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A NEW APPLICATION OF FUCHS-HAYMAN'S FUNCTION

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In this paper we use Fuchs-Hayman's function to solve a generalized "narrow" inverse problem of the Nevanlinna's theory in the class of analytic in the closed half-plane functions.

I. In 1962 W.H.J. Fuchs and W.K. Hayman [6] constructed the entire function $F(z)$ of infinite order with given finite deficient values and deficiencies satisfing the condition $\sum_n \delta_n \le 1$. In 1976 M.O. Girnyk [2] showed that the function $F\big(\frac{1+z}{1-z}\big)$ was a solution of the "narrow" inverse problem of Nevanlinna's theory in the class of analytic in the unit disc functions. Later on E.D. Fineberg [4] solved the same problem in the class of analytic in the half-plane functions of infinite order in terms of Tsuji characteristics. She also used the function $F(z)$.

Tsuji's characteristics and Nevanlinna's characteristics are usually used in the value distribution theory of functions which are analytic or meromorphic in the half-plane. Tsuji's characteristics naturally define the deficient values and the deficiencies in sense of Tsuji. Introduce these characteristics as in [1]:

$$\mathfrak{m}(r,f)=\frac{1}{2\pi}\int\limits_{\varkappa(r)}^{\pi-\varkappa(r)}\log^+|f(r\sin\theta e^{i\theta})|\frac{d\theta}{r\sin^2\theta},\quad \mathfrak{m}(r,a,f)\equiv\mathfrak{m}\Big(r,\frac{1}{f-a}\Big),$$

$$\mathfrak{N}(r,f)=\int\limits_1^r\frac{\mathfrak{n}(t,f)dt}{t^2},\quad \mathfrak{N}(r,a,f)\equiv\mathfrak{N}\Big(r,\frac{1}{f-a}\Big),$$

where $r>1$, $a\in\mathbb{C}$, $\varkappa(r)=\arcsin\frac{1}{r}$, $\mathfrak{n}(t,f)$ is a number of poles of the function $f(z)$ which belong to the set $\{z:|z|>1, |z-\frac{it}{2}|\le\frac{t}{2}\}$. Finally, $\mathfrak{T}(r,f)=\mathfrak{m}(r,f)+\mathfrak{N}(r,f)$. Further, we put

$$\rho_T[f]=\varlimsup_{r\to\infty}\frac{\log\mathfrak{T}(r,f)}{\log r},\quad \delta_T(a,f)=\varliminf_{r\to\infty}\frac{\mathfrak{m}(r,a,f)}{\mathfrak{T}(r,f)}.$$

Values $\rho_T[f]$ and $\delta_T(a,f)$ are called the order and the deficiency in the point a in the sense of Tsuji respectively.

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In 1981 M.O. Girnyk [3] proved a generalization of the defect relation

$$\sum_{a\neq\infty}\delta(a,f)+\sum_{k=1}^{\infty}\sum_{a\neq 0,\infty}\delta(a,f^{(k)})\leq 1 \tag{1.1}$$

which holds in the subclass of entire functions that satisfy the condition $\log T(2r,f^{(k)}) = o(T(r,f^{(k)})$ as $r\to\infty$, $k\in\mathbb{Z}_+$. Sharpness of relation (1.1) was shown.

In first part of this paper we shall prove the analog of Girnyk's result for the case of analytic in the closed half-plane functions. In the second part we generalize Fineberg's result.

Consider the class $\mathcal{K}$ of analytic in $\overline{\mathbb{C}}_+=\{z:\operatorname{Im} z\geq 0\}$ functions f that satisfy the condition

$$\log\mathfrak{T}(2r,f^{(k)})=o\big(\mathfrak{T}(r,f^{(k)})\big),\quad r\to\infty,\ k\in\mathbb{Z}_+. \tag{1.2}$$

If $f\in\mathcal{K}$ then (compare with [1], p.128)

$$\sum_{a\neq\infty}\delta_T(a,f^{(k)})\leq\delta_T(0,f^{(k+1)}),\quad k\in\mathbb{Z}_+. \tag{1.3}$$

Inequality (1.3) gives

$$(\forall f\in\mathcal{K})\quad \sum_{a\neq\infty}\delta_T(a,f)+\sum_{k=1}^{\infty}\sum_{a\neq 0,\infty}\delta_T(a,f^{(k)})\leq 1. \tag{1.4}$$

Sharpness of inequality (1.3) is established by the following theorem.

