УДК 515.12

THE STRONGLY UNIVERSAL PROPERTY IN CONVEX SETS

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T. Banakh. *The strongly universal property in convex sets*, Matematychni Studii, **10**(1998) 203–218.

It is shown that a separable linear metric AR-space L is strongly $\mathcal{C}$-universal, where $\mathcal{C}$ is a class of spaces, if and only if L contains a convex closed strongly $\mathcal{C}$-universal subspace.

We inspect also the strongly $\mathcal{C}$-universal property in convex sets containing a $\mathcal{C}$-universal closed subset of infinite codimension.

Т. Банах. *Свойство сильной универсальности в опуклых множествах* // Математичні Студії. – 1998. – Т.10, № 2. – С.203–218.

Показано, что сепарабельное линейное метрическое AR-пространство L сильно $\mathcal{C}$ универсально, где $\mathcal{C}$ — класс пространств, тогда и только тогда, когда L содержит опуклое замкнутое сильно $\mathcal{C}$-универсальное подпространство.

Мы также исследуем свойство сильной $\mathcal{C}$-универсальности в опуклых множествах, содержащих $\mathcal{C}$-универсальное замкнутое подмножество бесконечной коразмерности.

Introduction

The strongly universal property was introduced by M.Bestvina and J.Mogilski in [BM] to characterize topologically certain incomplete infinite-dimensional absolute retracts. This property was extended afterward onto pairs of spaces and eventually onto ordered systems of spaces. The strongly universal property turned to be a very powerful instrument in recognizing various topological spaces appearing naturally in topology, topological algebra, functional analysis, measure theory etc. (see the survey [Ca]).

In this paper we investigate the strongly universal property in convex sets. Two theorems stated below are the main results of this article. In sake of simplicity, we formulate here their space versions only (in full generality, these theorems treating systems of spaces, can be found in §5, 6).

Theorem 1. *A separable linear metric AR-space L is strongly $\mathcal{C}$-universal, where $\mathcal{C}$ is a class of spaces, if and only if L contains a closed convex strongly $\mathcal{C}$-universal AR-subset.*

In [Ca, 8.5], R.Cauty has proved that a separable normed space L is strongly $\mathcal{C}$-universal, provided it contains a $\mathcal{C}$-universal closed subspace of infinite codimension. He has also asked (in a private communication) if this result still holds for locally convex (non-normable) spaces. The following theorem gives a partial answer to this question.

1991 *Mathematics Subject Classification.* 57N17, 57N20.

Theorem 2. *Let $\mathcal{C}$ be a class of spaces, A a closed subset in a separable metrizable locally convex space L, and $X \subset L$ a convex set such that $A \subset X$ has infinite codimension in X. The space X is strongly $\mathcal{C}$-universal, provided one of the following conditions is satisfied*

1. *A is bounded and $\mathcal{C}$-universal;*
2. *the class $\mathcal{C}$ is $(0,1]$-stable and $\{*\}$-additive, and A is $\mathcal{C}$-universal;*
3. *A is a strongly $\mathcal{C}$-universal ANR.*

In fact, Theorem 2 is valid more generally for convex AR-sets satisfying the locally compact approximation property (briefly LCAP). We define a space X to satisfy LCAP if the identity map $\mathrm{id}: X \to X$ can be strongly approximated by maps $f: X \to X$ with locally compact closure $\mathrm{Cl}_X f(X)$. This property is introduced and studied in §2. In that section, relationship between LCAP and other known properties (like the strong discrete approximation property or the properties $(*)$ and $(*)^1$ of [Bo] and [Do]) is established as well. It is remarkable that every homotopy dense subspace in a locally compact ANR has LCAP. In particular, every convex set in a locally convex space satisfies LCAP.

In §3 we establish some properties of towers of subsets to be applied in §4 where our main technical result is proved. Because this result has also an independent value, we found it reasonable to state it here in introduction.

Theorem 3. *Let X be an ANR with LCAP and $\mathcal{C}$ be a class of spaces. The space X is strongly $\mathcal{C}$-universal provided it contains a tower $X_1 \subset X_2 \subset \dots \subset X_n \subset \dots X$ of Z-sets such that*

1. *each X_n is a strongly $\mathcal{C}$-universal ANR;*
2. *for given cover $\mathcal{U} \in \mathrm{cov}(X)$, a number $n \in \mathbb{N}$, and a map $f: A \to X$ of a finite-dimensional compactum, there exist an $m \in \mathbb{N}$ and a map $\bar{f}: A \to X_m$ such that $\bar{f}$ is $\mathcal{U}$-near to f and $\bar{f}|f^{-1}(X_n) = f|f^{-1}(X_n)$.*

1. Preliminaries.

All spaces considered in this paper are metrizable and separable, all maps are continuous.

For a metric space (X,d) and a subset $A \subset X$ by $O(x,\varepsilon)$ we denote the open ε-neighborhood of A and by $\bar{A}$ or $\mathrm{Cl}_X A$ its closure; $\mathrm{cov}(X)$ denotes the collection of all open covers of X. Let $\mathcal{U} \in \mathrm{cov}(X)$. We say that a family $\mathcal{F}$ of subsets of X is inscribed into $\mathcal{U}$ (denoted by $\mathcal{F} \prec \mathcal{U}$) if for every $F \in \mathcal{F}$ there is $U \in \mathcal{U}$ with $F \subset U$. Two maps $f, g: Y \to X$ are called $\mathcal{U}$-near (denoted by $(f,g) \prec \mathcal{U}$) if the family $\{\{f(y), g(y)\}\}_{y\in Y}$ is inscribed into $\mathcal{U}$. Let $\mathcal{S}t\,\mathcal{U} = \{\mathcal{S}t\,(U,\mathcal{U}) \mid U \in \mathcal{U}\}$, where for an $A \subset X$ $\mathcal{S}t\,(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid A \cap U \neq \varnothing\}$, and $\mathrm{mesh}\,\mathcal{U} = \sup\{\mathrm{diam}\,U \mid U \in \mathcal{U}\}$.

A map $f: Y \to X$ is defined to be *closed over a set* $A \subset X$ provided for every $x \in X$ and every neighborhood $U \subset Y$ of $f^{-1}(x)$ there is a neighborhood $V \subset X$ of x such that $f^{-1}(V) \subset U$.

A map $p: X \to Y$ is called *perfect* if it is closed and the preimage $p^{-1}(y)$ of any $y \in Y$ is compact. According to [En, 3.7.18], a map p is perfect if and only if the preimage $p^{-1}(K)$ of any compactum $K \subset Y$ is compact. We will use quite often the following known

1.1. Lemma. *If $p: K \to X$ is a perfect map of a locally compact space then*

1. *the image $p(K)$ is a closed locally compact set in X;*
2. *there is a cover $\mathcal{U} \in \mathrm{cov}(X)$ such that a map $p': K \to X$ is perfect whenever $(p', p) \prec \mathcal{U}$.*

A subset $X \subset Y$ is defined to be *homotopy dense in* Y if there exists a homotopy $h: Y \times [0,1] \to Y$ such that $h(Y \times (0,1]) \subset X$ and $h(y,0) = y$ for every $y \in Y$.

An embedding $e: X \to Y$ is called *homotopy dense* if $e(X)$ is a homotopy dense subset in Y. A subset $X \subset Y$ is called *homotopy negligible* if its complement $Y \setminus X$ is homotopy dense in X. The definition implies the following useful fact: if X is a homotopy negligible subset in Y then for every open $U \subset Y$ the set $U \cap X$ is homotopy negligible in U. According to [To], a subset X in an ANR-space Y is homotopy negligible if and only if it is locally homotopy negligible (that is every map $f: I^k \to Y$ of a finite-dimensional cube with $f(\partial I^k) \cap X = \varnothing$ can be approximated by a map $\bar{f}: I^k \to Y$ such that $\bar{f}|\partial I^k = f|\partial I^k$ and $\bar{f}(I^k) \cap X = \varnothing$). It is well known that if Y is a set in a linear metric space and X is a convex dense subset in Y then $Y \setminus X$ is locally homotopy negligible in Y. Consequently, if Y is an AR then X is homotopy dense in Y.

A space X is defined to satisfy *the strong discrete approximation property* (briefly SDAP) if given a cover $\mathcal{U} \in \mathrm{cov}(X)$ and a map $f: \oplus_{n\in\mathbb{N}} I^n \to X$ there exists a map $\bar{f}: \oplus_{n\in\mathbb{N}} I^n \to X$ such that $(\bar{f}, f) \prec \mathcal{U}$ and the collection $\{\bar{f}(I^n)\}_{n\in\mathbb{N}}$ is discrete in X. Accordingly to [DT], every infinite-dimensional linear metric space satisfies SDAP.

A set $A \subset X$ is defined to be a *(strong) Z-set* provided it is closed and given a cover $\mathcal{U} \in \mathrm{cov}(X)$ there exists a map $f: X \to X$ such that $(f, \mathrm{id}) \prec \mathcal{U}$ and $f(X) \cap A = \varnothing$ (resp. $\mathrm{Cl}_X(f(X)) \cap A = \varnothing$). An embedding $f: A \to X$ is called a *Z-embedding* if $f(A)$ is a Z-set in X. It is well known that for a locally finite collection $\{A_i\}_{i\in\mathcal{I}}$ of Z-sets in an ANR-space X the union $\bigcup_{i\in\mathcal{I}} A_i$ is a Z-set in X.

