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VECTORS OF EXPONENTIAL TYPE OF OPERATORS WITH DISCRETE SPECTRUM

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In this paper root vectors of a linear operator on Banach space with discrete spectrum are described in terms of vectors of exponential type. An application in the investigation of analyticity properties of the root functions of elliptic boundary value problems is given.

1. Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space over the field of complex numbers $\mathbb{C}$, $A\colon \mathcal{D}(A) \subset \mathcal{B} \to \mathcal{B}$ a closed unbounded linear operator with a dense domain $\mathcal{D}(A)$ in $\mathcal{B}$. Further, let $\sigma(A)$ be a spectrum of A, $(\xi I - A)^{-1}$ its resolvent determined for all $\xi \in \mathbb{C}\setminus\sigma(A)$ and I the unit operator on $\mathcal{B}$.

It is said (see e.g. [1]) that A is an operator with discrete spectrum, whenenever:

a) the spectrum $\sigma(A)$ is a sequence of eigenvalues $\{\xi_j\}_{j=1}^{\infty}$ of the operator A with a unique boundary point at infinity;

b) to every number $\xi_j \in \sigma(A)$ the finite-dimensional root subspace

$$\mathcal{R}_j(A) \equiv \big\{x \in \mathcal{B} : (\xi_j I - A)^{n_j} x = 0\big\},$$

is corresponded, where n_j is the index of the eigenvalue ξ_j ([2], VII, 1).

From the sequence of direct sums $\big\{\bigoplus_{j=1}^{n} \mathcal{R}_j(A)\big\}_{n=1}^{\infty}$ of the operator A with a discrete spectrum let's form the inductive limit

$$\mathcal{R}(A) \equiv \bigcup_{n=1}^{\infty} \bigoplus_{j=1}^{n} \mathcal{R}_j(A) = \lim_{n\to+\infty} \operatorname{ind} \bigoplus_{j=1}^{n} \mathcal{R}_j(A).$$

Following [3, 4], for an arbitrary number $\nu > 0$ we define the Banach space $\mathcal{D}^{\nu}(A) \equiv \big\{x \in \mathcal{D}(A) : \|x\|_{\nu} < \infty\big\}$ with the norm

$$\|x\|_{\nu} \equiv \sum_{k=0}^{\infty} \frac{\|A^k x\|}{\nu^k}.$$

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Every space $\mathcal{D}^\nu(A)$ is invariant with respect to the operator A and for the corresponding restriction $A_\nu \equiv A|\mathcal{D}^\nu(A)$ the inequality $\|A_\nu x\|_\nu \le \nu\|x\|_\nu$ holds for all $x \in \mathcal{D}^\nu(A)$.

In the case $\nu \le \gamma$ the inclusions $\mathcal{D}^\nu(A) \subset \mathcal{D}^\gamma(A)$ are continuous and we can determine the inductive limit

$$\mathcal{D}^{\{1\}}(A) \equiv \bigcup_{\nu>0} \mathcal{D}^\nu(A) = \liminf_{\nu\to+\infty} \mathcal{D}^\nu(A).$$

In addition, the spectrum of the restriction $A_{\{1\}} \equiv A|\mathcal{D}^{\{1\}}(A)$ is decomposed into union $\sigma(A_{\{1\}}) = \bigcup_{\nu>0}\sigma(A_\nu)$, where $\sigma(A_\nu)$ is a spectrum of A_ν on $\mathcal{D}^\nu(A)$.

Lemma 1. *If A is an operator with discrete spectrum then the equality $\sigma(A) = \sigma(A_{\{1\}}) = \bigcup_{\nu>0}\sigma(A_\nu)$ holds.*

Proof. For an arbitrary eigenvalue $\xi_j \in \sigma(A)$ the corresponding eigenvector x_j belongs to every space $\mathcal{D}^\nu(A)$, as $\nu > |\xi_j|$. From this we arrive at the inclusion $\sigma(A) \subset \sigma(A_{\{1\}})$.

On the other hand, for arbitrary $\xi \in \mathbb{C}\setminus\sigma(A)$ we have

$$\|(\xi I-A)^{-1}x\|_\nu = \sum_{k=0}^{\infty}\frac{\|A^k(\xi I-A)^{-1}x\|}{\nu^k} = \sum_{k=0}^{\infty}\frac{\|(\xi I-A)^{-1}A^kx\|}{\nu^k} \le \|(\xi I-A)^{-1}\|\|x\|_\nu.$$

Previous relations are correct, since the space $\mathcal{D}^\nu(A)$ belongs to the space $\mathcal{C}(A^\infty)$ of smooth vectors of the operator A (see [3] for a definition) and the equality $A^k(\xi I-A)^{-1}x = (\xi I-A)^{-1}A^kx$ holds for all $x \in \mathcal{C}(A^\infty)$.

