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ON LINEAR TOPOLOGICAL SPACES (LINEARLY) HOMEOMORPHIC TO $\mathbb{R}^\infty$

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We prove that every infinite-dimensional (locally convex) linear topological space that can be expressed as a direct limit of finite-dimensional metrizable compacta is (linearly) homeomorphic to the space $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$.

Given an increasing sequence of topological spaces

$$X_1 \subset X_2 \subset \dots \subset X_n \subset \cdots$$

we define the direct limit topology on $X = \bigcup_{n\in\mathbb{N}} X_n$ letting $U \subset X$ to be open if and only if $U \cap X_n$ is open for every $n \in \mathbb{N}$.

By $\mathbb{R}^\infty$ we denote the direct limit of the sequence

$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots ,$$

where the embeddings $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ are defined by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$.

The space $\mathbb{R}^\infty$ considered with the natural linear operations is a locally convex linear topological space. Denote by $\mathcal{C}^\infty_{fd}$ the class of spaces that are direct limits of finite-dimensional metrizable compacta.

In this note we prove the following

Theorem. *Any infinite-dimensional (locally convex) linear topological space $X \in \mathcal{C}^\infty_{fd}$ is (linearly) homeomorphic to $\mathbb{R}^\infty$.*

Remark. A metric counterpart of the space $\mathbb{R}^\infty$ is the linear subspace $l_2^f = \{(t_i)_{i=1}^\infty \in l_2 \mid t_i = 0$ for all but finitely many $i\}$ in the separable Hilbert space l_2. A part of Theorem (that dealing with homeomorphisms) has its metric analog: every infinite-dimensional linear metrizable space that can be represented as a countable union of finite-dimensional compacta is homeomorphic to the pre-Hilbert space l_2^f [CDM]. In the meantime, the other part of Theorem is specific for direct limit topologies and admits no generalization onto metric locally convex spaces: the linear span$([0,1])$ of a linearly independent arc $[0,1]$ in l_2 is a locally convex linear metric space (even pre-Hilbert space) which is a countable union of finite-dimensional compacta but is not isomorphic to the space l_2^f.

Theorem results from the following a little bit more general results.

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Proposition 1. *Any convex set $X \in \mathcal{C}^{\infty}_{fd}$ in a locally convex linear topological space L is affinely homeomorphic to a convex set in $\mathbb{R}^\infty$. Moreover, if $X = L$ then X is linearly homeomorphic to $\mathbb{R}^\infty$.*

Proof. If X is finite-dimensional then the statement is trivial, so we assume that X is infinite-dimensional. Write $X = \varinjlim X_n$, where each X_n is a finite-dimensional compactum. Without loss of generality, $0 \in X_1 \subset X_2 \subset \cdots \subset X$.

Claim. *For every compactum $C \subset X$ the convex hull* $\operatorname{conv}(C) \subset X$ *is finite-dimensional.*

Proof. Assume on the contrary that $\operatorname{conv}(C)$ is infinite-dimensional. Then for every $n \in \mathbb{N}$ there exists $x_n \in \frac{1}{n}\operatorname{conv}(C) \setminus X_n$ (recall that X_n's are finite-dimensional). It follows from the definition of the direct limit topology on X that the set $A = \{x_n \mid n \in \mathbb{N}\}$ is closed in X. Moreover, $0 \notin A$. Since the space L is locally convex, there is a convex neighborhood $U \subset L$ of the origin such that $U \cap A = \varnothing$. Since the set C is compact, $\frac{1}{n}C \subset U$ for some $n \in \mathbb{N}$. By convexity of U, $x_n \in \operatorname{conv}(\frac{1}{n}C) \subset U$. This contradicts to $x_n \in A$ and $U \cap A = \varnothing$. □

Applying Claim to the compacta X_n's, one can construct a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ of linearly independent vectors and a number sequence $\{m(n)\}_{n\in\mathbb{N}}$ such that $X_n \subset L_{m(n)} = \operatorname{span}\{x_1, \dots, x_{m(n)}\}$ for every $n \in \mathbb{N}$. Then the identity map $i\colon X \to \varinjlim L_{m(n)}$ is continuous. Now remark that for every $n \in \mathbb{N}$ the intersection $X \cap L_{m(n)}$ is a metrizable direct limit of compacta, and hence, $X \cap L_{m(n)}$ is locally compact. Using this fact, show that i is a topological embedding, and notice that $\varinjlim L_{m(n)}$ is affinely homeomorphic to $\mathbb{R}^\infty$.

If $X = L$ then $X = \bigcup_{n\in\mathbb{N}} L_{m(n)}$, and consequently, $i(X) = \varinjlim L_{m(n)}$. □

Proposition 2. *Every infinite-dimensional convex set $X \in \mathcal{C}^{\infty}_{fd}$ in a linear topological space $L \in \mathcal{C}^{\infty}_{fd}$ is homeomorphic to $\mathbb{R}^\infty$.*

To prove this proposition we need

Lemma. *Let X be a convex infinite-dimensional subset of a linear topological space $L \in \mathcal{C}^{\infty}_{fd}$. For every compact set $C \subset X$ there is an embedding $e\colon C \times [0,1] \to X$ such that $e(c, 0) = c$ for every $c \in C$.*

Proof. We will search the embedding e among the maps of the type $e(c,t) = (1 - \frac{t}{2})c + \frac{t}{2}x_0$, where $x_0 \in X$.