Let $(\Gamma, \leq)$ be an ordered set. Any order preserving collection $(X_\gamma)_{\gamma\in\Gamma}$ of subsets of a space X is called *a Γ-system in X*. *A Γ-system* is a pair $(X, X_\gamma)_{\gamma\in\Gamma}$ consisting of a space X and a Γ-system in X. We use german letters to denote Γ-systems (e.g., $\mathfrak{C}, \mathfrak{X}, \mathfrak{Y}$).

For a Γ-system $\mathfrak{X} = (X, X_\gamma)_{\gamma\in\Gamma}$, a subset $B \subset X$, a space Y, and maps $f: Z \to X$,
$g: X \to Y$ let $\bigcap \mathfrak{X} = \bigcap_{\gamma\in\Gamma} X_\gamma \subset X$, $B \cap \mathfrak{X} = (B, B \cap X_\gamma)_{\gamma\in\Gamma}$, $\mathfrak{X} \setminus B = (X \setminus B, X_\gamma \setminus B)_{\gamma\in\Gamma}$, $\mathfrak{X} \times Y = (X \times Y, X_\gamma \times Y)_{\gamma\in\Gamma}$, $f^{-1}(\mathfrak{X}) = (Y, f^{-1}(X_\gamma))_{\gamma\in\Gamma}$, and $g(\mathfrak{X}) = (Y, g(X_\gamma))_{\gamma\in\Gamma}$.

A Γ-system $\mathfrak{L} = (L, L_\gamma)_{\gamma\in\Gamma}$ in a linear space L is called *linear* if for every $\gamma \in \Gamma$ L_γ is a linear subspace in L.

Two Γ-systems $\mathfrak{X} = (X, X_\gamma)_{\gamma\in\Gamma}$ and $\mathfrak{Y} = (Y, Y_\gamma)_{\gamma\in\Gamma}$ are called homeomorphic provided there is a homeomorphism $h: X \to Y$ such that $h(\mathfrak{X}) = \mathfrak{Y}$. Homeomorphisms are denoted by the symbol $\cong$.

A class $\vec{\mathcal{C}}$ of Γ-systems is defined to be

(1) *T-stable*, where T is a space, if $T \times \mathfrak{C} \in \vec{\mathcal{C}}$ for every $\mathfrak{C} \in \vec{\mathcal{C}}$;
(2) *$\{*\}$-additive*, if for every $\mathfrak{C} = (C, C_\gamma)_{\gamma\in\Gamma} \in \vec{\mathcal{C}}$ and an embedding $C \subset X$ with $X = \{*\} \cup C$ we have $(X, \{*\} \cup C_\gamma)_{\gamma\in\Gamma} \in \vec{\mathcal{C}}$.

Let $\mathfrak{C} = (C, C_\gamma)_{\gamma\in\Gamma}$ and $\mathfrak{X} = (X, X_\gamma)_{\gamma\in\Gamma}$ be two Γ-systems. The Γ-system $\mathfrak{X}$ is defined to be *$\mathfrak{C}$-universal* if there exists a closed embedding $e: C \to X$ such that $e^{-1}(\mathfrak{X}) = \mathfrak{C}$; $\mathfrak{X}$ is defined to be *strongly $\mathfrak{C}$-universal*, if given a cover $\mathcal{U} \in \mathrm{cov}(X)$, a closed subset $B \subset C$ and a map $f: C \to X$ such that the restriction $f|B: B \to X$ is a Z-embedding with $(f|B)^{-1}(\mathfrak{X}) = B \cap \mathfrak{C}$, there is a Z-embedding $\bar{f}: C \to X$ such that $\bar{f}|B = f|B$, $(\bar{f}, f) \prec \mathcal{U}$, and $\bar{f}^{-1}(\mathfrak{X}) = \mathfrak{C}$.

A Γ-system $\mathfrak{X}$ is defined to be *(strongly) $\vec{\mathcal{C}}$-universal*, where $\vec{\mathcal{C}}$ is a class of Γ-systems, if $\mathfrak{X}$ is (strongly) $\mathfrak{C}$-universal for every $\mathfrak{C} \in \vec{\mathcal{C}}$.

In the sequel we will need the following results on strongly universal systems.

1.2. Lemma ([Ca, 4.2], [BC, 4.2]). *Let $\mathfrak{X} = (X, X_\gamma)_{\gamma\in\Gamma}$ and $\mathfrak{C} = (C, C_\gamma)_{\gamma\in\Gamma}$ be two*

Γ-systems. Suppose X is an ANR and $\mathfrak{X}$ is a strongly $\mathfrak{C}$-universal system. Then

(1) *for every open set $U \subset X$ and a closed set $B \subset C$ the system $U \cap \mathfrak{X}$ is strongly $B \cap \mathfrak{C}$-universal;*
(2) *for every open set $U \subset X$, a cover $\mathcal{U} \in \mathrm{cov}(U)$, and a map $f: C \to X$ there exists a Z-embedding $g: f^{-1}(U) \to U$ such that $(g, f|f^{-1}(U)) \prec \mathcal{U}$ and $g^{-1}(\mathfrak{X}) = f^{-1}(U) \cap \mathfrak{C}$.*

1.3. Lemma [Ca, 4.4]. *Let X be an ANR, Y a locally compact ANR such that every Z-set in $X \times Y$ is strong, and $\mathfrak{X} = (X, X_\gamma)_{\gamma\in\Gamma}$, $\mathfrak{C} = (C, C_\gamma)_{\gamma\in\Gamma}$ two Γ-systems. If $\mathfrak{X}$ is strongly $\mathfrak{C}$-universal then so is $\mathfrak{X} \times Y$.*

1.4. Lemma [Ca, 4.3]. *Let X be an ANR and $\mathfrak{X} = (X, X_\gamma)_{\gamma\in\Gamma}$, $\mathfrak{C} = (C, C_\gamma)_{\gamma\in\Gamma}$ be two Γ-systems. The Γ-system $\mathfrak{X}$ is strongly $\mathfrak{C}$-universal if and only if there is an open cover $\mathcal{U}$ of X such that for every $U \in \mathcal{U}$ the Γ-system $\mathcal{U} \cap \mathfrak{X}$ is strongly $\mathfrak{C}$-universal.*

1.5. Lemma [Ba$_1$]. *If $\{F_i\}_{i\in\mathcal{I}}$ is a locally finite collection of subsets in a space X then there exists a cover $\mathcal{U} \in \mathrm{cov}(X)$ such that the collection $\{\mathcal{S}t\,(F_i, \mathcal{U})\}_{i\in\mathcal{I}}$ is locally finite in X.*

2. The locally compact approximation property.

2.1. Definition. A space X is defined to satisfy *the locally compact approximation property* (briefly LCAP) if for every cover $\mathcal{U} \in \mathrm{cov}(X)$ there is a map $f: X \to X$ such that $(f, \mathrm{id}) \prec \mathcal{U}$ and the closure $\mathrm{Cl}_X(f(X))$ is locally compact.

Below we give a characterization of LCAP, and establish relationship between LCAP and other known properties.

2.2. Proposition. *An ANR X has LCAP if and only if for every cover $\mathcal{U} \in \mathrm{cov}(X)$ there are a locally finite simplicial complex K, a map $q: X \to K$, and a perfect map $p: K \to X$ such that $(p \circ q, id) \prec \mathcal{U}$.*

Proof. The "if" part is trivial since for a perfect map $p: K \to X$ of a locally compact space K, the image $p(K)$ is a closed locally compact set in X.

To prove the "only if" part fix an ANR X with LCAP, and a cover $\mathcal{U} \in \mathrm{cov}(X)$. Let $\mathcal{V} \in \mathrm{cov}(X)$ be a cover with $\mathcal{S}t\,\mathcal{V} \prec \mathcal{U}$. Since X has LCAP, there is a map $f: X \to X$ such that $(f, \mathrm{id}) \prec \mathcal{V}$ and $f(X) \subset F$, where F is a closed locally compact set in X. By Lemma 1.1, there is a cover $\mathcal{W} \in \mathrm{cov}(X)$, $\mathcal{W} \prec \mathcal{V}$, such that a map $g: F \to X$ is perfect whenever g is $\mathcal{W}$-near to the identity embedding $F \subset X$. Since X is an ANR, there are a locally finite simplicial complex K, and maps $q': F \to K$, $p: K \to X$ such that $(p \circ q', \mathrm{id}_F) \prec \mathcal{W}$. By [MU], we can assume that the map q' is surjective. Since $p \circ q'$ is perfect, so is the map p. Letting $q = q' \circ f$ we get the maps q, p and a simplicial complex K satisfying our requirements. □

2.3. Proposition. *If an ANR X satisfies LCAP then every open subspace $U \subset X$ satisfies LCAP too.*

Proof. Let X be an ANR with LCAP, $U \subset X$ an open subspace, and $\mathcal{U} \in \mathrm{cov}(U)$ a cover. We have to construct a map $f: U \to U$ such that $(f, \mathrm{id}_U) \prec \mathcal{U}$ and $\mathrm{Cl}_U\, f(U)$ is locally compact. This will be done by induction.