Theorem 1. *Let $(a_{k,n})$ be an arbitrary sequence of finite complex numbers such that $a_{k,i}\neq a_{k,j}$ as $j\neq i$, $a_{k,n}\neq 0$ as $k\geq 1$, $k\in\mathbb{Z}_+$, $1\leq n\leq N_k\leq\infty$, $(\delta_{k,n})$ a sequence of positive numbers such that $\sum_{k=0}^{\infty}\sum_{n=1}^{N_k}\delta_{k,n}\leq 1$. There is a function $f\in\mathcal{K}$ such that $\rho_T[f]=\infty$, $\delta_T(a_{k,n},f^{(k)})=\delta_{k,n}$ and $f^{(k)}$ has no other finite deficient values.*

To prove Theorem 1 we need the following lemma (analog of Lemma 4.3 [6], p.84).

Lemma 1. *Let $\varphi(t)$ be a non-negative, bounded, local-integrable on positive ray function such that*

$$\lim_{r\to+\infty}\frac{1}{r}\int_0^r\varphi(t)dt=l.$$

Then

$$I(r)=\frac{1}{\pi}\int_{\varkappa(r)}^{\pi/2}\frac{e^{r\sin^2\theta}\varphi(\frac{1}{2}r\sin 2\theta)d\theta}{r\sin^2\theta}=\frac{(l+o(1))e^r}{2\sqrt{\pi}r\sqrt{r}},\quad r\to+\infty,$$

where $\varkappa(r)=\arcsin\frac{1}{r}$.

Proof of Lemma 1. Let's consider the case $\varphi(\theta) \equiv 1$. Then

$$I(r) = \frac{1}{\pi} \int\limits_{\varkappa(r)}^{\pi/2} \frac{e^{r \sin^2 \theta} d\theta}{r \sin^2 \theta}.$$

Let $t = r \sin^2 \theta$ be a new variable of integration. This replacement yieds $\theta = \arcsin \sqrt{\frac{t}{r}}$ and $\theta'(t) = \frac{1}{2\sqrt{t(r-t)}}$. Hence, integrating by parts and using the equality $\int \frac{dt}{t^{3/2}\sqrt{r-t}} = -\frac{2}{r}\sqrt{\frac{r-t}{t}}$ we obtain

$$\frac{1}{\pi} \int\limits_{\varkappa(r)}^{\pi/2} \frac{e^{r \sin^2 \theta} d\theta}{r \sin^2 \theta} = \frac{1}{2\pi} \int\limits_{1/r}^{r} \frac{e^t dt}{t^{3/2}\sqrt{r-t}} = -\frac{e^t}{\pi r}\sqrt{\frac{r-t}{t}}\Bigg|_{1/r}^{r} + \frac{1}{\pi r}\int\limits_{1/r}^{r} e^t \sqrt{\frac{r-t}{t}} dt =$$

$$= \frac{e^{1/r}}{\pi}\sqrt{1-\frac{1}{r^2}} + \frac{1}{\pi r}\Big(\int\limits_{1/r}^{r-r^\alpha} e^t \sqrt{\frac{r-t}{t}} dt + \int\limits_{r-r^\alpha}^{r} e^t \sqrt{\frac{r-t}{t}} dt \Big) \equiv \frac{I_1 + I_2}{\pi r} + O(1).$$

Here $0 < \alpha < 1$. It is easy to see that

$$I_1 = \int\limits_{1/r}^{r-r^\alpha} e^t \sqrt{\frac{r-t}{t}} dt = O\Big(r^2 e^{r-r^\alpha}\Big), \quad r \to +\infty,$$

$$I_2 = \int\limits_{r-r^\alpha}^{r} e^t \sqrt{\frac{r-t}{t}} dt = \frac{(1+o(1))}{\sqrt{r}} \int\limits_{r-r^\alpha}^{r} e^t \sqrt{r-t} dt = \frac{(1+o(1))e^r}{\sqrt{r}} \int\limits_{r-r^\alpha}^{r} e^{t-r}\sqrt{r-t} dt =$$

$$= \frac{(1+o(1))e^r}{\sqrt{r}} \int\limits_{0}^{r^\alpha} e^{-u}\sqrt{u} du = \frac{(1+o(1))\sqrt{\pi}e^r}{2\sqrt{r}}, \quad r \to +\infty.$$

