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NONEXISTENCE OF ABSORBING SET FOR A TRANSFINITE EXTENSION OF COVERING DIMENSION

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R. Pol has shown that for every countable ordinal α there exists a universal space for separable metrizable spaces X with $\operatorname{ind} X \leq \alpha$. W. Olszewski has shown that for every countable limit ordinal λ there is no universal space for separable metrizable space with $\operatorname{Ind} X \leq \lambda$. We prove that for every countable limit ordinal there is no universal space for separable metrizable spaces with $\dim X \leq \alpha$, where dim is a transfinite extension of covering dimension introduced by P. Borst.

As an application, we show that there is no absorbing sets (in the sense of Bestvina and Mogilski) for the classes of spaces X with $\dim X \leq \alpha$ belonging to some absolute Borel class.

0. All spaces under the discussion are metrizable and separable, all ordinals are countable. Let us recall that a space X is universal in a class $\mathcal{C}$ of spaces if $X \in \mathcal{C}$ and every space of $\mathcal{C}$ is embeddable in X.

The transfinite dimensions ind and Ind are transfinite extensions of the classical Menger-Urysohn small inductive dimension ind and Brouwer-Čech large inductive dimension Ind, respectively (see [1] and [2]).

R. Pol has shown that for every countable ordinal α there exists a universal space for separable metrizable spaces X with $\operatorname{ind} X \leq \alpha$ [3]. W. Olszewski has shown that there is no universal space for separable metrizable spaces X with $\operatorname{Ind} X \leq \alpha$ for any limit ordinal α [4]. He has also shown that there is no universal space for compact metrizable spaces X with $\operatorname{ind} X \leq \alpha$ for any limit ordinal α [4].

P. Borst has introduced a transfinite extension of the covering dimension dim [5].

In this paper we prove that there is no universal space for separable metrizable spaces X with $\dim X \leq \alpha$ for any limit ordinal α. The proof is based on an idea of W. Olszewski.

The paper is organized as follows: in section 1 we give some known definitions and constructions, in section 2 we prove two lemmas, in section 3 we obtain the main result, and in section 4 we give an application to the theory of absorbing sets.

1. Let L be an arbitrary set. By $\operatorname{Fin} L$ we shall denote the collection of all finite, non-empty subsets of L. Let M be a subset of $\operatorname{Fin} L$. For $\sigma \in \{\varnothing\} \cup \operatorname{Fin} L$ we put

$$M^\sigma = \{\tau \in \operatorname{Fin} L \mid \sigma \cup \tau \in M \text{ and } \sigma \cap \tau = \varnothing\}.$$

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Let M^a abbreviate $M^{\{a\}}$.

Definition 1 ([5]). Define the ordinal number $\operatorname{Ord} M$ inductively as follows
$\operatorname{Ord} M = 0$ iff $M = \varnothing$,
$\operatorname{Ord} M \leq \alpha$ iff for every $a \in L$, $\operatorname{Ord} M^a < \alpha$,
$\operatorname{Ord} M = \alpha$ iff $\operatorname{Ord} M \leq \alpha$ and $\operatorname{Ord} M < \alpha$ is not true, and
$\operatorname{Ord} M = \infty$ iff $\operatorname{Ord} M > \alpha$ for every ordinal number α.

A finite sequence $\{(A_i, B_i)\}_{i=1}^m$ of pairs of disjoint closed sets in a space X is called *inessential* if we can find open sets O_i, $i = 1, \dots, n$ such that

$$A_i \subset O_i \subset \overline{O}_i \subset X \setminus B_i \text{ and } \bigcap_{i=1}^m \operatorname{Fr} O_i = \varnothing.$$

Otherwise it is called *essential*.

We have the following characterization of the covering dimension: $\dim X \leq n$ if and only if every sequence $\{A_i, B_i\}_{i=1}^{n+1}$ of pairs of disjoint closed sets in X is inessential [6; 3.2.6].

Let X be a space. Define

$$L(X) = \{(A, B) \mid A, B \subset X, \text{ closed, disjoint}\}$$

and

$$M_{L(X)} = \{\sigma \in \operatorname{Fin} L(X) \mid \sigma \text{ is essential in } X\}.$$

Definition 2 ([5]). For a space X we set $\dim X = \operatorname{Ord} M_{L(X)}$.

Let us recall that $\dim X$ exists if and only if X is weakly infinite-dimensional [5].

For an ordinal number α, we let $\lambda(\alpha)$ be a limit ordinal or zero and $n(\alpha)$ be finite ordinal such that $\alpha = \lambda(\alpha) + n(\alpha)$. We will use a shorter notation $\alpha = \lambda + n$ if it will cause no confusion.

