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ON CONDITIONAL INVARIANCE PRINCIPLE FOR RANDOM WALKS

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Functional central limit theorem is proved for a stochastic process constructed from a one-dimensional random walk with fixed endpoints of its trajectories. The limiting Gaussian measure corresponds to a Brownian bridge with orientation dependent parameters.

1. Introduction

Conditional limit theorems for random walks form an extensively studied and well understood area in the theory of random processes (see, e. g., [14], [8], [9], [13], [15], [4]). Besides their pure mathematical interest these results have a natural physical interpretation in the framework of the so-called one-dimensional Solid-On-Solid (1D SOS) model, which is the simplest interface model.

From the physical point of view the probability distribution of the n-step random walk $S_0 = 0$, $S_1 = \xi_1, \dots, S_n = \xi_1 + \dots + \xi_n$ (here ξ_i are i. i. d. random variables) conditioned by fixing $S_n = n \tan \varphi$ describes the statistical properties of the inclined interface (in the 1D SOS model) with the slope angle φ. It is known ([14]) that the vertical fluctuations of such an interface have asymptotically Gaussian behaviour. Their magnitude can be described in terms of the stiffness $\tau(\varphi) + \tau''(\varphi)$, where $\tau(\varphi)$ denotes the appropriately defined surface tension in the direction orthogonal to the orientation of the interface ([1], [3]).

Due to its simplicity, the 1D SOS model serves as a testing model for discussing questions arising in the "real" physical context. Nevertheless, in certain cases (like in the low-temperature 2D Ising model, see [6]) the 1D SOS model provides a good approximation to the "true" interface. From this point of view, the random walks with fixed endpoints describe the local behaviour of the phase boundary; therefore, it is more natural to study their fluctuations in the direction orthogonal to the local orientation of the corresponding interface. As it was predicted in [1] using thermodynamical arguments, the magnitude of such fluctuations depends on the direction through the stiffness.

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The aim of the present note is to prove the functional central limit theorem for the random process describing the fluctuations of random walks with fixed endpoints in the direction orthogonal to their slope line. The limiting Gaussian measure coincides with the Brownian bridge scaled in such a way that the variances of its local increments are inversely proportional to the stiffness.

The arguments used below are applicable also to other situations. In particular, combining the ideas presented with those of [5] one can describe the local behaviour of the phase separation line in the low-temperature 2D Ising model [11].

2. Results

Let integer valued random variables[1] $\xi_1, \xi_2, \dots$ be independent and have the same probability distribution $\mathbf{P}(\cdot)$ with finite expectation and variance

$$\mathbf{E}\,\xi \equiv a, \qquad \mathbf{D}\,\xi \equiv \mathbf{E}\left(\xi - \mathbf{E}\,\xi\right)^2 > 0. \tag{2.1}$$

Assume that the distribution $\mathbf{P}(\cdot)$ is concentrated on the integer lattice $\mathbb{Z}^1$, i. e., the greater common divisor of those values ξ having non-vanishing probabilities equals 1. Denote by $\mathcal{D}_\xi$ the set of all real numbers h such that

$$L(h) \equiv \log \mathbf{E}\, \exp\{h\xi\} < \infty \tag{2.2}$$

and suppose that the connected set $\mathcal{D}_\xi \subseteq \mathbb{R}^1$ contains some neighbourhood of the origin. Observe that due to (2.1) the function $L(\cdot)$ is convex in the interior $\mathcal{D}_\xi^\circ$ of the set $\mathcal{D}_\xi$. Moreover, it is analytical in some complex neighbourhood $\mathcal{U}(\mathcal{D}_\xi^\circ)$ of the interval $\mathcal{D}_\xi^\circ$.

Consider a random walk $S_0 = 0$, $S_k = \sum_{j=1}^k \xi_k$, $k = 1, 2, \dots$, generated by the random variables ξ_i. For any natural number n define a random polygonal function $x_n(t)$, $t \in [0,1]$, via[2]

$$x_n(t) \equiv S_{[nt]} + \{nt\}\xi_{[nt]+1},$$

The aim of the present paper is to study statistical properties of the random polygons $x_n(t)$ conditioned by fixing their endpoints.