Fix a metric d on X, and let $\mathcal{U}_0 \in \mathrm{cov}(U)$ be a cover such that $\mathcal{U}_0 \prec \mathcal{U}$ and $\mathcal{U}_0 \prec \{O(x, d(x, X \setminus U)/2) \mid x \in U\}$. Let $(\mathcal{U}_n)_{n=1}^\infty \subset \mathrm{cov}(U)$ be a sequence of covers such that for every $n \in \mathbb{N}$ $\mathcal{S}t\,\mathcal{U}_n \prec \mathcal{U}_{n-1}$ and $\mathrm{mesh}\,\mathcal{U}_n < 2^{-n}$. Consider the sets $U_n = \{x \in U \mid d(x, X \setminus U) > 2^{-n}\}$, $n \geq 2$, and let $U_0 = U_1 = \varnothing$.

Inductively, we shall construct a sequence of maps $\{f_n: U \to U\}_{n=1}^\infty$ satisfying for every $n \in \mathbb{N}$ the following conditions:

(1_n) $f_n|\bar U_{n-2} = f_{n-1}|\bar U_{n-2}$,
(2_n) $f_n|U \setminus U_{n+4} = \mathrm{id}|U \setminus U_{n+4}$,
(3_n) $(f_n, f_{n-1}) \prec \mathcal{U}_n$,
(4_n) $\mathrm{Cl}_U(f_n(U)) \cap \bar U_{n+1}$ is locally compact.

Let $f_0 = \mathrm{id}_U$ and assume that for an $n \in \mathbb{N}$ the maps f_k satisfying (1_k)–(4_k), $k < n$, have been constructed. By the inductive assumptions (3_k), $k < n$, and the choice of the covers $\mathcal{U}_n$'s, we have $(f_n, \mathrm{id}) \prec \mathcal{U}_0$, and consequently,

$$f_{n-1}(\bar U_{n-2}) \subset U_{n-1}. \tag{1}$$

By (4_{n-1}), the set $F = \mathrm{Cl}_U(f_{n-1}(U)) \cap \bar U_n$ is locally compact and closed in X. According to Lemma 1.1, there is a cover $\mathcal{V} \in \mathrm{cov}(X)$ such that a map $p\colon F \to X$ is perfect whenever $(p, \mathrm{id}_F) \prec \mathcal{V}$. Without loss of generality, $\mathrm{mesh}\,\mathcal{V} < 2^{-n-4}$ and

$$\{V \cap \bar U_{n+4}\}_{V \in \mathcal{V}} \prec \mathcal{U}_n. \tag{2}$$

Since X is an ANR, there is a cover $\mathcal{W} \in \mathrm{cov}(X)$, $\mathcal{W} \prec \mathcal{V}$, such that any two $\mathcal{W}$-near maps into X are $\mathcal{V}$-homotopic. Using the locally compact approximation property of X, find a map $g\colon X \to X$ such that $(g, \mathrm{id}) \prec \mathcal{W}$ and the closure $\mathrm{Cl}_X(g(X))$ is locally compact. Since $\mathrm{mesh}\,\mathcal{W} < 2^{-n-4}$ we have $g^{-1}(\bar U_{n-1}) \subset U_n$. Since the maps g and id are $\mathcal{V}$-homotopic, there is a map $h\colon X \to X$ such that

$$h|X \setminus U_{n+4} = \mathrm{id}|X \setminus U_{n+4}, \tag{3}$$
$$h|g^{-1}(\bar U_{n-1}) = \mathrm{id}|g^{-1}(\bar U_{n-1}), \tag{4}$$
$$h|\bar U_{n+3} \setminus U_n = g|\bar U_{n+3} \setminus U_n,$$
$$(h, \mathrm{id}) \prec \mathcal{V}. \tag{5}$$

Notice that by the choice of the cover $\mathcal{V}$, the set $h(F) \subset X$ is closed and locally compact.

We claim that the map $f_n = h \circ f_{n-1}$ satisfies the conditions (1_n)–(4_n). First notice that $\mathrm{mesh}\,\mathcal{V} < 2^{-n-4}$ and (3) imply $f_n(U) \subset U$. The condition (1_n) follows from (1) and (4), (2_n) from (2_{n-1}) and (3), and (3_n) from (2) and (5). To verify (4_n) notice that the conditions (3_k), $k \le n$, yield $(f_n, \mathrm{id}) \prec \mathcal{U}_0$ and hence $f_n^{-1}(\bar U_{n+1}) \subset \bar U_{n+2}$. Thus $\mathrm{Cl}_U(f_n(U)) \cap \bar U_{n+1} \subset \mathrm{Cl}_X(f_n(\bar U_{n+2}))$. Since $f_{n-1}(\bar U_{n+2}) \subset (\bar U_{n+3} \setminus U_n) \cup F$, we have $f_n(\bar U_{n+2}) \subset h(\bar U_{n+3} \setminus U_n) \cup h(F) = g(\bar U_{n+3} \setminus U_n) \cup h(F) \subset \mathrm{Cl}_X(g(X)) \cup h(F)$. Thus $\mathrm{Cl}_U(f_n(U)) \cap \bar U_{n+1}) \subset \mathrm{Cl}_X(f_n(\bar U_{n+2})) \subset \mathrm{Cl}_X(g(X)) \cup h(F)$ is locally compact and the condition (4_n) follows. The inductive step is over.

Let finally $f = \lim_{n\to\infty} f_n\colon U \to U$. Using the properties ($1_n$)–($4_n$), $n \in \mathbb{N}$, show that $(f, \mathrm{id}_U) \prec \mathcal{U}$ and the closure $\mathrm{Cl}_U(f(U))$ is locally compact. □

2.4. Proposition. *If X is an ANR with LCAP then every homotopy dense subspace $Y \subset X$ has LCAP too.*

Proof. Let Y be a homotopy dense set in X and $\mathcal{U} \in \mathrm{cov}(Y)$ be a cover of Y. For every $U \in \mathcal{U}$ pick up an open set $\tilde U \subset X$ with $\tilde U \cap Y = U$ and let $\tilde{\mathcal{U}} = \{\tilde U \mid U \in \mathcal{U}\}$ and $V = \bigcup \tilde{\mathcal{U}}$. Let $\mathcal{V} \in \mathrm{cov}(V)$ be a cover with $\mathcal{S}t\,\mathcal{V} \prec \tilde{\mathcal{U}}$. By Proposition 2.3, the open set $\tilde{\mathcal{U}}$ in X has LCAP. Hence there is a map $f\colon V \to V$ such that $(f, \mathrm{id}) \prec \mathcal{V}$ and the closure $F = \mathrm{Cl}_V(f(V))$ is locally compact. By Lemma 1.1, there is a cover $\mathcal{W} \in \mathrm{cov}(V)$, $\mathcal{W} \prec \mathcal{V}$, such that a map $g\colon F \to V$ is perfect whenever $(g, \mathrm{id}_F) \prec \mathcal{W}$. Since Y is homotopy dense in X, there is a map $g\colon F \to Y$ such that $(g, \mathrm{id}_F) \prec \mathcal{W}$.

Then g is perfect, and consequently, $g(F)$ is a closed locally compact set in V. Then the map $h = g \circ f|Y\colon Y \to Y$ has the following properties: $(h, \mathrm{id}) \prec \mathcal{U}$ and $h(Y)$ lies in the closed locally compact set $g(F) \subset Y$. Thus Y has LCAP. □

Evidently, every locally compact space satisfies LCAP. This remark and Proposition 2.4 imply

2.5. Proposition. *Every homotopy dense subspace in a locally compact ANR satisfies LCAP.*

It would be interesting to know whether the converse is also true.

2.6. Question. Let X be an ANR satisfying LCAP. Is there a homotopy dense embedding of X into a locally compact ANR?

By [Ba_1], every ANR with SDAP admits a homotopy dense embedding into a Hilbert cube manifold. This fact and Proposition 2.5 yield

2.7. Proposition. *If X is an ANR with SDAP then X satisfies LCAP.*

In fact, for an ANR-space X we have the following implications: SDAP⇒LCAP⇒ $(*) \Rightarrow (*)^1$, where $(*)$ and $(*)^1$ are the properties from [Bo] and [Do].