Then we obtain $\xi \in \mathbb{C}\setminus\sigma(A_{\{1\}})$, i.e., $\mathbb{C}\setminus\sigma(A) \subset \mathbb{C}\setminus\sigma(A_{\{1\}})$ or $\sigma(A_{\{1\}}) \subset \sigma(A)$. The lemma is proved. •

Theorem 1. *For an arbitrary number $\nu > 0$ the equality $\mathcal{D}^\nu(A) = \bigoplus_{|\xi_j|<\nu}\mathcal{R}_j(A)$ holds. In particular, the spaces $\mathcal{D}^\nu(A)$ are finite-dimensional.*

Proof. For $\xi \in \mathbb{C}\setminus\sigma(A)$ the inequality $\|(\xi I-A)^{-1}x\|_\nu \le \|(\xi I-A)^{-1}\|\|x\|_\nu$ is valid and the relations $(\xi I-A)^{-1}(\xi I-A)x = (\xi I-A)(\xi I-A)^{-1}x = x$ hold for all $x \in \mathcal{D}^\nu(A)$. Therefore, $(\xi I_\nu - A_\nu)^{-1} = (\xi I-A)^{-1}|\mathcal{D}^\nu(A)$ is the resolvent of the operator A_ν, I_ν the unit operator in $\mathcal{D}^\nu(A)$.

Next, let

$$P_\nu = (2\pi i)^{-1}\oint_{\Gamma_\nu}(\xi I-A)^{-1}d\xi$$

be a Riesz projector, in which the closed contour Γ_ν separates the set points $\sigma(A_\nu)$ from the rest part of $\sigma(A)$. Since

$$I_\nu = (2\pi i)^{-1}\oint_{\Gamma_\nu}(\xi I_\nu - A_\nu)^{-1}d\xi,$$

i.e., $I_\nu = P_\nu|\mathcal{D}^\nu(A)$, the inclusion $\mathcal{D}^\nu(A) \subset P_\nu(\mathcal{B})$ holds. For an arbitrary eigenvalue $\xi_j \in \sigma(A_\nu)$ the value of the norm $\|x_j\|_\nu = \sum_{k=0}^{\infty}\left(\frac{|\xi_j|}{\nu}\right)^k\|x_j\|$ on the corresponding eigenvector is finite, and therefore $\nu > \max\{|\xi_j| \mid \xi_j \in \sigma(A_\nu)\}$. From this and Theorem 5.14.3 [1] we have $P_\nu(\mathcal{B}) = \bigoplus_{|\xi_j|<\nu}\mathcal{R}_j(A)$ and $\mathcal{D}^\nu(A) \subset \bigoplus_{|\xi_j|<\nu}\mathcal{R}_j(A)$.

On the contrary, since the spectral radius of the restriction $T_\nu \equiv A|P_\nu(\mathcal{B})$ of A on the subspace $P_\nu(\mathcal{B})$ is less than ν, i.e. $\lim\limits_{n\to+\infty}\|T_\nu^n\|^{1/n}<\nu$, we obtain

$$\sum_{n=0}^{\infty}\frac{\|A^n x\|}{\nu^n}\le\sum_{n=0}^{\infty}\frac{\|T_\nu^n\|\|x\|}{\nu^n}<\infty$$

and $\bigoplus_{|\xi_j|<\nu}\mathcal{R}_j(A)\subset\mathcal{D}^\nu(A)$. The theorem is proved. •

Corollary 1. *Under the conditions of Theorem 1, the following topological isomorphism $\mathcal{R}(A)\simeq\mathcal{D}^{\{1\}}(A)$ is valid. In particular, the space $\mathcal{D}^{\{1\}}(A)$ is nuclear.*