So, let $C \subset X$ be a compactum. Then the set $D = [0,1]\cdot([1/2,1]C - [1/2,1]C) = \{t(\tau c - \tau' c') \mid t \in [0,1],\ \tau, \tau' \in [1/2,1],\ c, c' \in C\} \subset L$ is also compact. Moreover, since $L \in \mathcal{C}^{\infty}_{fd}$, the compactum D is finite-dimensional. Let $n = \dim(D)$. Since the convex set X is infinite-dimensional it contains $n+2$ linearly independent vectors $x_1, \dots, x_{n+2}$. Let $\Delta = \operatorname{conv}\{x_1, \dots, x_{n+2}\} \subset X$ be their convex hull. We claim that there is $x_0 \in \Delta$ such that $x_0 \cdot (0,1] \cap D = \varnothing$. Assuming the converse, we obtain that $\Delta = \bigcup_{k=1}^{\infty} \Delta_k$, where $\Delta_k = \{x \in \Delta \mid x \cdot [0, 1/k] \subset D\}$ (remark that the set D together with a point $d \in D$ contains the interval $[0,1]d$). Using the compactness of D prove that each set Δ_k is closed in Δ. Consequently, by the Baire Category Theorem, one of the sets Δ_k's (to say Δ_m) is somewhere dense in Δ. Then $\dim(\Delta_m) = \dim(\Delta) = n + 1$ and the map $f(x) = \frac{1}{m}x$ for $x \in \Delta_m$ determines the embedding $f\colon \Delta_m \to D$ of the $(n+1)$-dimensional compactum Δ_m

into the n-dimensional space D. The obtained contradiction shows that there is $x_0 \in \Delta \subset X$ such that $x_0 \cdot (0,1] \cap D = \varnothing$.

Let us show that the map $e\colon C \times [0,1] \to X$ defined by $e(x,t) = (1-\frac{t}{2})c + \frac{t}{2}x_0$ for $(c,t) \in C \times [0,1]$ is an embedding. Since the space C is compact, it suffices to prove that the map e is injective. Let (c,t) and (c',t') be distinct points in $C \times [0,1]$. If $t = t'$ then $c \neq c'$ and obviously $e(c,t) \neq e(c',t')$. If $t \neq t'$ then $\frac{t-t'}{2}x_0 \notin D = [0,1]([1/2,1]C - [1/2,1]C) \ni (1-\frac{t'}{2})c' - (1-\frac{t}{2})c$. Consequently, $\frac{t-t'}{2}x_0 \neq (1-\frac{t'}{2})c' - (1-\frac{t}{2})c$. This yields $(1-\frac{t}{2})c + \frac{t}{2}x_0 \neq (1-\frac{t'}{2})c + \frac{t'}{2}x_0$, i.e., the map e is injective. $\square$

Proof of Proposition 2. Let $X \in \mathcal{C}^\infty_{fd}$ be a convex infinite-dimensional subset in a linear topological space $L \in \mathcal{C}^\infty_{fd}$. To prove that the space X is homeomorphic to $\mathbb{R}^\infty$ we will apply Sakai's Characterizing Theorem [Sa]. According to this theorem, a space $X \in \mathcal{C}^\infty_{fd}$ is homeomorphic to $\mathbb{R}^\infty$ if and only if for every finite-dimensional compact pair $B \subset A$ any embedding $i\colon B \to X$ extends to an embedding $\bar{i}\colon A \to X$.

Fix a finite-dimensional compactum A, a closed subset B, and an embedding $i\colon B \to X$. At first we shall construct a map $\tilde{i}\colon A \to X$ extending the embedding i. For this fix any metric d on the compactum A and consider the cover $\mathcal{U} = \{O_d(a, d(a,B)/3) \mid a \in A \setminus B\}$ of the space $A \setminus B$. For every $U = O(a, d(a,B)/3) \in \mathcal{U}$ fix a point $b_a \in B$ with $d(a,b_a) = d(a,B)$. Let $\{\lambda_U\colon A\setminus B \to [0,1]\}_{U\in\mathcal{U}}$ be a partition of unity of order $\leq \dim(A)+1$ inscribed into the cover $\mathcal{U}$. It can be easily verified that the map $\tilde{i}\colon A \to X$ defined by

$$\tilde{i}(a) = \begin{cases} i(a), & a \in B \\ \sum_{U\in\mathcal{U}} \lambda_U(a) i(b_a), & a \in A \setminus B \end{cases}$$

is the required extension of the embedding $i\colon B \to X$.

Let $j\colon A \setminus B \to [0,1]^m$ be an embedding of the quotient space A/B such that $j(\{B\}) = \mathbf{0} = (0,\dots,0)$. It follows from Lemma that there is an embedding $e\colon \tilde{i}(A) \times [0,1]^m \to X$ such that $e(x,0) = x$ for $x \in \tilde{i}(A)$. Denote by $\pi\colon A \to A/B$ the quotient map. It is easily seen that the map $\bar{i} = e(\tilde{i}, j \circ \pi)\colon A \to X$ defined by $\bar{i}(a) = e(\tilde{i}(a), j \circ \pi(a))$, $a \in A$, is the required embedding extending the embedding $i\colon B \to X$. By [Sa], the space X is homeomorphic to $\mathbb{R}^\infty$. $\square$

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