Hence, we have $\frac{1}{\pi} \int\limits_{\varkappa(r)}^{\pi/2} \frac{e^{r \sin^2 \theta} d\theta}{r \sin^2 \theta} = \frac{(1+o(1))e^r}{2\sqrt{\pi} r \sqrt{r}}$ as $r \to +\infty$. It remains to show that

$$\int\limits_{\varkappa(r)}^{\pi/2} \frac{e^{r \sin^2 \theta}(\varphi(\frac{1}{2} r \sin 2\theta) - l) d\theta}{r \sin^2 \theta} = o\Big(\frac{e^r}{r\sqrt{r}}\Big), \quad r \to +\infty.$$

Putting $\varepsilon(r) = \frac{1}{r^{2/3}}$, we evidently have

$$\int\limits_{\pi/2-\varepsilon(r)}^{\pi/2} \frac{e^{r \sin^2 \theta} d\theta}{r \sin^2 \theta} = O\Big(\frac{e^r \varepsilon(r)}{r}\Big) = o\Big(\frac{e^r}{r\sqrt{r}}\Big) \quad \text{and}$$

$$\int_{\varkappa(r)}^{\pi/3} \frac{e^{r\sin^2\theta}d\theta}{r\sin^2\theta} = O\Big(e^{3r/4}r\Big) = o\Big(\frac{e^r}{r\sqrt{r}}\Big), \quad r\to+\infty.$$

Thus, it is enough to show that the relation

$$\int_{\pi/3}^{\pi/2-\varepsilon(r)} \frac{e^{r\sin^2\theta}(\varphi(\frac12 r\sin 2\theta)-l)d\theta}{r\sin^2\theta} = o\Big(\frac{e^r}{r\sqrt{r}}\Big) \tag{1.5}$$

holds. Following [1], p.168–169 we define for $\theta\in(\frac{\pi}{3},\frac{\pi}{2}-\varepsilon(r))$

$$\Phi(\theta) = \int_0^{\frac12 r\sin 2\theta} \varphi(t)dt - \frac{l}{2}r\sin 2\theta.$$

Then $\Phi'(\theta) = (\varphi(\frac12 r\sin 2\theta)-l)r\cos 2\theta$. The range of θ provides that there exists a constant $K_1>0$ such that inequality $r\sin 2\theta \ge K_1 r^{\frac13}$ holds uniformly on θ. Lemma's conditions give

$$\Phi(\theta) = o(r\sin 2\theta), \quad r\to+\infty \tag{1.6}$$

uniformly on θ. Let $\chi(\theta)=\frac{e^{r\sin^2\theta}}{r^2\sin^2\theta\cos 2\theta}$, then

$$\int_{\pi/3}^{\pi/2-\varepsilon(r)} \frac{e^{r\sin^2\theta}(\varphi(\frac12 r\sin 2\theta)-l)d\theta}{r\sin^2\theta} = \int_{\pi/3}^{\pi/2-\varepsilon(r)} \chi(\theta)d\Phi(\theta) =$$

$$= \chi(\theta)\Phi(\theta)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}-\varepsilon(r)} - \int_{\pi/3}^{\pi/2-\varepsilon(r)} \chi'(\theta)\Phi(\theta)d\theta. \tag{1.7}$$

A simple calculation yields $\chi'(\theta)<0$ for $\theta\in(\frac{\pi}{3},\frac{\pi}{2}-\varepsilon(r))$ if r is sufficiently large. Therefore relation (1.6) implies

$$\int_{\pi/3}^{\pi/2-\varepsilon(r)} \chi'(\theta)\Phi(\theta)d\theta = o(1)\int_{\pi/3}^{\pi/2-\varepsilon(r)} \chi'(\theta)r\sin 2\theta d\theta.$$

But integrating by parts we get

$$\int_{\pi/3}^{\pi/2-\varepsilon(r)} \chi'(\theta)r\sin 2\theta d\theta = \chi(\theta)r\sin 2\theta\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}-\varepsilon(r)} - 2\int_{\pi/3}^{\pi/2-\varepsilon(r)} \chi(\theta)r\cos 2\theta d\theta =$$

$$= 2\frac{e^{r\sin^2\theta}\cos\theta}{r\sin\theta\cos 2\theta}\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}-\varepsilon(r)} - 2\int_{\pi/3}^{\pi/2-\varepsilon(r)} \frac{e^{r\sin^2\theta}d\theta}{r\sin^2\theta} = -\frac{(2+o(1))\varepsilon(r)e^r}{r} + \frac{4e^{3/4r}}{\sqrt{3}r} -$$

$$-2\int\limits_{\pi/3}^{\pi/2-\varepsilon(r)} \frac{e^{r\sin^2\theta}d\theta}{r\sin^2\theta} = o\Big(\frac{e^r}{r\sqrt{r}}\Big) - 2\int\limits_{\pi/3}^{\pi/2-\varepsilon(r)} \frac{e^{r\sin^2\theta}d\theta}{r\sin^2\theta}. \tag{1.8}$$