Proof. The isomorphism follows from the simple properties of inductive limits. The nuclearity is established by the following arguments. The strong adjoint space $\mathcal{D}^{-\{1\}}(A)$ to the inductive limit $\mathcal{D}^{\{1\}}(A)$ has the form of the projective limit

$$\mathcal{D}^{-\{1\}}(A)=\lim_{\nu\to+\infty}\operatorname{pr}\mathcal{D}^{-\nu}(A),$$

where $\mathcal{D}^{-\nu}(A)$ are the strong adjoint of $\mathcal{D}^\nu(A)$ finite-dimensional spaces. Therefore, $\mathcal{D}^{-\{1\}}(A)$ is nuclear, by the known property of such a class of spaces ([5], III, 7). The strong adjoint space of a nuclear space is nuclear as well ([5], IV, 9), and since it coincides with $\mathcal{D}^{\{1\}}(A)$, the proof is complete. •

Corollary 2. *A system of root subspaces of an operator A with discrete spectrum is complete if and only if the density condition $\overline{\mathcal{D}^{\{1\}}(A)}=\mathcal{B}$ holds.*

2. Describe the vectors of exponential type of differentiation operators.

Let Ω be a bounded region of class C^∞ in $\mathbb{R}^n$ ([6], 3.2.1), $1<p<\infty$ and $L_p(\Omega)$ a space of summable functions $u(t)=u(t_1,\dots,t_n)$ with the usual norm

$$\|u\|_{L_p(\Omega)}=\Big(\int_\Omega|u(t)|^p dt\Big)^{1/p}.$$

Consider in the space $L_p(\Omega)$ the differentiation operators $\partial^\alpha=\frac{\partial^{|\alpha|}}{\partial t_1^{\alpha_1}\dots\partial t_n^{\alpha_n}}$ of order $|\alpha|=\alpha_1+\dots+\alpha_n$, where $\alpha=(\alpha_1,\dots,\alpha_n)$ are integer vectors with nonnegative coordinates. Further, let $C^\infty(\overline{\Omega})$ be a space of infinite-differentiable functions defined in the closure of Ω. For every number $\nu>0$ we define the subspace $\mathcal{D}^\nu(\partial)=\big\{u\in C^\infty(\overline{\Omega}):\|u\|_\nu<\infty\big\}$ with the norm

$$\|u\|_\nu=\sum_{k=0}^{\infty}\frac{1}{\nu^k}\Big(\sum_{|\alpha|=k}\|\partial^\alpha u\|^p_{L_p(\Omega)}\Big)^{1/p}$$

and form the inductive limit

$$\mathcal{D}^{\{1\}}(\partial)\equiv\bigcup_{\nu>0}\mathcal{D}^\nu(\partial)=\lim_{\nu\to+\infty}\operatorname{ind}\mathcal{D}^\nu(\partial).$$

Show that $\mathcal{D}^{\{1\}}(\partial)$ coincides with the restrictions on Ω of the space of entire analytical functions of n complex variables of the exponential type.

Lemma 2. *If the entire analytical function $u(z) = u(z_1, \dots, z_n)$ satisfies the inequality*

$$|u(z)| = |u(t+is)| \le C_1 e^{a(\|t\|+\|s\|)}, \tag{1}$$

where $z_j = t_j + is_j$, $j = 1, \dots, n$, $\|t\| = \left(\sum_{j=1}^n |t_j|^2\right)^{1/2}$, $\|s\| = \left(\sum_{j=1}^n |s_j|^2\right)^{1/2}$, and $a > 0$, i.e., $u(z)$ has an exponential type, then for all $|\alpha| = k$, $k = 0, 1, \dots$ the inequality

$$|\partial^\alpha u(t)| \le C_2 B^k e^{a\|t\|}$$

holds, where the constants C_2 and B are independent of k.

Proof. The partial derivatives of the function $u(z)$ may be calculated by the Cauchy formula

$$\partial^\alpha u(t) = \frac{\alpha!}{(2\pi i)^n} \int\limits_{\Gamma_1} \cdots \int\limits_{\Gamma_n} \frac{u(\xi_1, \dots, \xi_n) d\xi_1 \dots d\xi_n}{(\xi_1 - t_1)^{\alpha_1+1} \dots (\xi_n - t_n)^{\alpha_n+1}}, \tag{2}$$

where $\alpha! = \alpha_1!\alpha_2! \dots \alpha_n!$, Γ_j is a circle in the plane ξ_j with the centre in t_j and the radius R. From (1) and (2) we obtain

$$|\partial^\alpha u(t)| \le \frac{\alpha!}{(2\pi)^n R^k} \int_0^{2\pi} \cdots \int_0^{2\pi} |u(t_1 + Re^{i\varphi_1}, \dots, t_n + Re^{i\varphi_n})| d\varphi_1 \dots d\varphi_n \le$$
$$\le \frac{\alpha! C_1}{R^k} e^{a(nR + \|t'\|)}, \tag{3}$$

where $\xi_j - t_j = Re^{i\varphi_j}$, $t' = (t'_1, \dots, t'_n)$, t'_j a point between the values $t_j - R$ and $t_j + R$, in which the value $a|t_j|$ attains its maximum.