Definition 2.1. A real number r is called $(\xi-)$*admissible* if for some $h \in \mathcal{D}_\xi^\circ$ one has $L'(h) = r$.

Let b_n be a sequence of admissible numbers such that nb_n are integer and $b_n \to b$ as $n \to \infty$ with some admissible b. In this case $\mathbf{P}(S_n = nb_n) > 0$ for all sufficiently large n (indeed, for $b = a$ this follows from the local limit theorem [10]; in the opposite case the result follows from the large deviation principle in the strong form [7], [4]). Note that we can restrict our considerations to the case $b > 0$ only (in fact, the mapping $\xi \mapsto -\xi$ reduces the case $b < 0$ to $b > 0$; the situation with $b = 0$ describes "vertical fluctuations of horizontal interfaces" and was discussed in

[1] For simplicity we consider only the discrete case here. The generalization to the non-lattice situation is straightforward (see, e. g., [4, §2]).

[2] Here and below $[y]$ denotes the integral part of a real number y and $\{y\} = y - [y]$ is its fractional part.

[14]). Therefore, without loss of generality one can assume that all b_n's are positive numbers satisfying the inequality $\mathbf{P}\left(S_n = nb_n\right) > 0$ for all n. Fix such a sequence b_n for future considerations.

Let $\theta_n(t)$, $t \in [0,1]$, be the conditional random process

$$\theta_n(t) = \big(x_n(t) | S_n = nb_n\big).$$

Its stochastic behaviour is well known. Namely, denoting

$$\theta_n^*(t) = \frac{1}{\sqrt{n}}\Big(\theta_n(t) - nb_n t\Big) \tag{2.3}$$

one proves the following statement for the corresponding distributions μ_n^* in the space $\mathbf{C}[0,1]$ of continuous functions on $[0,1]$ (see, e. g., [14]; a close result can be found in [4]).

Proposition 2.1. *Let $\bar{h} = \bar{h}(b) \in \mathcal{D}_\xi^\circ$ be such that $L'(\bar{h}) = b$. Then the sequence of measures μ_n^* converges weakly in $\mathbf{C}[0,1]$ to the distribution $\bar{\mu}$ of the process*

$$\bar{\theta}(t) = \big(L''(\bar{h})\big)^{1/2} w_{1,0}(t), \qquad t \in [0,1], \tag{2.4}$$

where $w_{1,0}(t)$ denotes the Brownian bridge on $[0,1]$.

In particular, the distribution of the random process $\frac{1}{n}\theta_n(t)$ converges weakly in $\mathbf{C}[0,1]$ to the distribution concentrated on the deterministic function $e(t) = bt$, $t \in [0,1]$. Its graph γ is a segment having the slope angle φ, $\tan\varphi = b$. For any $t \in [0,1]$ the quantity $s = s(t) = t/\cos\varphi$ presents the length of the segment on the graph γ of $e(t)$ with the endpoints $(0,0)$ and (t, bt); denote the inverse mapping $s \mapsto t(s) = s\cos\varphi$ by t_s. The quantity $l = s(1) = (\cos\varphi)^{-1}$ gives the total length of γ. Similarly, denote by γ_n the segment connecting the points $(0,0)$ and $(1, b_n)$.