Thus, (1.6), (1.7) and (1.8) yield relation (1.5). The lemma is proved.

Proof of Theorem 1. Our proof follows closely that given in [3]. We write $\delta_{0,0} = 1 - \sum_{k=0}^{\infty}\sum_{n=1}^{N_k}\delta_{k,n}$. Define a bijective conformity between $(a_{k,n})$ and $\mathbb{N}$. Let $\varkappa_{k,n} \in \mathbb{N}$ corresponds to $a_{k,n}$. If S be an arbitrary subset of $\mathbb{Z}_+$ then the value

$$d(s) = \lim_{n\to\infty} \frac{\operatorname{card}(S\cap[0,n])}{n}$$

is called the density of S whenever the limit exists. According to Lemma 4.4 ([6], p.86), we can choose $S_{k,n}$ such that $\mathbb{Z}_+ = \bigcup S_{k,n}$, $S_{i,j}\cap S_{i',j'} = \varnothing$ as $i \neq i'$ or $j \neq j'$, $d(S_{k,n}) = \delta_{k,n}$ and $\nu > (1+|a_{k,n}|)2^{\varkappa_{k,n}}$ as $n > 0$, $\nu \in S_{k,n}$.

For every $\nu \in \mathbb{Z}$ we write

$$\psi_\nu(z) = \begin{cases} e^z, & |\nu| \in S_{0,0}, \\ a_{k,n}\frac{z^k}{k!}, & |\nu| \in S_{k,n}, n > 0. \end{cases}$$

The function we are looking for is defined by the equality

$$f(z) = \exp(-e^{-iz} + iz) \sum_{\nu=-\infty}^{+\infty} \psi_\nu(z) E_\nu(-iz),$$

where $E_\nu(z)$ is an entire function, $\nu \in \mathbb{Z}$, from Lemma 4.1 ([6], p.81–82).

Lemma 2. *There is an asymptotic representation*

$$f(z) = \psi_m(z) + O\big(|z|^2 e^{|z|} |\exp(-e^{-iz} + iz)|\big)$$
$$z \to \infty, \quad \operatorname{Re} z \in (-(2m+1)\pi, -(2m-1)\pi), \quad m \in \mathbb{Z}. \tag{1.9}$$

Lemma 2 follows from Lemma 1 [3].

For arbitrary $h \in (0, \frac{1}{2})$ we put

$$\varphi^h_{k,n}(t) = \begin{cases} \cos(|t+2\pi\nu|+h), & |t+2\pi\nu| < \frac{\pi}{2} - h, |\nu| \in S_{k,n}, \\ 0, & \text{otherwise.} \end{cases}$$

For $|\nu| \in S_{k,n}$ one obtains

$$\int\limits_{2\pi\nu-\pi}^{2\pi\nu+\pi} \cos(|t+2\pi\nu|+h)\,dt = \int\limits_{-\pi/2+h}^{\pi/2-h} \cos(|t|+h)\,dt = 2\int\limits_{0}^{\pi/2-h} \cos(t+h)\,dt = 2(1-\sin h).$$

It is obvious that $\lim_{r\to\infty}\frac{1}{r}\int_0^r \varphi_{k,n}^h(t)\,dt = \frac{\delta_{k,n}}{\pi}(1-\sin h)$. Let $z=x+iy$ and $|x+2\pi\nu| < \frac{\pi}{2}-h$, $|\nu|\in S_{k,n}$. We write $C_h=\{\zeta : |\zeta - z| = h\}$. Then, using Cauchy's Theorem and Lemma 2, we have $\forall k\in\mathbb{Z}_+$, $n>0$

$$|f^{(k)}(z)-a_{k,n}| = |f^{(k)}(z)-\psi_\nu^{(k)}(z)| = \Big|\frac{k!}{2\pi i}\int_{C_h}\frac{f(\zeta)-\psi_\nu(\zeta)}{(\zeta-z)^{k+1}}d\zeta\Big| \le$$
$$\le K_2\frac{k!}{h^k}(|z|+h)^2 e^{|z|+h}\exp\big(-e^{y-h}\varphi_{k,n}^h(x)\big),$$

where K_2 is a constant independent of z and h.