Choose the radius R so that the value $\frac{\exp(anR)}{R^k}$ attains its minimum. It is easy to check that it takes place for $R = \frac{k}{an}$, so that inequality (3), by Stirling's formula, may be reduced to the form

$$|\partial^\alpha u(t)| \le C_3 (2an)^k e^{a\|t'\|}, \tag{4}$$

where $C_3 = \text{const}$ is independent of k. Consider the last factor in inequality (4). For this, replace t'_j by $t_j + \theta_j R$, where $|\theta_j| \le 1$. Then, by the equivalence of norms in $\mathbb{C}^n$, we have $e^{a\|t'\|} = \exp\left(a \sum_{j=1}^n \|t_j + \theta R\|\right) \le \exp\left(a \sum_{j=1}^n \|t_j\|\right) e^{na\theta R} \le C_4 e^{a\|t\|}$. At last we obtain the following estimate

$$|\partial^\alpha u(t)| \le C_2 B^k e^{a\|t\|},$$

where $B = 2na$. •

Lemma 3. *The space $\mathcal{D}^{\{1\}}(\partial)$ coincides with the space of entire functions of n complex variables of exponential type, which we denote by $\mathcal{E}\text{xp}(\mathbb{C}^n)$.*

Proof. For an arbitrary number $\nu > 0$ we define the subsidiary spaces of the form

$$\mathcal{E}^\nu = \left\{ u \in C^\infty(\overline{\Omega}) : \exists c = c(u, \nu) > 0;\ \sup_{t \in \Omega} |\partial^\alpha u(t)| \le c\nu^k;\ |\alpha| = k;\ k = 0, 1, \dots \right\} \tag{5}$$

and form their union $\mathcal{E} = \bigcup_{\nu > 0} \mathcal{E}^\nu$.

Let $u(t) \in \mathcal{D}^\nu(\partial)$ and $n < p$. By Sobolev's theorem,

$$\sup_{t\in\Omega} |u(t)| \le c_1 \max_{|\alpha|\le 1} \|\partial^\alpha u\|_{L_p(\Omega)}.$$

From this we get

$$\sup_{t\in\Omega} |\partial^\alpha u(t)| \le c_1 \max\{1,\nu\}\nu^k \|u\|_\nu \le c_0\nu^k$$

for all $|\alpha| = k$, $k = 0, 1, \ldots$. Hence, $\mathcal{D}^\nu(\partial) \subset \mathcal{E}^\nu$ and $\mathcal{D}^{\{1\}}(\partial) \subset \mathcal{E}$.

Let's show that the space $\mathcal{E}$ coincides with the space of entire functions of exponential type.

Let $u(t) \in \mathcal{E}^\nu$. Write down $u(t+h)$, where $h = (h_1, h_2, \ldots, h_n)$ and $t + h \in \Omega$ in the form

$$u(t+h) = u(t) + \frac{du(t)}{1!} + \frac{d^2u(t)}{2!} + \cdots + \frac{d^k u(t+\theta h)}{k!}, \qquad 0 \le \theta \le 1,$$

where $du(t)$ is a total differential of the function $u(t)$. A remainder of the series satisfies the condition

$$\Big|\frac{d^k u(t+\theta h)}{k!}\Big| = \Big|\sum_{|\alpha|=k} \frac{1}{\alpha!}\partial^\alpha u(t+\theta h)h^\alpha\Big| \le c\nu^k \sum_{|\alpha|=k} \frac{h^\alpha}{\alpha!} \to 0, \qquad k\to\infty,$$

where $h^\alpha = h_1^{\alpha_1} h_2^{\alpha_2} \ldots h_n^{\alpha_n}$. Therefore, the functions from $\mathcal{E}^\nu$ are analytical in Ω and are decomposed into the convergent series $\sum_\alpha \frac{1}{\alpha!}\partial^\alpha u(t)h^\alpha$. By estimate (5), for all $h \in \mathbb{C}^n$ such series determines the extension of $u(t)$ as an entire function on $\mathbb{C}^n$. From the same estimate for all $t \in \Omega$ and all $s = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ we have