Recall that a space X satisfies *the discrete n-cells property* if for every cover $\mathcal{U} \in \mathrm{cov}(X)$ and a map $f\colon I^n \times \mathbb{N} \to X$ there exists a map $\bar{f}\colon I^n \times \mathbb{N} \to X$ such that $(\bar{f}, f) \prec \mathcal{U}$ and the collection $\{\bar{f}(I^n \times \{k\})\}_{k\in\mathbb{N}}$ is discrete in X. Repeating the arguments of Lemma 3.1 [Bo] we can prove

2.8. Proposition. *A Polish ANR X with LCAP satisfies SDAP if and only if for every $n \in \mathbb{N}$ X satisfies the discrete n-cells property.*

According to [BBMW, p.66], there exists a Polish ANR-space which satisfies the discrete n-cells property for every $n \in \mathbb{N}$ but fails to satisfy the SDAP. By Proposition 2.8 this space fails to satisfy LCAP. Hence, we have

2.9. Remark. There exists a complete-metrizable separable ANR-space without LCAP.

2.10. Proposition. *Let X be an ANR with LCAP and $Y \subset X$ be a set which is both homotopy dense and homotopy negligible in X. Then Y satisfies SDAP.*

Proof. Fix a cover $\mathcal{U} \in \mathrm{cov}(Y)$ and a map $f\colon \oplus_{n=1}^{\infty} I^n \to Y$. For every $U \in \mathcal{U}$ fix an open set $\tilde{U} \subset X$ such that $\tilde{U} \cap Y = U$, and consider the set $W = \cup\{\tilde{U} \mid U \in \mathcal{U}\}$ and its cover $\tilde{\mathcal{U}} = \{\tilde{U} \mid U \in \mathcal{U}\}$. Let $\mathcal{W} \in \mathrm{cov}(W)$ be a cover such that $\mathcal{S}t\,\mathcal{W} \prec \tilde{\mathcal{U}}$. By Proposition 2.3, the open subspace $W \subset X$ has LCAP. Thus there exists a map $g\colon W \to W$ such that $(g, \mathrm{id}_W) \prec \mathcal{W}$ and the closure $F = \mathrm{Cl}_W(g(W))$ is locally compact. Since the set Y is homotopy negligible in $X \in$ ANR, Y is homotopy negligible in W. According to Lemma 1.1, without loss of generality, we can assume $F \cap Y = \varnothing$. Since the set Y is homotopy dense in X, there is a map $\bar{f}\colon \oplus_{n=1}^{\infty} I^n \to Y$ such that $(\bar{f}, g \circ f) \prec \mathcal{W}$ and for every $n \in \mathbb{N}$ $\bar{f}(I^n) \subset O(F, \frac{1}{n})$. It is easily seen that $(\bar{f}, f) \prec \mathcal{U}$ and the collection $\{\bar{f}(I^n)\}_{n\in\mathbb{N}}$ is locally finite in Y. By Lemma 4 of [Ba_1], Y satisfies SDAP. □

LCAP shares the following property of SDAP.

2.11. Proposition. *Every Z-set in a space with LCAP is strong.*

Proof. Let A be a Z-set in a space X having LCAP. To prove that the Z-set A is strong, fix a cover $\mathcal{U} \in \mathrm{cov}(X)$. Let $\mathcal{V} \in \mathrm{cov}(X)$ be a cover with $\mathcal{S}t\,\mathcal{V} \prec \mathcal{U}$. Since X has LCAP, there is a map $f\colon X \to X$ such that $(f, \mathrm{id}) \prec \mathcal{V}$, and the closure $F = \mathrm{Cl}(f(X))$ is locally compact. By Lemma 1.1, there is a cover $\mathcal{W} \in \mathrm{cov}(X)$,

$\mathcal{W} \prec \mathcal{V}$, such that a map $g: F \to X$ is perfect whenever $(g, \mathrm{id}_F) \prec \mathcal{W}$. Let $g: X \to X$ be a map with $(g, \mathrm{id}) \prec \mathcal{W}$ and $g(X) \cap A = \varnothing$. Consider the composition $h = g \circ f: X \to X$, and notice that $h(X) \subset g(F)$ and $g(F) \subset X$ is a closed set with $g(F) \cap A = \varnothing$. Hence $\mathrm{Cl}(h(X)) \cap A = \varnothing$, i.e., A is a strong Z-set in X. □

Finally, let us consider LCAP in convex sets.

2.12. Conjecture. Every convex AR-set in a linear metric space has LCAP.

The following statement confirms partly this conjecture.

2.13. Proposition. *Let X be a convex AR-set in a linear metric space L, and $\bar{X}$ be the completion of X with respect to any invariant metric on L. The space X satisfies LCAP if either $\bar{X}$ is an AR or $\bar{X}$ is not locally compact.*

Proof. If $\bar{X}$ is not locally compact then it satisfies SDAP [DM]. Since X is convex and dense in $\bar{X}$, $\bar{X} \setminus X$ is locally homotopy negligible in $\bar{X}$. Hence X satisfies SDAP too. By Proposition 2.7, X has LCAP.

If $\bar{X}$ is a locally compact AR then X has LCAP by Proposition 2.5. □

3. Some properties of towers of subsets.

Under a tower in a space X we understand any increasing sequence $X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \subset X$ of subsets of X.

3.1. Definition. A tower $(X_n)_{n\in\mathbb{N}}$ of subsets of a space X is defined to satisfy

(1) *the flattening property in X* if given a cover $\mathcal{U} \in \mathrm{cov}(X)$ there is a map $f: X \to X$ such that $(f, \mathrm{id}) \prec \mathcal{U}$ and every point $x \in X$ has a neighborhood $W \subset X$ with $W \cap f(X) \subset X_n$ for some $n \in \mathbb{N}$;
(2) *the strong flattening property in X* if for every open set $U \subset X$ the tower $(U \cap X_n)_{n\in\mathbb{N}}$ has the flattening property in U;
(3) *the mapping absorption property for finite-dimensional compacta* (MAPFDC) if given a cover $\mathcal{U} \in \mathrm{cov}(X)$, a pair of finite-dimensional compacta $B \subset A$, and a map $f: A \to X$ with $f(B) \subset X_n$ for some $n \in \mathbb{N}$, there are an $m \in \mathbb{N}$ and a map $\bar{f}: A \to X_m$ such that $(\bar{f}, f) \prec \mathcal{U}$ and $\bar{f}|B = f|B$.

The following proposition establishes relationship between the defined above conceptions and LCAP.

3.2. Proposition. *Let X be an ANR with LCAP and $(X_n)_{n\in\mathbb{N}}$ be a tower of subsets in X. If $(X_n)_{n\in\mathbb{N}}$ satisfies MAPFDC then it enjoys the strong flattening property.*

Proof. Suppose the tower $(X_n)_{n\in\mathbb{N}}$ has MAPFDC. We show first that it has the flattening property. Fix a cover $\mathcal{U} \in \mathrm{cov}(X)$.

Let $\mathcal{U}' \in \mathrm{cov}(X)$ be a cover with $\mathcal{S}t\,\mathcal{U}' \prec \mathcal{U}$. By Proposition 2.2, there are a locally finite simplicial complex K, a map $q: X \to K$, and a perfect map $p_0: K \to X$ such that $(p_0 \circ q, \mathrm{id}) \prec \mathcal{U}'$. By Lemma 1.1, there is a cover $\mathcal{V} \in \mathrm{cov}(X)$, $\mathcal{S}t\,\mathcal{V} \prec \mathcal{U}'$, such that a map $p: K \to X$ is perfect whenever $(p, p_0) \prec \mathcal{S}t\,\mathcal{V}$. Write $K = \bigcup_{n=1}^{\infty} K_n$, where each K_n is an open set in K with the compact finite-dimensional closure $\bar{K}_n \subset K_{n+1}$.

Inductively, we shall construct a number sequence $\{m(n)\}_{n\in\mathbb{N}}$ and a sequence of maps $\{p_n: K \to X\}_{n\in\mathbb{N}}$ such that

$$p_n|(K \setminus K_{n+1}) \cup \bar{K}_{n-1} = p_{n-1}|(K \setminus K_{n+1}) \cup \bar{K}_{n-1}, \; p_n(\bar{K}_n) \subset X_{m(n)}, \; (p_n, p_{n-1}) \prec \mathcal{V}. \tag{$*_n$}$$

Let $K_0 = \varnothing$, $m(0) = 1$, and assume that the map p_{n-1} and the number $m(n-1)$ have been constructed. Since X is an ANR, there exists a cover $\mathcal{W} \in \operatorname{cov}(X)$ such that any $\mathcal{W}$-near to $p_{n-1}|\bar{K}_n$ map $g: \bar{K}_n \to X$ extends to a map $p_n: K \to X$ such that $(p_n, p_{n-1}) \prec \mathcal{V}$ and $p_n|K \setminus K_{n+1} = p_{n-1}|K \setminus K_{n+1}$. Since the sequence $(X_k)_{k\in\mathbb{N}}$ has MAPFDC, there is a number $m(n) \in \mathbb{N}$ and a map $g: \bar{K}_n \to X_{m(n)}$ such that $(g, p_{n-1}|\bar{K}_n) \prec \mathcal{W}$ and $g|\bar{K}_{n-1} = p_{n-1}|\bar{K}_{n-1}$. Extend g to a map $p_n: K \to X$ with the properties $p_n|K \setminus K_{n+1} = p_{n-1}|K \setminus K_{n+1}$ and $(p_n, p_{n-1}) \prec \mathcal{V}$. Evidently, the map p_n and a number $m(n)$ satisfy $(*_n)$. The inductive step is over.