Thus,

$$\log^+\Big|\frac{1}{f^{(k)}(z)-a_{k,n}}\Big| \ge e^{y-h}\varphi_{k,n}^h(x) + O(|z|), \quad z\to\infty.$$

Putting $z = r\sin\theta e^{i\theta}$, we get $x=\frac{1}{2}r\sin 2\theta$, $y = r\sin^2\theta$. Now using Lemma 1 one obtains

$$\mathfrak{m}\Big(r, \frac{1}{f^{(k)}(z)-a_{k,n}}\Big) \ge (1+o(1))\delta_{k,n}\frac{1-\sin h}{e^h}\frac{e^r}{2\sqrt{\pi^3}r\sqrt{r}}, \quad r\to\infty \tag{1.10}$$

Futhermore, we have (see Lemma 3 for the details)

$$\sum_{\nu=-\infty}^{+\infty}\frac{|\psi(z)|}{1+\nu^2} = O\big(e^{|z|}\big), \quad z\to\infty. \tag{1.11}$$

From (1.9) and (1.11) it is easy to deduce that $|\psi(z)| \le K_3|z|^2e^{|z|}$, where K_3 is a constant independent of ν. Using Lemma 2 we give an estimate $\log^+|f(z)| \le e^y(-\cos x)^+ + O(|z|)$. Thus, by Lemma 1

$$\mathfrak{T}(r,f) = \mathfrak{m}(r,f) = \frac{1}{2\pi}\int_{\varkappa(r)}^{\pi-\varkappa(r)}\frac{\log^+|f(r\sin\theta e^{i\theta})|d\theta}{r\sin^2\theta} + O(r) \le (1+o(1))\frac{e^r}{2\pi^{\frac{3}{2}}r\sqrt{r}}. \tag{1.12}$$

From the definition of the class $\mathcal{K}$ and the logarithmic derivative lemma ([1], p.141) we conclude that

$$(\forall k\in\mathbb{N})\quad \mathfrak{T}(r,f^{(k)}) \le (1+o(1))\mathfrak{T}(r,f^{(k-1)}), \quad r\to\infty. \tag{1.13}$$

Using inequalities (1.10) and (1.12) and tending h to the zero, we obtain

$$\delta_T(a_{k,n}, f^{(k)}) \ge \delta_{k,n}, \quad n>0, k\in\mathbb{Z}_+. \tag{1.14}$$

Similarily one can show that

$$\delta_T(0, f-e^z) \ge \delta_{0,0}. \tag{1.15}$$

To complete the proof it is sufficient to show that

$$\sum_{a\ne\infty}\delta_T(a,f) + \sum_{k=1}^{\infty}\sum_{a\ne 0,\infty}\delta_T(a,f^{(k)}) + \delta_T(0,f-e^z) \le 1. \tag{1.16}$$

Let $s \in \mathbb{N}$, $q_1, \dots, q_s \in \mathbb{N}$. Using the inequality

$$\sum_{a \neq \infty} \mathfrak{m}(r, a, f) \leq \mathfrak{m}(r, 0, f') + \mathfrak{Q}(r, f),$$

where $\mathfrak{Q}(r, f) = o\big(\mathfrak{T}(r, f)\big)$, outside, perhaps, a set E, with mes$E < \infty$, we yield

$$\sum_{j=0}^{s} \sum_{n=1}^{q_j} \mathfrak{m}(r, 0, f - a_{j,n}) + \mathfrak{m}(r, 0, f - e^z) \leq$$
$$\leq \mathfrak{m}(r, 0, f^{(s+1)}) + \mathfrak{m}(r, 0, f^{(s+1)} - e^z) + \mathfrak{Q}(r, f). \quad (1.17)$$