We will need the construction of compacta D_λ^σ from [4].

Here $\lambda < \omega_1$ is a limit ordinal. Let Λ denote the set of all non-limit ordinals α such that $0 < \alpha < \lambda$ and let $\sigma \colon \mathbb{N} \to \Lambda$ be a sequence.

We consider the standart metric d on the square I^2.

Let $L_n = \{(i, n) \mid i = 1, \dots, 2^n\}, n = 1, 2, \dots$ and $L = \{0\} \cup \bigcup_{n=1}^\infty L_n$.

Let $Z_0 = \{(x, y) \in I^2 \mid y = 0\}$ and, for each $s = (i, n) \in L_n$, let $a_s = ((2i - 1)/2^{n+1}, 0) \in I^2$ and $Z_s = \{z \in I^2 \mid d(z, a_s) = 1/2^{n+1}\}$.

The set $Z = \bigcup\{Z_s \mid s \in L\}$ is a closed subspace of I^2.

Let $z_0 = (0, 0)$, $z_1 = (1, 0)$ and $z_i^n = (i/2^n, 0)$ for $n \in \mathbb{N}$ and $i = 0, 1, \dots, 2^n$.

Let S_α be the Smirnov cube [2].

Let α be an ordinal, $\alpha = \lambda + n$. We denote by a^λ the limit point in S_λ. Put $s_1^\alpha = \{a^\lambda\} \times \{(0, \dots, 0)\}$, $s_2^\alpha = \{a^\lambda\} \times \{(0, \dots, 0, 1)\} \in S_\alpha$.

Since $\sigma(n) \in \Lambda$, we have $S_{\sigma(n)} = S_{\lambda(n)} \times I^{k(n)}$, where $\lambda(n)$ is a limit ordinal. Set $X_s = S_{\sigma(n)-1}$ and $b_s = (a^{\lambda(n)}, 0, \dots, 0) \in S_{\sigma(n)-1}$ for $s \in L_n$. We identify Z with the subspace of $Z \times \prod\{X_s \mid s \in L \setminus \{0\}\}$ consisting of all points $\bigl(x_0, \{x_s \mid s \in L \setminus \{0\}\}\bigr)$ such that $x_s = b_s$ for all $s \in L \setminus \{0\}$. For $t \in L \setminus \{0\}$, let D_t be the subspace of the Cartesian product consisting of all $\bigl(x_0, \{x_s \mid s \in L \setminus \{0\}\}\bigr)$ such that $x_0 \in Z_t$ and $x_s = b_s$ for $s \neq t$.

Since Z_t is an arc, D_t is homeomorphic to $S_{\sigma(n)-1} \times I = S_{\sigma(n)}$; moreover, there exists a homeomorphism h of D_t onto $S_{\sigma(n)}$ with $h(z^n_{i-1}) = s^\alpha_1$, $h(z^n_i) = s^\alpha_2$, where $t = (i, n)$.

The space $D^\sigma_\lambda = Z \cup \left(\bigcup\{D_t \mid t \in L \setminus \{0\}\}\right)$ is a closed subset of $Z \times \prod\{X_s \mid s \in L \setminus \{0\}\}$.

It is shown in [4] that $\operatorname{Ind} D^\sigma_\lambda \leq \lambda$. Since $\dim X \leq \operatorname{Ind} X$ [5], we have $\dim D^\sigma_\lambda \leq \lambda$.

2. Let $\alpha = \lambda + n$ be a non-limit ordinal. Then we have $S_\alpha = S_\lambda \times I^n$. Put $A_i = S_\lambda \times I^{i-1} \times \{0\} \times I^{n-i}$ and $B_i = S_\lambda \times I^{i-1} \times \{1\} \times I^{n-i}$ where $i \in \{1, \dots, n\}$. Let $\nu = \{(A_1, B_1), \dots, (A_n, B_n)\}$.

Lemma 1. *For each $\gamma \subset \nu$ such that $|\gamma| = k \leq n$ we have* $\operatorname{Ord} M^\gamma_{L(S_\alpha)} \geq \lambda + n - k$.

Proof. For $\lambda = 0$ the lemma is a well-known result of dimension theory [6]. We will prove the lemma by transfinite induction for limit ordinals λ. Assume that we have proved the lemma for all limit ordinals $\beta < \lambda$.