Letting $p = \lim_{n\to\infty} p_n: K \to X$, we see that $(p, p_0) \prec \mathcal{S}t\,\mathcal{V} \prec \mathcal{U}'$, and consequently, the map p is perfect. Let us show that the map $f = p \circ q$ satisfies our requirements. Evidently, $(f, \mathrm{id}) \prec \mathcal{S}t\mathcal{U}' \prec \mathcal{U}$. Since K is locally compact, and p is perfect, $p(K)$ is a closed locally compact set in X. Moreover, $p(\bar{K}_n) \subset X_{m(n)}$, $n \in \mathbb{N}$. Let $x \in X$. If $x \notin p(K)$ then there is nothing to do (just let $W = X \setminus p(K)$ and notice that $W \cap f(X) = \varnothing$). So further we assume that $x \in p(K) \supset f(X)$. Since the map p is perfect, $p^{-1}(x)$ is compact, and hence, $p^{-1}(x) \subset K_n$ for some $n \in \mathbb{N}$. By closeness of p, there is a neighborhood W of x such that $p^{-1}(W) \subset K_n$. Then $W \cap f(X) \subset W \cap p(K) \subset p \circ p^{-1}(W) \subset p(K_n) \subset X_{m(n)}$. Therefore, the tower $(X_n)_{n\in\mathbb{N}}$ has the flattening property.

Now let us show that $(X_n)_{n\in\mathbb{N}}$ has the strong flattening property in X. Fix any open set $U \subset X$. It is easily verified that the tower $(U \cap X_n)_{n\in\mathbb{N}}$ has MAPFDC in U. By Proposition 2.3, U has LCAP. Then by the proved above, $(U \cap X_n)_{n\in\mathbb{N}}$ has the flattening property in U, i.e. $(X_n)_{n\in\mathbb{N}}$ has the strong flattening property in X. □

The previous proposition is of importance because on practice it is much easier to verify the MAPFDC than the strong flattening property. In particular, we have the following statement which can be proved by analogy with [CDM, 3.2] (in the proof one should use a well known fact asserting that any convex set is an absolute extensor for the class of finite-dimensional metrizable spaces [Hu, Ch.V]).

3.3. Proposition. *Let X be a set in a linear metric space and $(X_n)_{n\in\mathbb{N}}$ be a tower of convex subsets in X such that $\bigcup_{n\in\mathbb{N}} X_n$ is dense in X. Then the tower $(X_n)_{n\in\mathbb{N}}$ has the mapping absorption property for finite-dimensional compacta.*

4. The main technical result.

4.1. Theorem. *Let X be an ANR such that every Z-set in X is strong, and $\mathfrak{X} = (X, X_\gamma)_{\gamma\in\Gamma}$, $\mathfrak{C} = (C, C_\gamma)_{\gamma\in\Gamma}$ be two Γ-systems. The Γ-system $\mathfrak{X}$ is strongly $\mathfrak{C}$-universal, provided there exists a tower $X_1 \subset X_2 \subset \dots X_n \subset \dots \subset X$ of Z-sets such that*

1. *for every $n \in \mathbb{N}$ X_n is an ANR such that the Γ-system $X_n \cap \mathfrak{X}$ is strongly $\mathfrak{C}$-universal;*
2. *the tower $(X_n)_{n\in\mathbb{N}}$ has the strong flattening property in X.*

Proof. Suppose a space X, systems $\mathfrak{X}$, $\mathfrak{C}$, and a tower $(X_n)_{n=1}^\infty$ satisfy the assumptions of the theorem.

Fix a cover $\mathcal{U} \in \operatorname{cov}(X)$, a closed subset $B \subset C$, and a map $f: C \to X$ which restriction $f|B: B \to X$ is a Z-embedding with $(f|B)^{-1}(\mathfrak{X}) = B \cap \mathfrak{C}$.

To prove the theorem, it is necessary to construct a Z-embedding $\bar{f}: C \to X$ such that $\bar{f}|B = f|B$, $(\bar{f}, f) \prec \mathcal{U}$, and $\bar{f}^{-1}(\mathfrak{X}) = \mathfrak{C}$.

By our assumptions, every Z-set in X is strong. In particular, $f(B)$ is a strong Z-set in X. Therefore, using [BM, 1.1] and changing, if necessary, f by a near map, we can assume that $f(C \setminus B) \cap f(B) = \varnothing$ and the map f is closed over the set $f(B)$.

Let $C' = C \setminus B$, $X' = X \setminus f(B)$, and $\mathcal{U}' \in \operatorname{cov}(X')$ be a cover such that $\mathcal{S}t\,\mathcal{U}' \prec \mathcal{U}$ and $\mathcal{S}t\,\mathcal{U}' \prec \{O(x, d(x, f(B))/2) \mid x \in X'\}$ (here d is any metric on X). Since the tower $(X_n)_{n\in\mathbb{N}}$ has the strong flattening property in X, there is a map $p\colon X' \to X'$ such that $(p, \mathrm{id}) \prec \mathcal{U}'$ and every point $x \in X'$ has a neighborhood $W \subset X'$ such that $W \cap p(X') \subset X_n$ for some $n \in \mathbb{N}$. Let $F = \mathrm{Cl}_{X'}(p(X'))$, and write $F = \bigcup_{n\in\mathbb{N}} W_n$, where

$$W_n = \{x \in F \mid \text{ there is a neighborhood } W \subset F \text{ of } x \text{ such that } W \subset X_n\}, \quad n \in \mathbb{N}.$$

Let $U_1 \subset U_2 \subset \cdots \subset F$ be a tower of open sets in F such that $F = \bigcup_{n\in\mathbb{N}} U_n$ and for every $n \in \mathbb{N}$ $U_n \subset \bar{U}_n \subset U_{n+1}$, $\bar{U}_n \subset X_n$, and $\bar{U}_n \subset \{x \in X \mid d(x, f(B) \geq 2^{-n}\}$. Since the collection $\{F \setminus U_n\}_{n\in\mathbb{N}}$ is locally finite in X', by Lemma 1.5, there exists a cover $\mathcal{V} \in \operatorname{cov}(X')$, $\mathcal{S}t\,\mathcal{V}' \prec \mathcal{U}'$, such that the collection $\{\mathcal{S}t\,(F \setminus U_n, \mathcal{V})\}_{n\in\mathbb{N}}$ is still locally finite in X'. Let $\mathcal{V}' \in \operatorname{cov}(X')$ be a cover such that $\mathcal{S}t\,\mathcal{V}' \prec \mathcal{V}$.

By the choice of the cover $\mathcal{U}'$, the map $\bar{p} = p \cup \mathrm{id}\colon X = X' \cup f(B) \to X$ is continuous. Remark that the set $\tilde{F} = F \cup f(B)$ is closed in X. For every $n \in \mathbb{N}$ let $C_n = (\bar{p} \circ f)^{-1}(U_n)$ and $\tilde{C}_n = (\bar{p} \circ f)^{-1}(\bar{U}_n)$, and notice that C_n and $\tilde{C}_n$ are respectively open and closed sets in C with $C_n \subset \tilde{C}_n \subset C' = C \setminus B$.

To produce the required Z-embedding $\bar{f}\colon C \to X$ we shall construct inductively a sequence $\{f_n\colon C \to X\}_{n=0}^{\infty}$ of maps which satisfy the following conditions

- (1_n) $f_n|\tilde{C}_{n-1} \cup (C \setminus C_{n+1}) = f_{n-1}|\tilde{C}_{n-1} \cup (C \setminus C_{n+1})$;
- (2_n) $f_n(\tilde{C}_{n+1}) \subset X_{n+1}$;
- (3_n) $f_n(C_{n+1}) \cap (\tilde{F} \setminus U_{n+1}) = \varnothing$;
- (4_n) $\mathrm{Cl}_X(f_n(\tilde{C}_{n+1})) \cap (\tilde{F} \setminus U_{n+2}) = \varnothing$;
- (5_n) $f_n|C_{n+1}\colon C_{n+1} \to X \setminus (\tilde{F} \setminus U_{n+1})$ is a closed embedding;
- (6_n) $f_n(\tilde{C}_n)$ is closed in X;
- (7_n) $(f_n|C', f_{n-1}|C') \prec \mathcal{V}'$;
- (8_n) $f_n^{-1}(\mathfrak{X}) \cap C_{n+1} = \mathfrak{C} \cap C_{n+1}$.

Let $f_{-1} = \bar{p} \circ f$ and $C_{-1} = \tilde{C}_{-1} = C_0 = \tilde{C}_0 = \varnothing$. Assume that the map $f_{n-1}\colon C \to X$ satisfying (1_{n-1})–(8_{n-1}) has been constructed.

The inductive assumptions (1_k), $k < n$, imply $f_{n-1}(\tilde{C}_{n+1} \setminus C_n) = \bar{p} \circ f(\tilde{C}_{n+1} \setminus C_n) \subset \bar{U}_{n+1} \subset X_{n+1}$. This and (2_{n-1}) yield $f_{n-1}(\tilde{C}_{n+1}) \subset X_{n+1}$. Let $D = f_{n-1}(\tilde{C}_{n-1}) \cup (\tilde{F} \setminus U_{n+1})$.