$$|u(t+is)| \le \sum_\alpha \Big|\frac{\partial^\alpha u(t)}{\alpha!}(is)^\alpha\Big| \le \sum_\alpha \frac{c\nu^{|\alpha|}}{\alpha!}|s_1|^{\alpha_1}\cdot\ldots\cdot|s_n|^{\alpha_n} = ce^{\nu(|s_1|+\cdots+|s_n|)} \le c_2 e^{\nu\|s\|}.$$

Considering that $0 \in \Omega$, from the previous inequality, it follows that

$$|u(t+is)| = |u(z)| \le c_2 e^{\nu\|z\|} \tag{6}$$

for all $t, s \in \mathbb{R}^n$. Hence, $u(t) \in \mathcal{E}\mathrm{xp}(\mathbb{C}^n)$, i.e., $\mathcal{E} \subset \mathcal{E}\mathrm{xp}(\mathbb{C}^n)$.

On the contrary, let the entire function $u(z)$ satisfy estimate (6). Then, by Lemma 2, for $k = 0, 1, \ldots$ and $|\alpha| = k$

$$\begin{aligned} |\partial^\alpha u(t)| &\le C_2(2n\nu)^k e^{\nu\|z\|} \quad \text{for all} \quad t \in \mathbb{R}^n, \quad \text{or} \\ \sup_{t\in\Omega} |\partial^\alpha u(t)| &\le C_3(2n\nu)^k, \qquad k = 0, 1, \ldots \end{aligned} \tag{7}$$

by boundedness of the region Ω. It means that $u(z) \in \mathcal{E}^{2n\nu}$ and the equality $\mathcal{E} = \mathcal{E}\mathrm{xp}(\mathbb{C}^n)$ is proved.

It remains to show the inclusion $\mathcal{E} \subset \mathcal{D}^{\{1\}}(\partial)$. Indeed, by virtue of (7), we obtain $\|\partial^\alpha u\|_{L_p(\Omega)} \le c_3(2n\nu)^k$ and $\sum_{|\alpha|=k} \|\partial^\alpha u\|_{L_p(\Omega)} \le c_3(2n^2\nu)^k$, since the quantity of members of the sum on the left-hand side is equal to n^k. Further, from the inequality

$$\sum_{k=0}^{\infty} \frac{1}{(4n^2\nu)^k} \Big(\sum_{|\alpha|=k} \|\partial^\alpha u\|^p_{L_p(\Omega)} \Big)^{1/p} \le 2 \sup_k \frac{\big(\sum_{|\alpha|=k} \|\partial^\alpha u\|^p_{L_p(\Omega)}\big)^{1/p}}{(2n^2\nu)^k}$$

it follows that $u \in \mathcal{D}^{4n^2\nu}(\partial)$. Since $\mathcal{D}^{\{1\}}(\partial) = \bigcup_{\nu>0} \mathcal{D}^\nu(\partial)$, we have $u \in \mathcal{D}^{\{1\}}(\partial)$ and the lemma is proved. •

3. Using Theorem 1 and the results from section 2 we will show that the root functions of elliptic boundary value problems in a bounded region under the constant coefficients of equation and arbitrary coefficients in the boundary conditions assume the extension in the complex space to the entire functions of exponential type.

Let a collection of the differential operator

$$(Lu)(t) = \sum_{|\alpha|\le 2m} a_\alpha(t)\partial^\alpha u(t), \qquad a_\alpha(t) \in C^\infty(\overline{\Omega})$$

and the boundary operators

$$(B_j u)(t) = \sum_{|\alpha|\le m_j} b_{j,\alpha}(t)\partial^\alpha u(t), \qquad b_{j,\alpha}(t) \in C^\infty(\Gamma),$$

$j = 1, \dots, m$, $0 \le m_1 \le \dots \le m_j \le 2m-1$, where Γ is a boundary of Ω, be given. Assume that this collection is regular elliptic in a sense of the definition 4 from ([6], 5.2.1).