One writes $h(z) = e^{-z} f^{(s+1)}(z) + 1$. We note that $\mathfrak{T}(r, e^z) = O(\log r)$. Then

$$\mathfrak{m}(r, 1, h) + \mathfrak{m}(r, 2, h) = \mathfrak{m}(r, 0, f^{(s+1)}) + \mathfrak{m}(r, 0, f^{(s+1)} - e^z) + O\big(\mathfrak{T}(r, e^z)\big) \leq$$
$$\leq \mathfrak{T}(r, h) + O(\log r) \leq \mathfrak{T}(r, f^{(s+1)}) + O\big(\log r\big). \quad (1.18)$$

It follows from (1.17) that

$$(\forall s \in \mathbb{N}) \quad \mathfrak{T}(r, h) = \mathfrak{T}(r, f^{(s+1)}) + O(\log r).$$

Thus, taking into account (1.13), (1.17) and the definition of $h(z)$ we give (1.16). The statement of Theorem 1 follows from (1.14) and (1.15).

The theorem is proved.

Remark. The relation (1.4) holds for functions from the class $\mathcal{K}'$ of analytic in $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ and in a some neighborhood of the zero functions f satisfing condition (2). Theorem 1 reminds correct if we require $f \in \mathcal{K}'$ instead of $f \in \mathcal{K}$.

II. Theorem 2. *Let $f(z)$ be an analytic and not constant in the closed half-plane, $\psi_k(z)$, $k = 1, \dots, p$, $p \geq 2$ distinict analytic functions such that $\mathfrak{T}(r, \psi_k) = o\big(\mathfrak{T}(r, f)\big)$, $k = 1, \dots, p$. Then*

$$\sum_{k=1}^{p} \mathfrak{m}(r, \psi_k, f) \leq \mathfrak{T}(r, f) + S(r), \quad (2.1)$$

where $S(r) = O\big(\log \mathfrak{T}(r, f) + \log r\big)$ outside, perharps, a set E of finite measure in the case when f has an infinite order.

In the case f is entire this is a result of Chuang Chi-tai [5]. His proof used elementary properties of the characteristics, the basic theorems and the logarithmic derivative lemma. It can be see that applying the scheme of the proof from [5] one deduces Theorem 2.

Relation (2.1) implies

$$\sum_{k} \delta_T(\psi_k, f) \leq 1, \quad (2.2)$$

where ψ_k are functions defined for $k \in \mathbb{N}$. In order to state our next theorem, we must introduce a notation. Let $M(r, g) = \max\{g(\sin\theta e^{i\theta}) : \theta \in [0, \pi]\}$, where $g(z)$ is analytic in the closed half-plane.

Theorem 3. *Let $\alpha(z)$ be an analytic in $\overline{\mathbb{C}}_+$ function such that $\log^+ M(r,\alpha) = o\big(\frac{e^r}{r^{3/2}}\big)$, (a_k) a sequence of analytic in $\overline{\mathbb{C}}_+$ functions satisfing $\log^+ M(r,a_k) = o\big(\log^+ M(r,\alpha)\big)$, $r\to\infty$, (δ_k) a sequence of positive numbers such that $\sum_k \delta_k \le 1$ holds. There exists an analytic in $\overline{\mathbb{C}}_+$ function $f(z)$ satisfing $\delta_T(a_k,f)=\delta_k$. For any analytic in $\overline{\mathbb{C}}_+$ function $b(z)$ satisfing $M(r,b) = o\big(M(r,\alpha)\big)$, $r\to\infty$ and $b(z)\not\equiv a_k(z)$, $k\in\mathbb{N}$, the equality $\delta_T(b,f)=0$ holds.*

Proof. We put $\delta_0 = 1-\sum_{k\ge1}\delta_k$. Using the properties of $a_k(z)$ we can write for $z = r\sin\theta e^{i\theta}$, $\theta\in[0,\pi]$

$$(\forall a_k(z)) \quad (\exists b_k\in\mathbb{R}_+) : \forall r>1 \quad M(r,a_k)\le b_k M(r,\alpha).$$

According to Lemma 4.4 [6] there are sets S_k with the following properties: i) $\bigcup_k S_k = \mathbb{Z}_+$; ii) $S_i\cap S_j=\varnothing$ iff $i\ne j$; iii) $d(S_k)=\delta_k$, $k\in\mathbb{Z}_+$; iv) $k>0$, $\nu\in S_k \Rightarrow \nu > (1+b_k)2^k$.