Definition 2.2. Let $L^*(\cdot)$ be the Legendre transformation of the logarithmic moment generating function $L(\cdot)$ from (2.2),

$$L^*(p) \equiv \sup_{h\in\mathbb{R}^1}\big(ph - L(h)\big),$$

and $\varphi \in \big(-\pi/2, \pi/2\big)$ be the slope angle of γ. The *surface tension* $\tau(\varphi)$ in the direction orthogonal to γ is

$$\tau(\varphi) \equiv L^*(\tan\varphi)\cos\varphi. \tag{2.5}$$

Fix any point $\sigma = (t_s, b_n t_s) \in \gamma_n$, denote by $\mathfrak{n}_{\sigma,n}(t)$ the normal line to γ_n at σ,

$$\mathfrak{n}_{\sigma,n}(t) = \frac{t_s - t}{b_n} + b_n t_s, \qquad t \in \mathbb{R}^1, \tag{2.6}$$

and consider a (random) trajectory of the process $\frac{1}{n}\theta_n(t)$. Several (random) points of their intersection can appear (but no less than one); we choose two extremal of them, the most upper and the most lower one, and denote their abscissas by $\tilde{t}^+ = \tilde{t}^+_{\sigma,n}$ and $\tilde{t}^- = \tilde{t}^-_{\sigma,n}$ respectively, $\tilde{t}^+ \le \tilde{t}^-$.

Let a number $\alpha \in (0,1)$ be fixed. For any natural n consider points $\sigma_j = (t_j, b_n t_j)$, $t_j = j/[n^\alpha]$, $j \in J_\alpha \equiv \{1, \dots, [n^\alpha]-1\}$, and determine (random) numbers $\tilde{t}_j^\pm = \tilde{t}_{\sigma_j,n}^\pm$ as above. Define the continuous random processes $\tilde{\zeta}_n^\pm(s)$ on $[0,l]$ such that (recall that $s = t/\cos\varphi$)

$$\tilde{\zeta}_n^\pm(s_j) = \frac{t_j - \tilde{t}_j^\pm}{\sin\varphi}\sqrt{n} = \theta_n^*(\tilde{t}_j^\pm)\cos\varphi, \qquad j = 1, \dots, [n^\alpha]-1, \tag{2.7}$$

and which are linearly interpolated elsewhere (we put by definition $\tilde{\zeta}_n^\pm(0) = \tilde{\zeta}_n^\pm(l) = 0$). Observe that $|t_j - \tilde{t}_j^+|/\sin\varphi$ presents the distance between $\sigma \in \gamma_n$ and the most upper common point of graphs of $\frac{1}{n}\theta_n(t)$ and $\mathfrak{n}_{\sigma,n}(t)$ (recall (2.6)).

The main result of the present paper is given by the following statement.

Theorem. *The distribution $\tilde{\nu}_n^+$ of the process $\tilde{\zeta}_n^+(s)$ converges weakly in $\mathbf{C}[0,l]$ to the distribution of the process*

$$\bar{\zeta}(s) = \big(\tau(\varphi) + \tau''(\varphi)\big)^{-1/2} w_{l,0}(s), \qquad s \in [0,l], \tag{2.8}$$

with $w_{l,0}(s) = w(s) - \frac{s}{l}w(l)$ denoting the Brownian bridge on $[0,l]$. The same is true for the distribution $\tilde{\nu}_n^-$ of $\tilde{\zeta}_n^-(s)$. Moreover, the process $\tilde{\zeta}_n^+(s) - \tilde{\zeta}_n^-(s)$ vanishes in probability as $n \to \infty$.

3. Preliminaries

We collect here some technical results to be used in the proof of Theorem.

Lemma 3.1. *Let $\alpha \in (0,1)$ be as fixed above and $t_1, t_2 \in [0,1]$ satisfy the following conditions: $t_2 \ge n^{-\alpha}$ and $n^{-(\alpha+1)/2} \le t_1 \le t_2/2$. Then there exist positive numbers C_1, C_2, α_1, α_2, δ, and n_0 such that for all $n \ge n_0$ one has*

$$\mathbf{P}\left(|x_n(t_1) - nb_n t_1| > \rho\sqrt{nt_1} \;\middle|\; x_n(t_2) = nb_n t_2\right) \le g_n(\rho),$$

where

$$g_n(\rho) = \begin{cases} C_1 \exp\{-\alpha_1\rho^2\}, & \text{if } |\rho| \le \delta\sqrt{nt_1}, \\ C_2 \exp\{-\alpha_2\sqrt{nt_1}|\rho|\}, & \text{if } |\rho| > \delta\sqrt{nt_1}. \end{cases}$$

Proof of this statement can be obtained by literal repetition of that of estimate (6.16) from [4, Lemma 6.3].