Consider in $L_p(\Omega)$ the closed operator A defined by the relations

$$Au = Lu \quad \text{and} \quad \mathcal{D}(A) = W^{2m}_{p,\{B_j\}}(\Omega),$$

where $W^{2m}_{p,\{B_j\}}(\Omega) = \big\{u : u \in W^{2m}_p(\Omega);\quad B_j u|\Gamma = 0;\quad j = 1, \dots, m\big\}$ and $W^{2m}_p(\Omega)$ is the Sobolev space of $2m$-th order.

Theorem 2. *The following equality*

$$\mathcal{D}^{\{1\}}(A) = \liminf_{\nu\to+\infty} \mathcal{D}^\nu(L, B_j), \tag{8}$$

is valid, where $\mathcal{D}^\nu(L, B_j) =$

$$= \Big\{u \in W^{2m}_p(\Omega) : \sum_{k=0}^{\infty} \frac{\|L^k u\|_{L_p(\Omega)}}{\nu^k} < \infty;\ B_j L^k u|\Gamma = 0;\ j = 1, \dots, m;\ k = 0, 1, \dots\Big\}.$$

Proof. By Theorem 5.4.1 [6],

$$\bigcap_{k=0}^{\infty} D(L^k) = \bigcap_{k=0}^{\infty} W^{2mk}_p(\Omega) = C^\infty(\overline{\Omega}).$$

Further, by Theorem 1 ([6], 5.4.4), the space of smooth vectors $\mathcal{C}(A^\infty)$ of the operator A coincides with the closed subspace

$$C^\infty_{L,B_j}(\overline{\Omega}) = \{u : u \in C^\infty(\overline{\Omega});\ B_j L^k u| = 0;\ j = 1,\dots,m;\ k = 0,1,\dots\}$$

of the locally convex space $C^\infty(\overline{\Omega})$ endowed with the topology which is determined by the seminorms $\sup_{t\in\Omega} |\partial^\alpha u(t)|, 0 \le |\alpha| < \infty$. From this and from the definition of the space $\mathcal{D}^{\{1\}}(A)$ we obtain (8). •

Let the resolvent set $\rho(A)$ of A be nonempty and $0 \in \rho(A)$. Then the following statement holds.

Theorem 3. *If the coefficients $a_\alpha(t)$ of the operator L are constants, then*

$$\mathcal{R}(A) = \{u : u \in \mathcal{D}^{\{1\}}(\partial), B_j L^k u|\Gamma = 0, j = 1,\dots,m, k = 0,1,\dots\}.$$

Proof. By Theorem 5.4.3 [6], there exist positive numbers c_1 and $c_1^{'}$ such that

$$c_1\|u\|_{W_p^{2m}(\Omega)} \le \|Au\|_{L_p(\Omega)} \le c_1^{'}\|u\|_{W_p^{2m}(\Omega)} \quad \text{for all} \quad u \in D(A).$$

Further, suppose that for some $i > 1$ there exist positive numbers c_i and $c_i^{'}$ such that

$$c_i\|u\|_{W_p^{2mi}(\Omega)} \le \|A^i u\|_{L_p(\Omega)} \le c_i^{'}\|u\|_{W_p^{2mi}(\Omega)}, \quad u \in D(A^i). \tag{10}$$

Then

$$\|A^{i+1}u\|_{L_p(\Omega)} = \|A^i(Au)\|_{L_p(\Omega)} \le c_i^{'}\|Au\|_{W_p^{2mi}(\Omega)} = c_i^{'}\Big(\sum_{|\alpha|\le 2mi} \|\partial^\alpha Au\|^p_{L_p(\Omega)}\Big)^{1/p} =$$

$$= c_i^{'}\Big(\sum_{|\alpha|\le 2mi} \|A\partial^\alpha u\|^p_{L_p(\Omega)}\Big)^{1/p} \le c_i^{'}c_1^{'}\Big(\sum_{|\alpha|\le 2mi} \|\partial^\alpha u\|^p_{W_p^{2m}(\Omega)}\Big)^{1/p} = c_{i+1}^{'}\|u\|_{W_p^{2m(i+1)}(\Omega)},$$

$$\|A^{i+1}u\|_{L_p(\Omega)} = \|A^i(Au)\|_{L_p(\Omega)} \ge c_i\|Au\|_{W_p^{2mi}(\Omega)} = c_i\Big(\sum_{|\alpha|\le 2mi} \|\partial^\alpha Au\|^p_{L_p(\Omega)}\Big)^{1/p} =$$

$$= c_i\Big(\sum_{|\alpha|\le 2mi} \|A\partial^\alpha u\|^p_{L_p(\Omega)}\Big)^{1/p} \ge c_i c_1\Big(\sum_{|\alpha|\le 2mi} \|\partial^\alpha u\|^p_{W_p^{2m}(\Omega)}\Big)^{1/p} = c_{i+1}\|u\|_{W_p^{2m(i+1)}(\Omega)}.$$

Thus, inequalities (10) are valid for all $i \ge 1$ and the following equalities for coefficients $c_i^{'} = (c_1^{'})^i$, $c_i = (c_1)^i$ hold.