Claim. $(f_{n-1}|\tilde{C}_{n+1})^{-1}(X_{n+1} \setminus D) = C_{n+1} \setminus \tilde{C}_{n-1}$.

Proof. In fact, we have to show that $f_{n-1}(C_{n+1} \setminus \tilde{C}_{n-1}) \subset X_{n+1} \setminus D$ and $f_{n-1}((\tilde{C}_{n+1} \setminus C_{n+1}) \cup \tilde{C}_{n-1}) \subset D$.

The second inclusion is obvious because $f_{n-1}(\tilde{C}_{n+1} \setminus C_{n+1}) = \bar{p} \circ f(\tilde{C}_{n+1} \setminus C_{n+1}) \subset \bar{U}_{n+1} \setminus U_{n+1}$. The first one will be proven if we show

$$f_{n-1}(C_{n+1} \setminus \tilde{C}_{n-1}) \cap (\tilde{F} \setminus U_{n+1}) = \varnothing \tag{1}$$

and

$$f_{n-1}(C_{n+1} \setminus \tilde{C}_{n-1}) \cap f_{n-1}(\tilde{C}_{n-1}) = \varnothing. \tag{2}$$

Indeed, by (3_{n-1}),

$$f_{n-1}(C_n) \cap (\tilde{F} \setminus U_n) = \varnothing. \tag{3}$$

The conditions (1_k), $k < n$, yield $f_{n-1}(C_{n+1} \setminus C_n) = \bar{p} \circ f(C_{n+1} \setminus C_n) \subset U_{n+1} \setminus U_n$. This together with (3) just implies (1) and

$$f_{n-1}(C_{n+1} \setminus C_n) \cap f_{n-1}(C_n) = \varnothing. \tag{4}$$

Since the map $f_{n-1}|C_n$ is injective, we have

$$f_{n-1}(C_n \setminus \tilde{C}_{n-1}) \cap f_{n-1}(\tilde{C}_{n-1}) = \varnothing. \tag{5}$$

The conditions (4) and (5) obviously imply (2). □

By (4_{n-1}), $\mathrm{Cl}_X(f_{n-1}(\tilde{C}_n)) \cap (\tilde{F} \setminus U_{n+1}) = \varnothing$. Let W_1 be an open neighborhood of $\mathrm{Cl}_X(f_{n-1}(\tilde{C}_n))$ such that $\overline{W}_1 \cap (\tilde{F} \setminus U_{n+1}) = \varnothing$.

Let us show that we have also

$$\mathrm{Cl}_X(f_{n-1}(\tilde{C}_{n+1})) \cap (\tilde{F} \setminus U_{n+2}) = \varnothing. \tag{6}$$

Indeed, $\mathrm{Cl}_X(f_{n-1}(\tilde{C}_{n+1})) = \mathrm{Cl}_X(f_{n-1}(\tilde{C}_{n+1} \setminus \tilde{C}_n)) \cup \mathrm{Cl}_X(f_{n-1}(\tilde{C}_n))$. But $\mathrm{Cl}_X(f_{n-1}(\tilde{C}_{n+1} \setminus \tilde{C}_n)) = \mathrm{Cl}_X(\bar{p} \circ f(\tilde{C}_{n+1} \setminus \tilde{C}_n)) \subset \bar{U}_{n+1}$. Since $\bar{U}_{n+1} \cap (F \setminus U_{n+2}) = \varnothing$, (6) follows. Let $W_2 \subset X$ be an open neighborhood of $\mathrm{Cl}_X(f_{n-1}(\tilde{C}_{n+1}))$ such that

$$\overline{W}_2 \cap (\tilde{F} \setminus U_{n+2}) = \varnothing. \tag{7}$$

Let $\mathcal{W} \in \mathrm{cov}(X_{n+1} \setminus D)$ be a cover such that $\mathcal{W} \prec \mathcal{V}'$,

$$\mathcal{W} \prec \{W_1, X \setminus \mathrm{Cl}_X(f_{n-1}(\tilde{C}_n))\}, \tag{8}$$

$$\mathcal{W} \prec \{W_2, X \setminus \mathrm{Cl}_X(f_{n-1}(\tilde{C}_{n+1}))\}, \text{ and} \tag{9}$$

$$\mathcal{W} \prec \{O(x, d(x, D)/2) \mid x \in X_{n+1} \setminus D\}. \tag{10}$$

By Lemma 1.2(1), the Γ-system $X_{n+1} \cap \mathfrak{X}$ is strongly $\tilde{C}_{n+1} \cap \mathfrak{C}$-universal. Then, we can apply Lemma 1.2(2) to produce a Z-embedding $e \colon C_{n+1} \setminus \tilde{C}_{n-1} \to X_{n+1} \setminus D$ such that

$$(e, f_{n-1}|C_{n+1} \setminus \tilde{C}_{n-1}) \prec \mathcal{W} \tag{11}$$

and $e^{-1}(\mathfrak{X}) = (C_{n+1} \setminus \tilde{C}_{n-1}) \cap \mathfrak{C}$.

Because of (10) and (11), the embedding e extends to a continuous map $f_n \colon C \to X$ such that $f_n|\tilde{C}_{n-1} \cup (C \setminus C_{n+1}) = f_{n-1}|\tilde{C}_{n-1} \cup (C \setminus C_{n+1})$.

Let us show that the map f_n satisfies the conditions (1_n)–(8_n). Verification of (1_n), (2_n), (3_n), (7_n) and (8_n) is easy and is left to the reader.

Because of (9) and (11), $f_n(\tilde{C}_{n+1}) \subset W_2$, and hence, $\mathrm{Cl}_X(f_n(\tilde{C}_{n+1})) \subset \overline{W}_2$. This together with (7) yields (4_n).

To show (5_n), notice at first that the map $f_n|C_{n+1} \colon C_{n+1} \to X_{n+1} \setminus (\tilde{F} \setminus U_{n+1})$ is injective and continuous. To verify that it is an embedding, fix a point $x_0 \in C_{n+1}$ and a sequence $(x_k)_{k=1}^{\infty} \subset C_{n+1}$ such that $\lim_{k\to\infty} f_n(x_k) = f_n(x_0)$. If $x_0 \notin \tilde{C}_{n-1}$ then $f_n(x_0) \notin f_n(\tilde{C}_{n-1})$. Since $f_n(\tilde{C}_{n-1}) = f_{n-1}(\tilde{C}_{n-1})$ is closed in X, without loss of generality, $f_n(x_k) \notin f_n(\tilde{C}_{n-1})$, $k \in \mathbb{N}$. Then $f_n(x_0) = e(x_0)$ and $f_n(x_k) = e(x_k)$, $k \in \mathbb{N}$. Since e is an embedding, the sequence $(x_k)_{k\in\mathbb{N}}$ converges to x_0. Now let us consider the other variant: $x_0 \in \tilde{C}_{n-1}$. Then by (10) and (11), $\lim_{k\to\infty} f_n(x_k) = f_n(x_0) \in f_n(\tilde{C}_{n-1})$ implies $\lim_{k\to\infty} f_{n-1}(x_k) = f_{n-1}(x_0)$. It follows from (5_{n-1}) and (6_{n-1}) that $f_{n-1}(\tilde{C}_{n-1})$ is a closed subset in X such that $f_{n-1}(\tilde{C}_{n-1})\cap(\tilde{F}\setminus U_n) = \varnothing$. Since $f_{n-1}(C\setminus C_n) = \bar{p}\circ f(C\setminus C_n) \subset \tilde{F}\setminus U_n$, the equality $\lim_{k\to\infty} f_{n-1}(x_k) = f_{n-1}(x_0) \in f_{n-1}(\tilde{C}_{n-1})$ implies that $x_k \in C_n$ for almost all $k \in \mathbb{N}$. Since $f_{n-1}|C_n$ is an embedding, $(x_k)_{k\in\mathbb{N}}$ converges to x_0. Therefore, $f_n|C_{n+1}$ is an embedding. Since $f_n(\tilde{C}_{n-1}) = f_{n-1}(\tilde{C}_{n-1})$ is closed in $X_{n+1} \setminus (\tilde{F} \setminus U_{n+1})$ and $f_n(C_{n+1} \setminus \tilde{C}_{n-1}) = e(C_{n+1} \setminus \tilde{C}_{n-1})$ is closed in $(X_{n+1} \setminus (\tilde{F} \setminus U_{n+1})) \setminus f_{n-1}(\tilde{C}_{n-1})$, the union $f_n(\tilde{C}_{n-1}) \cup f_n(C_{n+1} \setminus \tilde{C}_{n-1}) = f_n(C_{n+1})$ is closed in $X_{n+1} \setminus (\tilde{F} \setminus U_{n+1})$. Hence (5_n) holds.

By (8) and (11), we have $f_n(\tilde{C}_n) \subset \overline{W}_1$. Since $f_n(\tilde{C}_n)$ is closed in $X_{n+1} \setminus (\tilde{F} \setminus U_{n+1})$ and $f_n(\tilde{C}_n) \subset \bar{W}_1$, we get $f_n(\tilde{C}_n)$ is closed in $\overline{W}_1 \cap (X_{n+1} \setminus (\tilde{F} \setminus U_{n+1})) = X_{n+1} \cap \overline{W}_1$ (recall that $\overline{W}_1 \cap (\tilde{F} \setminus U_{n+1}) = \varnothing$). Since $X_{n+1} \cap \overline{W}_1$ is closed in X, $f_n(\tilde{C}_n)$ is closed in X, i.e., (6_n) holds. The inductive step is over.