Lemma 3.2. *Let $t_2 \in [0,1]$ be as in Lemma 3.1, t_1 satisfy the condition $0 < t_1 \le t_2/2$, and the numbers $\tilde{h} \in \mathcal{D}_\xi^\circ$, $\tilde{b}$ be related via $L'(\tilde{h}) = \tilde{b}$. Then there exists $C = C(\tilde{b}) > 0$ such that for all real h and sufficiently large n one has*

$$\mathbf{P}\left(S_{[nt_1]} \ge nt_1\tilde{b} + y \;\middle|\; S_{[nt_2]} = nt_2\tilde{b}\right) \le C\sqrt{nt_2}\exp\{-hy + nt_1\tilde{f}(h)\}, \tag{3.1}$$

where

$$\tilde{f}(h) \equiv L(\tilde{h} + h) - L(\tilde{h}) - h\tilde{b}.$$

Proof. For any random element ξ with the distribution $\mathbf{P}(\cdot)$ and the logarithmic moment generating function $L(\cdot)$ denote by $\mathbf{P}_h(\cdot)$ the $(h-)$tilted distribution,

$$\mathbf{P}_h(\xi = x) \equiv \exp\{hx - L(h)\}\mathbf{P}(\xi = x), \tag{3.2}$$

and by $\langle\cdot\rangle_h$ the operator of mathematical expectation with respect to $\mathbf{P}_h(\cdot)$.

Denote $n_1 = [nt_1]$, $n_2 = [nt_2]$ and consider the random vector $\Lambda_n = (S_{n_1}, S_{n_2})$. Its logarithmic moment generating function $L_\Lambda(H)$, $H = (h_1, h_2)$ is given by

$$L_\Lambda(H) \equiv \log \mathbf{E} \exp\bigl\{(H, \Lambda_n)\bigr\} = n_1 L(h_1 + h_2) + (n_2 - n_1) L(h_2).$$

For $H_0 = (0, \tilde{h})$ one easily obtains

$$\nabla_H L_\Lambda(H)\bigr|_{H=H_0} = (n_1\tilde{b}, n_2\tilde{b}),$$

where ∇_H denotes the gradient with respect to H. Then (cf. (3.2))

$$\begin{aligned}\mathbf{P}\bigl(S_{n_2} = n_2\tilde{b}, S_{n_1} \ge n_1\tilde{b} + y\bigr) &= \\ &= \exp\{-\tilde{h}n_2\tilde{b} + L_\Lambda(H_0)\}\mathbf{P}_{H_0}\bigl(S_{n_2} = n_2\tilde{b}, S_{n_1} \ge n_1\tilde{b} + y\bigr) \le \\ &\le \exp\{-n_2(\tilde{h}\tilde{b} + L(\tilde{h}))\}\mathbf{P}_{H_0}\bigl(S_{n_1} \ge n_1\tilde{b} + y\bigr).\end{aligned} \tag{3.3}$$

Due to the Markov inequality,

$$\mathbf{P}_{H_0}\bigl(S_{n_1} \ge n_1\tilde{b} + y\bigr) \le \langle\exp\{hS_{n_1}\}\rangle_{H_0} \exp\{-h(n_1\tilde{b} + y)\}, \tag{3.4}$$

where

$$\langle\exp\{hS_{n_1}\}\rangle_{H_0} = \exp\bigl\{L_\Lambda(\tilde{h}, h) - L_\Lambda(\tilde{h}, 0)\bigr\}. \tag{3.5}$$

