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SOME OPEN PROBLEMS IN THE THEORY OF TOPOLOGICAL GROUPS AND SEMIGROUPS

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Throughout the paper all topological spaces are assumed to be Hausdorff. A group G provided with a topology is called *semitopological* if the multiplication in G is separately continuous, and the inversion $x \mapsto x^{-1}$ is continuous. A group G provided with a topology is called *paratopological* if the multiplication in G is continuous.

Reznichenko [R_1] proved that any Tychonoff pseudocompact group with continuous multiplication is a topological group. Pfister [Pf] proved that any countable compact regular paratopological group is a topological group. Bokalo and the author [BG] recently showed that any sequentialy compact Hausdorff paratopological group is a topological group. Reznichenko [R_2] proved that if G is a semitopological group and G is a Čech-complete space, then G is a topological group.

Problem 1. *Is there a countable compact paratopological group (not necessarily regular) which is not a topological group?*

Even the following is not known.

Problem 2. *Let G be a countable compact paratopological group. Is G totally bounded?*

Problem 3. *Is every semitopological group whose space is Baire a topological group?*

A (para-)topological group G is said to be *minimal* if the topology of G is a minimal element of the set of all Hausdorff group (para-)topologies on G. A topological space X without isolated points is called *maximal* if X has an isolated point in any stronger topology. A topological (paratopological) group is called *maximal* if it is maximal as a topological space. For more details on minimal topological groups see the book [DPS].

I. Protasov recently showed that any maximal paratopological group is a topological group. Now, the following questions are self-suggesting.

Problem 4. *Is there a minimal paratopological group which is not a topological group?*

The following question stands better chances to get a positive answer than the previous one:

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Problem 5. *Let (G,τ) be a paratopological group. Is there a group topology $\tau' \leq \tau$ such that (G,τ') is a topological group?*

The notion of free topological group $F(X)$ over a topological space X was introduced by A.A. Markov [M]. He proved that for any Tychonoff space X a free topological group $F(X)$ exists and is unique, algebraically free and Tychonoff.

A topological inverse semigroup is, by definition, a Hausdorff topological space S equipped with a continuous binary associative operation $(\cdot): S \times S \to S$ such that every element $x \in S$ has a unique inverse element x^* and the map $(\cdot)^*: S \to S$ assigning to each $x \in X$ its inverse x^* is continuous.

A free topological inverse semigroup over a topological space X is a pair $\big(I(X), i\big)$ consisting of a topological inverse semigroup $I(X)$ and a topological embedding $i: X \to I(X)$ such that for every map $f: X \to S$ into a topological inverse semigroup S there exists a unique homomorphism $\tilde{f}: I(X) \to S$ such that $f = \tilde{f} \circ i$.

Problem 6. *Is there semigroup $I(X)$ Tychonoff for a Tychonoff space X?*

In the paper [BGG] it is proved that if X is a functionally Hausdorff space then the free topological inverse semigroup $I(X)$ is functionally Hausdorff and algebraically free. In the same place it is shown that if X is a totally disconnected space then so is the space $I(X)$. The following questions remain open.

Problem 7. *Is $I(X)$ zero-dimensional for zero-dimensional X?*

Problem 8. *Describe classes $\mathcal{P}$ of topological spaces having the following property: if $I(X)$ is homeomorphic to $I(Y)$ and $X \in \mathcal{P}$, then $Y \in \mathcal{P}$.*

A general problem we consider in the theory of the topological inverse semigroups can be formulated as follows:

Problem 9. *Let S be a topological inverse semigroup, $E(S)$ the subsemigroup of idempotents of S, $\mathcal{H} = \{H(e) \mid e \in E(S)\}$ the family of maximal subgroups of S and $\mathcal{P}_1, \mathcal{P}_2$ some topological properties. If $E(S)$ has the property $\mathcal{P}_1$ and each $H(e)$ has the property $\mathcal{P}_2$ than what topological properties has the semigroup S?*

In particular,

Problem 10. *Let $\mathcal{P}_1 = \mathcal{P}_2 =$ "metrizability" and let S be a compact semigroup. Is then S metrizable?*

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