We claim that $\bar{f} = \lim_{n\to\infty} f_n: C \to X$ is the required Z-embedding. Indeed, by (1_n), (7_n) and (8_n), $n \in \mathbb{N}$, we have $\bar{f}|B = \bar{p} \circ f|B = f|B$, $(\bar{f}|C', \bar{p} \circ f|C') \prec \mathcal{S}t\,\mathcal{V}'$, and $(\bar{f}|C')^{-1}(\mathfrak{X}) = C' \cap \mathfrak{C}$. Since $\mathcal{S}t\,\mathcal{V}' \prec \mathcal{U}'$, $\mathcal{S}t\mathcal{U}' \prec \mathcal{U}$, $(\bar{p} \circ f, f) \prec \mathcal{U}'$, and $(\bar{f}|B)^{-1}(\mathfrak{X}) = B \cap \mathfrak{C}$, we obtain $(\bar{f}, f) \prec \mathcal{U}$ and $\bar{f}^{-1}(\mathfrak{X}) = \mathfrak{C}$.

Let us show that $\bar{f}$ is a closed embedding. Since $\bar{f}$ is closed over $f(B)$, it suffices to verify that $\bar{f}|C': C' \to X'$ is a closed embedding. By $(5_n), (6_n)$, $n \in \mathbb{N}$, for every $n \in \mathbb{N}$ the restriction $\bar{f}|\tilde{C}_{n+1}: \tilde{C}_{n+1} \to X'$ is a closed embedding. Since $C' = \bigcup_{n\in\mathbb{N}} \tilde{C}_{n+1}$ this gives that the map $\bar{f}|C'$ is injective. Since $(\bar{f}|C', \bar{p}\circ f|C') \prec \mathcal{S}t\,\mathcal{V}' \prec \mathcal{V}$ and $\bar{p} \circ f(\tilde{C}_{n+1} \setminus C_{n-1}) \subset F \setminus U_{n-1}$, we have $\bar{f}(\tilde{C}_{n+1} \setminus C_{n-1}) \subset St(F \setminus U_{n-1}, \mathcal{V})$. By the choice of the cover $\mathcal{V}$, the collection $\{\mathcal{S}t\,(F \setminus U_n, \mathcal{V})\}_{n\in\mathbb{N}}$ is locally finite in X'. Hence the collection $\{\bar{f}(\tilde{C}_{n+1} \setminus C_{n-1})\}_{n\in\mathbb{N}}$ is locally finite in X' as well. Since $\bar{f}$ is injective and for every $n \in \mathbb{N}$ $\bar{f}|\tilde{C}_{n+1} \setminus C_{n-1}: \tilde{C}_{n+1} \setminus C_{n-1} \to X'$ is a closed embedding, we get that $\bar{f}: C' \to X'$ is a closed embedding, and consequently, $\bar{f}: C \to X$ is a closed embedding too.

To see that $\bar{f}(C)$ is a Z-set in X, remark that $\bar{f}(C) = f(B) \cup \bigcup_{n\in\mathbb{N}} \bar{f}(\tilde{C}_{n+1} \setminus C_{n-1})$, $f(B)$ is a Z-set in X, and $\{\bar{f}(\tilde{C}_{n+1}, C_{n-1})\}_{n\in\mathbb{N}}$ is a locally finite collection of Z-sets in $X \setminus f(B)$ (recall that each $X_{n+1} \supset \bar{f}(\tilde{C}_{n+1} \setminus C_{n-1})$ is a Z-set in X).□

5. The strongly universal property in star-shaped sets.

Let L be a linear space. Recall that a subset $X \subset L$ is defined to be *star-shaped with respect to a point* $x \in X$ if for every $y \in X$ the segment $[y, x]$ connecting x and y lies in X. *The kernel* $\mathrm{Ker}(X)$ of X is the set of all points $x \in X$ such that X is star-shaped with respect to x. It is well known [Va] that the kernel $\mathrm{Ker}(X)$ is a convex set in L. We say that X is *a star-shaped set (with dense kernel)* if $\mathrm{Ker}(X) \neq \varnothing$ ($\mathrm{Ker}(X)$ is dense in X). Let us remark that every convex set X is star-shaped with dense kernel $\mathrm{Ker}(X) = X$. On the other hand, for every star-shaped set $X \subset L$ with dense kernel we have $\mathrm{Ker}(X) \subset X \subset \bar{X} = \overline{\mathrm{Ker}(X)}$, and thus the closure $\bar{X}$ is a convex set.

A subset $A \subset L$ is defined to be *algebraically infinite-dimensional* if it contains an infinite linearly independent subset $S \subset A$. For a set $A \subset L$ denote by $\pi_A: L \to L/\operatorname{span} A$ the quotient map. We shall say that a set $A \subset L$ has *infinite codimension in an* $X \subset L$ if the quotient set $\pi_A(X) \subset L/\operatorname{span} A$ is algebraically infinite-dimensional.

Recall that a subset $A \subset L$ of a linear topological space is *bounded* if for every neighborhood $U \subset L$ of the origin there is an $n \in \mathbb{N}$ such that $A \subset n \cdot U$.

5.1. Theorem. *Let A be a closed set in a linear metric space L, $\vec{\mathcal{C}}$ a class of Γ-systems, and $\mathfrak{L} = (L, L_\gamma)_{\gamma\in\Gamma}$ a linear Γ-system in L. Suppose $X \subset L$ is a star-shaped AR-set with LCAP such that $\mathrm{Ker}(X) \cap (\bigcap \mathfrak{L})$ is dense in X and $A \subset X$ has infinite codimension in $\mathrm{Ker}(X) \cap (\bigcap \mathfrak{L})$. The Γ-system $X \cap \mathfrak{L}$ is strongly $\vec{\mathcal{C}}$-universal, provided one of the following conditions is satisfied*

1. *A is bounded and the Γ-system $A \cap \mathfrak{L}$ is $\vec{\mathcal{C}}$-universal;*
2. *the class $\vec{\mathcal{C}}$ is $(0,1]$-stable and $\{*\}$-additive, and $A \cap \mathfrak{L}$ is $\vec{\mathcal{C}}$-universal;*
3. *A is an ANR and the Γ-system $A \cap \mathfrak{L}$ is strongly $\vec{\mathcal{C}}$-universal.*

Proof. To verify that the Γ-system $X \cap \mathfrak{L}$ is strongly $\vec{\mathcal{C}}$-universal, fix a cover $\mathcal{U} \in \mathrm{cov}(X)$, a Γ-system $\mathfrak{C} = (C, C_\gamma)_{\gamma\in\Gamma} \in \vec{\mathcal{C}}$, a closed subset $B \subset C$, and a map $f: C \to X$ such that $f|B: B \to X$ is a Z-embedding with $(f|B)^{-1}(\mathfrak{L}) = B \cap \mathfrak{C}$.

Since X is an AR with LCAP, every Z-set in X is strong. In particular, $f(B)$ is a strong Z-set in X. Then by [BM, 1.1], without loss of generality, we can assume that $f(C \setminus B) \cap f(B) = \varnothing$ and f is closed over the set $f(B)$.

Let $C' = C \setminus B$, $X' = X \setminus f(B)$, and $\mathcal{U}' \in \mathrm{cov}(X')$ be a cover such that $\mathcal{S}t\mathcal{U}' \prec \mathcal{U}$ and $\mathcal{S}t\mathcal{U}' \prec \{O(x, d(x, f(B))/2) \mid x \in X'\}$ (here d is an invariant metric on L). By Proposition 2.3, the open subspace X' in X has LCAP. Thus, there is a map $p: X' \to X'$ such that $(p, \mathrm{id}) \prec \mathcal{U}'$ and the closure $F = \mathrm{Cl}_{X'}(p(X'))$ is locally compact. By the choice of the cover $\mathcal{U}'$, the map $\bar{p} = p \cup \mathrm{id}: X' \cup f(B) \to X$ is continuous.

Since A has infinite codimension in $\mathrm{Ker}(X) \cap (\bigcap \mathfrak{L})$, one can find a countable dense subset $S \subset \mathrm{Ker}(X) \cap (\bigcap \mathfrak{L})$ such that $\mathrm{span}(A) \cap \mathrm{span}(S) = \{0\}$ and $\mathrm{span}(S \cup A)$ has infinite codimension in $\mathrm{Ker}(X) \cap (\bigcap \mathfrak{L})$. Let $x_0 \in \mathrm{Ker}(X) \cap (\bigcap \mathfrak{L}) \setminus \mathrm{span}(S \cup A)$ be any point. The set $\mathrm{conv}\, S$, being convex and dense, is homotopy dense in X, see [BRZ, §1.2, Ex,12 and 13].