On the other hand, the local limit theorem for S_{n_2} (combined with the large deviation technique in the strong form if necessary, see, e. g., [7, §2], [4, §4] implies

$$\begin{aligned}\mathbf{P}\,(S_{n_2} = n_2\tilde{b}) &= \frac{1}{\sqrt{2\pi n_2 L''(\tilde{h})}} \exp\{-n_2(\tilde{h}\tilde{b} - L(\tilde{h}))\}(1 + o(1)) \ge \\ &\ge \frac{1}{C\sqrt{n_2}} \exp\{-n_2(\tilde{h}\tilde{b} - L(\tilde{h}))\}\end{aligned} \tag{3.6}$$

for all sufficiently large n (recall that $n_2 = [nt_2] \ge [n^{1-\alpha}] \to \infty$ as $n \to \infty$).

Finally, (3.1) follows immediately from (3.3)–(3.6). □

Now, combining Lemmas 3.1 and 3.2 one deduces

Lemma 3.3. *There exist positive constants C_1 and C_2 such that for all sufficiently large n one has*

$$\mathbf{P}\Bigl(\sup_{t\in[0,1]} \Bigl|\frac{1}{n}\theta_n(t) - bt\Bigr| > n^{-\alpha/2}\Bigr) \le C_1 \exp\{-C_2 n^{1-\alpha}\}. \tag{3.7}$$

Moreover, for any $\varepsilon > 0$ there exists $A > 0$ such that

$$\mathbf{P}\Bigl(\sup_{t\in[0,1]} \Bigl|\frac{1}{n}\theta_n(t) - bt\Bigr| > \frac{A}{\sqrt{n}}\Bigr) \le \varepsilon$$

provided n is sufficiently large.

Remark 3.3.1. According to (3.7) one has the uniform in $s \in [0, l]$ inequality

$$\mathbf{P}\bigl(|\tilde{t}_s^{\pm} - t_s| > n^{-\alpha/2}\bigr) \le C_1 \exp\{-C_2 n^{1-\alpha}\} \tag{3.8}$$

with some constants C_1, $C_2 > 0$ if only n is sufficiently large.

4. Proof of Theorem

Consider the random process (recall (2.3))

$$\zeta_n(s) \equiv \theta_n^*(s\cos\varphi)\cos\varphi \tag{4.1}$$

and denote its distribution in $\mathbf{C}\,[0,l]$ by ν_n.

Lemma 4.1. *The sequence of measures ν_n converges weakly in $\mathbf{C}\,[0,l]$ to the distribution of the random process (recall (2.8))*

$$\bar\zeta(s) = \bigl(\tau(\varphi)+\tau''(\varphi)\bigr)^{-1/2} w_{l,0}(s), \qquad s\in[0,l],$$

where $w_{l,0}(s)$ stands for the Brownian bridge in $[0,l]$.

Proof. Recall that $L'(\bar h) = b = \tan\varphi$. Therefore,

$$\tau(\varphi)+\tau''(\varphi) = L^{*''}\bigl(\tan\varphi\bigr)\cos^{-3}\varphi = \frac{1}{L''(\bar h)\cos^3\varphi}, \tag{4.2}$$

where the first equality follows from definition (2.5) and the second one is implied by the duality relations for the Legendre transformation (see, e. g., Property A.1 in [4]). Changing the variables $t\mapsto s$ in (2.4) one immediately deduces the statement of the lemma from Proposition 2.1 and relation (4.2). □

Remark 4.1.1. Definition (4.1) induces the one-to-one correspondence ω between $\mathbf{C}\,[0,1]$ and $\mathbf{C}\,[0,l]$,

$$\omega\colon f(t)\to g(s) = f(s\cos\varphi)\cos\varphi.$$

Observe that ω induces a bijection between compact sets in these spaces.

