{G}_T$, $h \in \mathfrak{H}$, $\lambda \in \mathfrak{H}^*$ and $z \in \mathbb{C}$ set $(x+h)(z) = \lambda(h)z$. In such a way we define a structure of $U(\mathfrak{H} \oplus \mathfrak{G}_T)$-module on $\mathbb{C}$. The module

$$M(\lambda) = M(T, \lambda) = U(\mathfrak{G}) \bigotimes_{U(\mathfrak{H} \oplus \mathfrak{G}_T)} \mathbb{C}$$

will be called the Verma-type module associated with T and λ.

We will say that a Borel subset T is standard provided for any $a, b \in T$, $b - a \in T$ there exists $n \in \mathbb{N}$ such that $na - b \in T$. The Verma-type modules associated with standard Borel subsets will be called the Verma modules. The following lemma is obvious.

Lemma 2. *Let B be a subgroup of A and $T \subset A$ a standard Borel subset. Then $S = B \cap T$ is a standard Borel subset in B.*

The standard properties of Verma modules are described in the following lemma:

Lemma 3.
1) $M(\lambda)$ *is a weight module with* $\operatorname{supp} M(\lambda) = \lambda \cup \lambda - T$.
2) $M(\lambda)$ *has a unique maximal submodule* $N(\lambda)$ *and a unique irreducible quotient* $L(\lambda) \simeq M(\lambda)/N(\lambda)$.
3) If V *is a weight* $\mathfrak{G}$*-module generated by an element* v *such that* $hv = \lambda(h)v$ *for all* $h \in \mathfrak{H}$ *and* $\mathfrak{G}_T v = 0$ *then there exists a canonical epimorphism* $\psi: M(\lambda) \to V$ *with* $\psi(1 \otimes 1) = v$.
4) If $M(\lambda)$ *is a Verma module then*

$$\dim M(\lambda)_\mu = \begin{cases} 1, & \mu = \lambda; \\ \infty, & \mu \in \lambda - T. \end{cases}$$

Proof is standard.

3. Structure of Verma modules

In the case $A \simeq \mathbb{Z}^n$ we will call algebra $\mathfrak{G}$ higher rank Virasoro algebra ([6]). Remark that $\mathfrak{G} \simeq \mathfrak{W}$ if and only if $n = 1$. In the present section we classify and investigate the submodule structure of Verma modules over our algebras. It would be convenient to assume that $\mathfrak{G} \not\simeq \mathfrak{W}$. Submodule structure of Verma modules over $\mathfrak{W}$ is described completely in [1].

First of all we give alternating geometrical construction of standard Borel subset for finitely-generated A. Assume that $A \simeq \mathbb{Z}^n \subset \mathbb{R}^n$, the Euclidian vector-space with inner product $(\cdot,\cdot): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. A Borel subset T will be called geometrical if there exists some vector $\vec{f} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ such that $T = \big\{a \in A : (\vec{f}, a) > 0\big\}$.

Lemma 4. *The following statements are equivalent*
1) T *is geometrical.*
2) T *defines a linear order on* A*, satisfying the Archimedes law:* for every $0 < a < b$ there exists $n \in \mathbb{N}$ such that $na > b$.
3) T *is standard.*

Proof. 2)⇔3) is obvious. 2)⇔1) follows from the description of Archimedes orders of $\mathbb{Z}^n$ ([10]).

We are back to the case of arbitrary torsion-free A. Let T be a standard Borel subset of A and $M(\lambda)$ be a Verma module associated with T and λ. Obviously, there exists a continuum many non-conjugated standard Borel subsets for A (the reader can find the criteria for two such subsets to be conjugated in [10]). For each standard Borel subset we have defined a one-parameter family of Verma modules with respect to this Borel subset. It happens that the submodule structure of Verma modules over our algebra is rather simple and does not depend on the standard Borel subset T:

Theorem 1.
1) If $\lambda \neq 0$ *then* $M(\lambda)$ *is irreducible.*
2) If $\lambda = 0$ *then* $N(0) = \{v \in M(0)\,,\ \operatorname{supp} v \cap 0 = \varnothing\}$*, and* $M(0)/N(0) \simeq \mathbb{C}$.
3) $N(0)$ *is irreducible.*

Proof. First of all we will prove the second part of the theorem. In the case $\lambda = 0$ there exists, according to Lemma 3, an epimorphism $\varphi : M(0) \to \mathbb{C}$, where $\mathbb{C}$ is a trivial $\mathfrak{G}$-module. Hence

$$N(0) = \{v \in M(\lambda)\,,\, \operatorname{supp} v \cap 0 = \varnothing\}$$

and $M(0)/N(0) \simeq \mathbb{C}$.

To prove the first part consider an element $v \in M(\lambda)_\mu$, $v \neq 0$. We can write it in the form

$$v = \sum_{i=1}^{k} a_i u_i v_\lambda,$$

where v_λ denotes a canonical generator of $M(\lambda)$, $a_i \in \mathbb{C}$, $1 \leq i \leq k$ and each u_i is of the form $u_i = e_{x_1} e_{x_2} \dots e_{x_{k_i}}$.

For every $i = 1, 2, \dots, k$ denote $P(u_i) = \{x_1, x_2, \dots, x_{k_i}\}$.