Let $u \in \mathcal{D}^\nu_0(\partial) = \{u : u \in \mathcal{D}^\nu(\partial);\ B_j L^k u|\Gamma = 0;\ j = 1,\dots,m;\ k = 0,1,\dots\}$ and $\nu > 1$. By Theorem 3.2.5 [6], the formula

$$\|u\|^*_{W_p^{2mk}(\Omega)} = \Big(\sum_{|\alpha|=2mk} \|\partial^\alpha u\|^p_{L_p(\Omega)} + \|u\|^p_{L_p(\Omega)}\Big)^{1/p}$$

gives an equivalent norm in the space $W_p^{2mk}(\Omega)$. From this and inequalities (10) we obtain

$$\|A^k u\|_{L_p(\Omega)} \leq c_k^{'}\Big(\sum_{|\alpha|=2mk}\|\partial^\alpha u\|^p_{L_p(\Omega)} + \|u\|^p_{L_p(\Omega)}\Big)^{1/p} \leq$$
$$\leq (c_1^{'})^k\Big(\sum_{|\alpha|=2mk}\|\partial^\alpha u\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)}\Big) \leq$$
$$\leq (c_1^{'})^k\Big(n^{2mk}\Big(\sum_{|\alpha|=2mk}\|\partial^\alpha u\|^p_{L_p(\Omega)}\Big)^{1/p} + \|u\|_{L_p(\Omega)}\Big),$$

$$\sum_{k=0}^{\infty}\frac{\|A^k u\|_{L_p(\Omega)}}{(c_1^{'}(n\nu)^{2m})^k} \leq c^{'}\sum_{k=0}^{\infty}\frac{1}{\nu^{2mk}}\Big(\sum_{|\alpha|=2mk}\|\partial^\alpha u\|^p_{L_p(\Omega)}\Big)^{1/p} \leq c^{'}\sum_{k=0}^{\infty}\frac{1}{\nu^k}\Big(\sum_{|\alpha|=k}\|\partial^\alpha u\|^p_{L_p(\Omega)}\Big)^{1/p}.$$

Thus, the inclusion

$$\mathcal{D}_0^\nu(\partial) \subset \mathcal{D}^\gamma(A) \tag{11}$$

holds, where $\gamma = c_1^{'}(n\nu)^{2m}$.

Let $u \in \mathcal{D}^\nu(A)$. Since $\|u\|_{W_p^{2mk}(\Omega)} = \big(\sum_{|\alpha|\leq 2mk}\|\partial^\alpha u\|^p_{L_p(\Omega)}\big)^{1/p}$, we obtain

$$\|A^k u\|_{L_p(\Omega)} \geq c_k\Big(\sum_{|\alpha|=k}\|\partial^\alpha u\|^p_{L_p(\Omega)}\Big)^{1/p} = (c_1)^k\Big(\sum_{|\alpha|=k}\|\partial^\alpha u\|^p_{L_p(\Omega)}\Big)^{1/p},$$

$$\sum_{k=0}^{\infty}\frac{\|A^k u\|_{L_p(\Omega)}}{\nu^k} \geq \sum_{k=0}^{\infty}\frac{1}{(\nu c_1^{-1})^k}\Big(\sum_{|\alpha|=k}\|\partial^\alpha u\|^p_{L_p(\Omega)}\Big)^{1/p}.$$

From this there follows the inclusion

$$\mathcal{D}^\nu(A) \subset \mathcal{D}_0^{\nu c_1^{-1}}(\partial). \tag{12}$$

Because $\mathcal{D}^{\{1\}}(\partial) = \bigcup_{\nu>0}\mathcal{D}^\nu(\partial)$ and $\mathcal{D}^{\{1\}}(A) = \bigcup_{\nu>0}\mathcal{D}^\nu(A)$, by (11), (12) and Theorem 1, we arrive at (9). The theorem is proved. •

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