Proof of Theorem. We prove first that for every $\varepsilon>0$

$$\mathbf{P}\,\bigl(\max_{j\in J_\alpha}|\theta_n^*(\tilde t_j^\pm)-\theta_n^*(t_j)|\ge\varepsilon\bigr)\to 0 \tag{4.3}$$

as $n\to\infty$, where as before J_α stands for the set $\bigl\{1,\dots,[n^\alpha]-1\bigr\}$. Indeed, fix arbitrary $\eta>0$ and consider any compact set $\mathcal{K}\subset\mathbf{C}\,[0,1]$ such that

$$\mu_n^*\bigl(\mathbf{C}\,[0,1]\backslash\mathcal{K}\bigr)<\frac{\eta}{2} \tag{4.4}$$

for all $n\ge n_0$ with sufficiently large n_0 (such $\mathcal{K}$ always exists due to the weak compactness of the sequence μ_n^*). According to Arzelà's theorem [2, App. 1], all $f\in\mathcal{K}$ are equicontinuous:

$$\lim_{\delta\to0}\ \sup_{f\in\mathcal{K}}\ \sup_{t',t''\in[0,1],\,|t'-t''|<\delta}\bigl|f(t')-f(t'')\bigr| = 0. \tag{4.5}$$

Let $\delta>0$ be such that $\sup_{|t'-t''|<\delta}\bigl|f(t')-f(t'')\bigr|<\varepsilon$ for all $f\in\mathcal{K}$. Then

$$\Bigl\{\max_{j\in J_\alpha}\bigl|\theta_n^*(\tilde t_j^\pm)-\theta_n^*(t_j)\bigr|\ge\varepsilon\Bigr\}\subset\Bigl\{\theta_n^*(\cdot)\in\mathbf{C}\,[0,1]\backslash\mathcal{K}\Bigr\}\cup\bigcup_{j\in J_\alpha}\Bigl\{\bigl|\tilde t_j^\pm-t_j\bigr|\ge\delta\Bigr\} \tag{4.6}$$

and (4.3) follows directly from (4.6), (4.4), and (3.8). As a result (recall definitions (2.7) and (4.1)), for any positive ε, η the inequality

$$\mathbf{P}\left(\max_{j\in J_\alpha}\bigl|\tilde\zeta_n^\pm(s_j)-\zeta_n(s_j)\bigr|\ge\varepsilon/2\right)<\eta/4 \tag{4.7}$$

holds provided only $n\ge n_0=n_0(\varepsilon,\eta,\varphi)$.

Next, let us check the convergence of finite dimensional distributions of the random processes $\tilde\zeta_n^\pm(s)$ to that of $\bar\zeta(s)$ from (2.8). Due to Lemma 4.1 it is enough to prove that

$$\sup_{s\in[0,l]}\bigl|\tilde\zeta_n^\pm(s)-\zeta_n(s)\bigr|\to 0 \tag{4.8}$$

in probability as $n\to\infty$. To do this, fix arbitrary $\varepsilon>0$, $\eta>0$ and consider any compact set $\mathcal{K}\subset\mathbf{C}\,[0,1]$ satisfying (4.4). For $s\in[0,l]$, denote by ρ_1, $\rho_2\in\bigl\{jl/[n^\alpha], j\in J_\alpha\bigr\}$ the numbers such that $s\in[\rho_1,\rho_2]$ and $|\rho_1-\rho_2|=l/[n^\alpha]$. Let $\lambda=\lambda(s)\in[0,1]$ be such that $s=\lambda\rho_1+(1-\lambda)\rho_2$. Now, find $\bar\delta>0$ with the property

$$\sup_{t',t''\in[0,1],|t'-t''|<\bar\delta}\bigl|f(t')-f(t'')\bigr|<\varepsilon/2 \tag{4.9}$$

uniformly in $f\in\mathcal{K}$ (recall (4.5)). Without loss of generality one can assume that $n\ge n_0$ with $\bar\delta[(n_0)^\alpha]>1$. According to the definition of $\tilde\zeta_n^\pm(\cdot)$, one has

$$\tilde\zeta_n^\pm(s)=\lambda\tilde\zeta_n^\pm(\rho_1)+(1-\lambda)\tilde\zeta_n^\pm(\rho_2). \tag{4.10}$$