Let $N = U(\mathfrak{G})v$. Our goal is to prove that $N = M(\lambda)$. Set $K(v) = \max_i k_i$. Denote

$$I = \big\{i \in \{1, 2, \dots, k\} \,:\, k_i = K(v)\big\} \quad \text{and} \quad P(I) = \bigcup_{i \in I} P(u_i).$$

Choose $x \in P(I)$ such that $x <_T y$ for every $x \neq y \in P(I)$. Then one can choose $y \in P$, $x <_T y$, $y <_T z$ for any $z \in P(I)$, $z \neq x$ such that $e_{-y} v \neq 0$. Continuing this procedure if necessary we can assume that $|I| = 1$ and $0 \notin \operatorname{supp} U(\mathfrak{G}_T)N_\mu$. Then at the same way one can choose an element $y \in A$ such that $e_{-y} v \neq 0$ and $K(e_{-y} v) < K(v)$. We obtain that $e_z v_\lambda \in N$ for some $z \in A$ and thus $v_\lambda \in N$, because $\lambda \neq 0$.

The last part follows from the proof of the first part.

4. Lie algebras defined by additive group of rationales

Throughout this section we assume $A \simeq (\mathbb{Q}, +)$.

Proposition 1.
1) If $\mathfrak{G}(\varphi)$ and $\mathfrak{G}(\psi)$ are simple then $\mathfrak{G}(\varphi) \simeq \mathfrak{G}(\psi)$.
2) There exist two Borel subsets in A: $T_1 = \{a \in A \,:\, a > 0\}$ and $T_2 = -T_1$.

Proof. To prove 1) it is sufficient to show that $\mathfrak{G}(\varphi) \simeq \mathfrak{G}(\theta)$, where $\theta(a) = a$ for all $a \in A$.

One can see that $\varphi(a) = a\varphi(1)$ for all $a \in A$. Let e_a, $a \in A$ be the standard base of $\mathfrak{G}(\varphi)$ and f_a, $a \in A$ be the standard base of $\mathfrak{G}(\theta)$. One can check by direct calculation that the map $F\colon \mathfrak{G}(\theta) \to \mathfrak{G}(\varphi)$ defined by $F(f_a) = \varphi(1)^{-1} e_a$ is an isomorphism.

To prove 2) suppose that T is a Borel subset and $1 \in T$. Since T is a subsemigroup, we obtain $\frac{1}{n} \in T$ for all $n \in \mathbb{N}$ and thus $T = T_1$. Observation that $\{\pm 1\} \cap T \neq \varnothing$ completes the proof.

Remark that both T_1 and T_2 are standard. This completes the description of Verma modules over the simple Lie algebra defined by $(\mathbb{Q}, +)$.

The following theorem describes the possible structure of the support of irreducible module over $\mathfrak{G}$. An analogue of this result for serial simple finite-dimensional Lie algebras was obtained in [2].

Theorem 2. *Let V be a non-trivial irreducible weight module over $\mathfrak{G}$. Then one of the following holds:*
1) $\operatorname{supp} V = \lambda + \operatorname{spec}(\varphi)$ *for some* $\lambda \in \mathfrak{H}^*$.
2) $\operatorname{supp} V = \lambda + \operatorname{spec}(\varphi) \setminus \{\mu\}$ *for some* $\lambda, \mu \in \mathfrak{H}^*$.

3) $\operatorname{supp} V \subset \lambda + \varphi(T_i)$ *for some* $\lambda \in \mathfrak{H}^*$ *and* $i \in \{1, 2\}$.

Proof. Clearly, $\operatorname{supp} V \subset \lambda + \operatorname{spec}(\varphi)$ for some $\lambda \in \mathfrak{H}^*$. Suppose that $|\lambda + \operatorname{spec}(\varphi) \setminus \operatorname{supp} V| > 1$. To prove the theorem it is sufficient to show that in this case $\operatorname{supp} V \subset \lambda' + \varphi(T_i)$ for some $\lambda' \in \lambda + \operatorname{spec}(\varphi)$ and $i \in \{1, 2\}$.

The standard order on $\mathbb{Q}$ induces an order $<$ on $\lambda + \operatorname{spec}(\varphi)$. Let $\mu_1, \mu_2 \in \lambda + \operatorname{spec}(\varphi) \setminus \operatorname{supp} V$ be two distinct elements. Suppose that $\mu_1 < \mu_2$. If for any $\mu \in \lambda + \operatorname{spec}(\varphi)$ such that $\mu_1 < \mu < \mu_2$ holds $\mu \notin \operatorname{supp} V$ then one can see that for any $\tau \in \operatorname{supp} V$ either $\tau < \mu_1$ or $\tau > \mu_2$. If there exist some elements $\mu \in \operatorname{supp} V$ such that $\mu_1 < \mu < \mu_2$ the one can easily obtain that some subalgebra $\mathfrak{G}_1 \simeq \mathfrak{W} \subset \mathfrak{G}$ has a non-trivial finite-dimensional representation. The latter is impossible, because $\mathfrak{W}$ is simple infinite-dimensional. This completes the proof.

We remark that all possibilities given by Theorem 2 really occur. Indeed, modules from intermediate theorem give us the examples for the first and the second types of the support and Verma modules give us the examples for the third type of the support.

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