Define $\psi_\nu(z)$, $\nu\in\mathbb{Z}$ and $f(z)$ by

$$\psi_\nu(z) = \begin{cases} \alpha(z), & |\nu|\in S_0,\\ a_k(z), & |\nu|\in S_k,\end{cases}$$

$$f(z) = \exp(-e^{-iz}+iz)\sum_{\nu=-\infty}^{+\infty}\psi_\nu(z)E_\nu(-iz).$$

Lemma 3. *There is an asymptotic representation*

$$f(z) = \psi_m(z) + O\big(|z|^2 M(r,\alpha)|\exp(-e^{-iz}+iz)|\big),$$
$$z = r\sin\theta e^{i\theta}\to\infty, \quad \operatorname{Re} z\in(-(2m+1)\pi, -(2m-1)\pi), \quad m\in\mathbb{Z}. \tag{2.3}$$

Proof of Lemma 3. We prove the relation

$$\sum_{-\infty}^{+\infty}\frac{|\psi_\nu(z)|}{1+\nu^2} = O\big(M(r,\alpha)\big). \tag{2.4}$$

Simple calculations yield ($z = r\sin\theta e^{i\theta}$),

$$\sum_{-\infty}^{+\infty}\frac{\psi_\nu(z)}{1+\nu^2} = \Big(\sum_{|\nu|\in S_0} + \sum_{n\ge1}\sum_{|\nu|\in S_n}\Big)\frac{|\psi_\nu(z)|}{1+\nu^2} \le 2M(r,\alpha)\sum_{\nu=0}^{+\infty}\frac{1}{1+\nu^2} +$$
$$+\sum_{\nu\ge1}\sum_{|\nu|\in S_n}\frac{|a_n(z)|}{1+\nu^2} \le 6M(r,\alpha) + 2M(r,\alpha)\sum_{n\ge1}\sum_{\nu>(1+b_n)2^n}\frac{b_n}{1+\nu^2}\le$$
$$\le 6M(r,\alpha)+2M(r,\alpha)\sum_{n\ge1}\frac{|b_n|}{2^n|b_n|} = O\big(M(r,\alpha)\big).$$

Further, arguing as in Lemma 4.2 ([6],p.83–84), we obtain the assertion of Lemma 3.

We put

$$\varphi(t) = (-\cos t)^+, \quad \varphi_k(t) = \begin{cases} \cos t, & -(2m+\frac{1}{2})\pi \le t \le -(2m-\frac{1}{2})\pi, m \in S_k, \\ 0, & \text{otherwise.} \end{cases}$$

Evidently,

$$\lim_{r\to+\infty} \frac{1}{r}\int_0^r \varphi(t)\,dt = \frac{1}{\pi}, \quad \lim_{r\to+\infty} \int_0^r \varphi_k(t)\,dt = \frac{\delta_k}{\pi}.$$

It is easy to conclude from (2.4) and Lemma 2 that $|\psi_\nu(z)| \le K_4|z|^2 M(r,\alpha)$, here K_4 is a constant independent from ν, $z = r\sin\theta e^{i\theta}$.

Thus, using (2.3) we obtain

$$\log^+|f(z)| \le e^y(-\cos t)^+ + O\big(M(r,\alpha)\big).$$

Then applying Lemma 1 one has

$$\mathfrak{T}(r,f) \le \frac{(1+o(1))e^r}{2\pi^{3/2}r\sqrt{r}} + O\big(\log^+ M(r,\alpha)\big) = \frac{(1+o(1))e^r}{2\pi^{3/2}r\sqrt{r}}, \quad r\to+\infty. \tag{2.5}$$

Analogously, $z = r\sin\theta e^{i\theta}$,

$$|f(z) - a_k(z)| \le K_5\big(|z|^2 M(r,\alpha)|\exp(-e^{-iz}+iz)|\big),$$

and

$$\log^+\left|\frac{1}{f(z)-a_k(z)}\right| \ge \mathrm{Re}(e^{-iz}-iz) + O\big(\log|z|^2 + \log M(r,\alpha)\big) =$$
$$= e^y\varphi_k(x) + O\big(\log r + \log^+ M(r,\alpha)\big).$$

Using Lemma 1 again we obtain $\delta_T(a_k,f) \ge \delta_k$. Similarly one can prove that $\delta_T(\alpha,f) \ge \delta_0$. Therefore from (2.2) we give $\delta_T(a_k,f) = \delta_k$ and $\delta_T(b,f) = 0$ when $M(r,b) = o\big(M(r,\alpha)\big)$.

Theorem 1 is proved.

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