Taking into account definition (4.1), rewrite

$$\begin{aligned}\bigl|\tilde\zeta_n^\pm(s)-\zeta_n(s)\bigr|\le\lambda\bigl|\tilde\zeta_n^\pm(\rho_1)-\zeta_n(\rho_1)\bigr|+(1-\lambda)\bigl|\tilde\zeta_n^\pm(\rho_2)-\zeta_n(\rho_2)\bigr|+\\+\lambda\bigl|\zeta_n(\rho_1)-\zeta_n(s)\bigr|+(1-\lambda)\bigl|\zeta_n(\rho_2)-\zeta_n(s)\bigr|.\end{aligned} \tag{4.11}$$

Then, the simple inclusion

$$\begin{aligned}\Bigl\{\bigl|\tilde\zeta_n^\pm(s)-\zeta_n(s)\bigr|\ge\varepsilon\Bigr\}\subset\Bigl\{\bigl|\tilde\zeta_n^\pm(\rho_1)-\theta_n^*(\rho_1\cos\varphi)\cos\varphi\bigr|\ge\varepsilon/2\Bigr\}\cup\Bigl\{\bigl|\rho_1-s\bigr|\ge\bar\delta\Bigr\}\cup\\ \cup\Bigl\{\bigl|\tilde\zeta_n^\pm(\rho_2)-\theta_n^*(\rho_2\cos\varphi)\cos\varphi\bigr|\ge\varepsilon/2\Bigr\}\cup\Bigl\{\bigl|\rho_2-s\bigr|\ge\bar\delta\Bigr\}\cup\\ \cup\Bigl\{\theta_n^*(\cdot)\in\mathbf{C}\,[0,1]\setminus\mathcal{K}\Bigr\},\end{aligned}$$

relations (4.4), (4.7), and (4.9)–(4.11) imply that for n under considerations the inequality

$$\mathbf{P}\left(\sup_{s\in[\rho_1,\rho_2]}\bigl|\tilde\zeta_n^\pm(s)-\zeta_n(s)\bigr|\ge\varepsilon\right)<\eta$$

holds. Observing that the last estimate is independent of $[\rho_1,\rho_2]$, one immediately deduces (4.8).

It remains to establish the weak compactness of the measures ν_n^+ (the case of ν_n^- is analogous). Note that due to Theorem 8.2 [2] we need to prove that for any $\varepsilon>0$ and $\eta>0$ one can find $\delta>0$ and n_0 such that for all $n\ge n_0$

$$\mathbf{P}\Bigl\{\sup_{|s'-s''|<\delta}\bigl|\tilde\zeta_n^+(s')-\tilde\zeta_n^+(s'')\bigr|\ge\varepsilon\Bigr\}\le\eta.$$

To do this we observe that

$$\left|\tilde{\zeta}_n^+(s') - \tilde{\zeta}_n^+(s'')\right| \le \left|\tilde{\zeta}_n^+(s') - \zeta_n(s')\right| + \left|\zeta_n(s') - \zeta_n(s'')\right| + \left|\zeta_n(s'') - \tilde{\zeta}_n^+(s'')\right|$$

and therefore

$$\mathbf{P}\left\{ \sup_{|s'-s''|<\delta} \left|\tilde{\zeta}_n^+(s') - \tilde{\zeta}_n^+(s'')\right| \ge \varepsilon \right\}$$
$$\le \mathbf{P}\left\{ \sup_{|s'-s''|<\delta} \left|\zeta_n(s') - \zeta_n(s'')\right| \ge \varepsilon/3 \right\} + 2\mathbf{P}\left\{ \sup_{s\in[0,l]} \left|\tilde{\zeta}_n^+(s) - \zeta_n(s)\right| \ge \varepsilon/3 \right\}. \tag{4.12}$$

Finally, the weak compactness of ν_n^+ follows from that of ν_n (recall Lemma 4.1), relations (4.8), and (4.12). □

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