Suppose the contrary: there exists a set $\gamma \subset \nu$ such that $|\gamma| = k \leq n$ and $\operatorname{Ord} M^\gamma_{L(S_\alpha)} < \lambda + n - k$. Then choose $i \in \{1, \dots, n\}$ such that $(A_i, B_i) \notin \gamma$. We have

$$\operatorname{Ord} M^{\gamma \cup \{(A_i,B_i)\}}_{L(S_\alpha)} = \operatorname{Ord}(M^\gamma_{L(S_\alpha)})^{(A_i,B_i)} < \lambda + n - k - 1.$$

Continuing we obtain $\operatorname{Ord} M^\nu_{L(S_\alpha)} = \beta < \lambda$. Since λ is a limit ordinal, we have $\beta + 1 \leq \lambda$. We have $S_{\beta+1} \subset S_\lambda$, hence $S_{\beta+n+1} = S_{\beta+1} \times I^n \subset S_\alpha$. Then we have $\operatorname{Ord} M_{L(S_{\beta+n+1})})^{\nu|S_{\beta+n+1}} \leq \operatorname{Ord} M^\nu_{L(S_\alpha)} = \beta$. But $A_i|S_{\beta+1} \times I^n = S_{\beta+1} \times I^{i-1} \times \{0\} \times I^{n-i}$, $B_i|S_{\beta+1} \times I^n = S_{\beta+1} \times I^{i-1} \times \{1\} \times I^{n-i}$ and, by inductive assumption,

$$\operatorname{Ord} M^{\nu|S_{\beta+n+1}}_{L(S_{\beta+n+1})} \geq \beta + 1 + n - n = \beta + 1.$$

We have obtained the contradiction.

Since $\dim S_\alpha \leq \operatorname{Ind} S_\alpha = \alpha$, we have:

Corollary. *For each countable ordinal α we have* $\dim S_\alpha = \alpha$.

Lemma 2. *Let (Y, d) be a totally bounded metric space with $\dim Y \leq \beta$ where β is a limit ordinal. Then for each $\varepsilon > 0$ there exists an ordinal $\nu < \beta$ such that for each non-limit ordinal α and embedding $i\colon S_\alpha \to Y$ with $d(i(s^\alpha_1), i(s^\alpha_2)) \geq \varepsilon$ we have $\alpha \leq \nu$.*

Proof. Let $O_1, \dots, O_n$ be an open finite cover of Y with $\operatorname{diam} O_i < \frac{\varepsilon}{6}$. Put $F_i = \operatorname{cl} O_i$ and $\mathcal{B} = \{(i,j) \in \{1, \dots, n\} \times \{1, \dots, n\} \mid F_i \cap F_j = \varnothing\}$. Since $\dim Y = \beta$, we have

$$\operatorname{Ord} M^{(F_i,F_j)}_{L(Y)} = \beta_{ij} < \beta$$

for each $(i,j) \in \mathcal{B}$. Put $\nu = \max\{\beta_{ij} \mid (i,j) \in \mathcal{B}\} + 1$. Since β is a limit ordinal, we have $\nu < \beta$.

Let α be a non-limit ordinal, $\alpha = \lambda + n$ and there exists an embedding $i\colon S_\alpha \to Y$ with $d(i(s^\alpha_1), i(s^\alpha_2)) \geq \varepsilon$. Then there exists $(i,j) \in \mathcal{B}$ such that $i(s^\alpha_1) \in O_i$ and $i(s^\alpha_2) \in O_j$. We can assume that $i(A_n) \subset O_i$ and $i(B_n) \subset O_j$. Then we have

$$\alpha - 1 \leq \operatorname{Ord} M^{(A_1,B_1)}_{L(S_\alpha)} \leq \operatorname{Ord} M^{(F_i,F_J)}_{L(Y)} = \beta_{ij} \leq \nu - 1.$$

Hence $\alpha \leq \nu$. The lemma is proved.

3. Theorem 1. *Let $\lambda < \omega_1$ be a limit ordinal. There exists no universal space for separable metrizable spaces X with $\dim X \le \lambda$.*

Proof. Assume the contrary. Let Y be a universal space for separable metrizable spaces X with $\dim X \le \lambda$. Let ρ be an arbitrary totally bounded metric on X. It follows from Lemma 2 that for every $n \in \mathbb{N}$ there exists an ordinal $\alpha_n < \lambda$ such that for each non-limit ordinal α and embedding $i: S_\alpha \to Y$ with $d(i(s_1^\alpha), i(s_2^\alpha)) \ge 2^{-2n}$ we have $\alpha \le \alpha_n$.

Set $\sigma(n) = \alpha_n + 2$ for $n \in \mathbb{N}$. Since Y is a universal space and $\dim D_\lambda^\sigma \le \lambda$, there exists an embedding $h: D_\lambda^\sigma \to Y$. Since there exists a homeomorphism j of D_s, $s \in L_n$ onto $S_{\sigma(n)}$ with $j(s_1^{\sigma(n)}) = z_{i-1}^n$, $j(s_2^{\sigma(n)}) = z_i^n$, it follows from the choice of σ and from Lemma 2 that $\rho(h(z_{i-1}^n), h(z_i^n)) < 2^{-2n}$ for each $(i,n) \in S_n$.

Hence, by the triangle inequality

$$\rho(h(z_0), h(z_1)) \le \sum_{i=1}^{2^n} \rho(h(z_{i-1}^n), h(z_i^n)) < 2^n 2^{-2n} = 2^{-n}.$$

Since n is an arbitrary natural number, we conclude that $h(z_0) = h(z_1)$, which contradicts to the assumption that h is a homeomorphism. The theorem is proved.

Remark 1. In Theorem 1 we have proved a stronger statement: for each limit ordinal λ there exists no metrizable separable space Y with $\dim Y \le \lambda$ which contains all metrizable compacta X with $\dim X \le \lambda$.

4. Recall briefly some necessary definitions of the theory of absorbing sets (see [7] for details).

Two maps $f, g: X \to Y$ are said to be $\mathcal{U}$-close, where $\mathcal{U}$ is a cover of Y, if for each $x \in X$ the set $\{f(x), g(x)\}$ is contained in an element of $\mathcal{U}$.

A closed subset X of Y is called a Z-set if every open cover $\mathcal{U}$ of Y there exists the map $f: Y \to Y$ which is $\mathcal{U}$-close to Id_Y and $f(Y) \cap X = \varnothing$. If additionally $f(Y)$ is closed in Y we say that X is a strong Z-set in Y. An embedding into Y is called a Z-embedding if its image is a Z-set in Y.

Let $\mathcal{C}$ be a class of spaces. A space X is strongly $\mathcal{C}$-universal if for every map $f: C \to X$ from a space $C \in \mathcal{C}$, for every closed subset $D \subset C$ such that $f|D: D \to X$ is a Z-embedding and for every $\mathcal{U} \in \mathrm{cov}(X)$, there exists a Z-embedding $h: C \to X$ such that $h|D = f|D$ and h is $\mathcal{U}$-close to f.

Finally, a space $X \in AR$ is called $\mathcal{C}$-absorbing set if $X = \bigcup_{i=1}^\infty X_i$ where each X_i is a strong Z-set in X, $X_i \in \mathcal{C}$, and X is strongly $\mathcal{C}$-universal.

We denote by $\mathcal{M}_\alpha$ (respectively $\mathcal{A}_\alpha$) the absolute multiplicative (respectively, additive) Borel class of order α (see [7]).

For each ordinal ξ and each class $\mathcal{C}$ of spaces let $\mathcal{C}(\dim, \xi) = \{X \in \mathcal{C} \mid \dim X \le \xi\}$.

Theorem 2. *For each countable limit ordinal ξ there is no $\mathcal{C}(\dim, \xi)$-absorbing set, where $\mathcal{C} \in \{\mathcal{M}_\alpha \mid \alpha < \omega_1\} \cup \{\mathcal{A}_\alpha \mid \alpha < \omega_1\}$.*

Proof. Assume the contrary and let X be a $\mathcal{C}(\dim, \xi)$-absorbing set. Given a representation $X = \bigcup_{i=1}^\infty X_i$, where $X_i \in \mathcal{C}(\dim, \xi)$ are closed subsets of X, take compactifications $Y_i \in \mathcal{C}(\dim, \xi)$ of X_i [8] and define X' as a one-point compactification of the disjoint topological sum of Y_i. It is easy to see that $\dim X' \le \xi$.

Now let Y be a compact space with $\dim Y \leq \xi$. Applying a construction of V. Chatyrko (see [9]) we obtain a compact space $\widehat{Y}$ satisfying the properties:

(i) $\dim \widehat{Y} \leq \xi$;

(ii) each open subset of $\widehat{Y}$ contains a copy of Y.

Since $\widehat{Y} \in \mathcal{C}(\dim, \xi)$, there exists an embedding $\widehat{Y} \to X$. By Baire Category Theorem, some X_i contains a copy of Y and, consequently, X' contains a copy of Y.

Thus, X' contains a copy of each compactum of dimension $\leq \xi$ which contradicts to Remark 1. Theorem is proved.

Remark 2. The same arguments applied to Olszewski's result show that there are no absorbing sets in the classes $\mathcal{C}(\mathrm{Ind}, \xi) = \{X \in \mathcal{C} \mid \mathrm{Ind}\, X \leq \xi\}$, for every countable limit ordinal ξ.

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