Changing, if necessary, p by a near map, without loss of generality, we can assume that $F \subset \mathrm{conv}\, S \subset \mathrm{Ker}(X)$. Since $F \subset X'$ is closed and locally compact, there is a locally finite cover $\mathcal{W} \in \mathrm{cov}(X')$, $\mathcal{W} \prec \mathcal{U}'$, such that for every $W \in \mathcal{W}$ the intersection $\overline{W} \cap F$ is compact.

Now we are going to construct a Z-embedding $f': C' \to X'$ such that $(f', p \circ f|C') \prec \mathcal{W}$ and $(f')^{-1}(\mathfrak{L}) = \mathfrak{C} \setminus B$. We consider separately three cases.

I). The set A is bounded and $A \cap \mathfrak{L}$ is $\vec{\mathcal{C}}$-universal. Let $e: C \to A$ be a closed embedding with $e^{-1}(\mathfrak{L}) = \mathfrak{C}$. By continuity of linear operations, there is a continuous function $\varepsilon: F \to (0,1]$ such that every $x \in F \subset \mathrm{Ker}(X)$ has a neighborhood $W \in \mathcal{W}$ such that $(1 - \varepsilon(x))x + \frac{\varepsilon(x)}{2} x_0 + \frac{\varepsilon(x)}{2} A \subset W$.

Define a map $f': C' \to X'$ by the formula

$$f'(c) = (1 - \varepsilon \circ p \circ f(c))\, p \circ f(c) + \frac{\varepsilon \circ p \circ f(c)}{2} x_0 + \frac{\varepsilon \circ p \circ f(c)}{2} e(c), \quad c \in C'. \quad (1)$$

II). A is an ANR and $A \cap \mathfrak{L}$ is strongly $\vec{\mathcal{C}}$-universal. Without loss of generality, the space A is not discrete (otherwise, $|C| \le 1$ and the theorem is trivial). Then there exists an embedding $i: [0,1] \to A$ such that $d(i(t), i(0)) \le t$ for every $t \in [0,1]$. Let $a_0 = i(0)$.

Using continuity of linear operations on X, construct a continuous function $\varepsilon: X \to [0,1]$ such that $\varepsilon^{-1}(0) = f(B)$ and every $x \in F$ has a neighborhood $W \in \mathcal{W}$ such that $(1 - \varepsilon(x))x + \frac{\varepsilon(x)}{2} x_0 + \frac{\varepsilon(x)}{2} O(a_0, 2\varepsilon(x)) \subset W$.

Consider the map $g = i \circ \varepsilon \circ \bar{p} \circ f: C \to A$ and notice that $g^{-1}(A \setminus \{a_0\}) = C'$. By Lemma 1.2, there is a closed embedding $e: C' \to A \setminus \{a_0\}$ such that $d(e(c), g(c)) < d(g(c), a_0)/2$, $c \in C'$, and $e^{-1}(\mathfrak{L}) = \mathfrak{C} \setminus B$.

Define a map $f': C' \to X'$ by the formula

$$f'(c) = (1 - \varepsilon \circ p \circ f(c))\, p \circ f(c) + \frac{\varepsilon \circ p \circ f(c)}{2} x_0 + \frac{\varepsilon \circ p \circ f(c)}{2} e(c), \quad c \in C'. \quad (2)$$

III). The class $\vec{\mathcal{C}}$ is $(0,1]$-stable and $\{*\}$-additive, and the Γ-system $A \cap \mathfrak{L}$ is $\vec{\mathcal{C}}$-universal. Let $\bar{C}$ be a compactification of C, $\alpha\tilde{C} = \{*\} \cup \bar{C} \times (0,1]$ be the Aleksandrov compactification of the space $\bar{C} \times (0,1]$, and $\tilde{C} = \{*\} \cup C \times (0,1]$. Since the class $\vec{\mathcal{C}}$ is $(0,1]$-stable and $\{*\}$-additive, the Γ-system $\tilde{\mathfrak{C}} = (\tilde{C}, \{*\} \cup C_\gamma \times (0,1])_{\gamma \in \Gamma}$ belongs to the class $\vec{\mathcal{C}}$. Let $e: \tilde{C} \to A$ be a closed embedding such that $e^{-1}(\mathfrak{L}) = \tilde{\mathfrak{C}}$ and put $a_0 = e(*)$. One can easily construct a continuous function $\delta: (0,1] \to (0,1]$ such that $e(C \times (0, \delta(t)]) \subset O(a_0, t)$ for every $t \in (0,1]$. Let $\varepsilon: X \to [0,1]$ be the function from the case II.

Define a map $f': C' \to X'$ by the formula

$$f'(c) = (1 - \varepsilon \circ p \circ f(c))\, p \circ f(c) + \frac{\varepsilon \circ p \circ f(c)}{2} x_0 + \frac{\varepsilon \circ p \circ f(c)}{2} e(c, \delta \circ \varepsilon \circ p \circ f(c)), \quad c \in C'. \quad (3)$$

Let us show that the map f' defined by the formula (1), (2), or (3) is a Z-embedding such that $(f', p \circ f|C') \prec \mathcal{W}$ and $(f')^{-1}(\mathfrak{L}) = \mathfrak{C} \setminus B$. In fact, the last two properties easily follow from the definition of f', and our task now is to show that the map f' is a Z-embedding. By the choice of x_0 and S, the equality $f'(c) = f'(c')$, where $c, c' \in C'$, implies $\varepsilon \circ p \circ f(c) = \varepsilon \circ p \circ f(c')$ and $e(c) = e(c')$ ($e(c, \delta \circ \varepsilon \circ p \circ f(c)) = e(c', \delta \circ \varepsilon \circ p \circ f(c'))$ in the case III). Since e is an embedding, this yields $c = c'$, i.e., the map f' is injective.

To show that f' is a closed embedding, it now suffices to verify that $f': C' \to X'$ is perfect. Fix a compactum $K \subset X'$. We have to show that the preimage $K^- = (f')^{-1}(K) \subset C'$ is compact. Since f is closed over $f(B)$ and $(f', f|C') \prec \mathcal{U}'$, the set K^- is closed not only in C' but also in C. Since $(f', p \circ f|C') \prec \mathcal{W}$, we have $p \circ f(K^-) \subset \mathcal{S}t(K, \mathcal{W})$. By the choice of the cover $\mathcal{W}$, the set $M = \mathrm{Cl}_X(\mathcal{S}t(K, \mathcal{W})) \cap F$ is compact. Let $\varepsilon_0 = \min\{\varepsilon(x) \mid x \in M\}$ and remark that the set $D = [1, \frac{2}{\varepsilon_0}](K - [0,1]M - [0,1]x_0) \subset L$ is compact.

Now we consider separately the cases I, II, and III.

I). It follows from (1) that for every $c \in K^-$

$$e(c) = \frac{2}{\varepsilon \circ p \circ f(c)}(f'(c) - (1 - \varepsilon \circ p \circ f(c))\, p \circ f(c) - \frac{\varepsilon \circ p \circ f(c)}{2} x_0) \subset D.$$

Since e is a closed embedding, $e^{-1}(D)$ is compact, and hence $K^- \subset e^{-1}(D)$ is compact too.

II). Notice that for every $c \in K^-$ we have $\varepsilon \circ p \circ f(c) \geq \varepsilon_0$. Hence, $\varepsilon \circ p \circ f(K^-) \subset [\varepsilon_0, 1]$. Let $\delta = \min\{d(a_0, i(t)) \mid t \in [\varepsilon_0, 1]\}$. By the choice of e, for every $c \in K^-$, we have $d(e(c), a_0) \geq \delta/2$. Hence $e(K^-)$ is closed not only in $A \setminus \{a_0\}$ but also in A. By (2), for every $c \in K^-$ we get $e(c) \in D$. Since the map $e|K^-: K^- \to L$ is a closed embedding, the set $K^- = (e|K^-)^{-1}(D)$ is compact.

III). By (3), for every $c \in K^-$ we get $\delta \circ \varepsilon \circ p \circ f(c) \in [\delta(\varepsilon_0), 1]$ and $e(c, \delta \circ \varepsilon \circ p \circ f(c)) \in D$. Since K^- is a closed set in C, the map $\mathrm{id} \times (\delta \circ \varepsilon \circ p \circ f): K^- \to C \times$

$[\delta(\varepsilon_0), 1] \subset \tilde{C}$ is a closed embedding. Then the map $\tilde{e} = e(\mathrm{id} \times \delta \circ \varepsilon \circ p \circ f)\colon K^- \to A$ is a closed embedding too. Since $\tilde{e}(K^-) \subset D$, we have that $K^- = \tilde{e}^{-1}(D)$ is compact.

Therefore, in all the three cases, we have shown that $f'\colon C' \to X'$ is a closed embedding. By the choice of S and x_0, and by the definition of f', the set $f'(C') \subset \operatorname{span}(A \cup S \cup \{x_0\})$ has infinite codimension in $\operatorname{Ker}(X)$. By the standard arguments (see e.g. [Ba$_2$]), one can show that $f'(C')$ is a Z-set in X'.

Letting finally $\bar{f}|B = f|B$ and $\bar{f}|C \setminus B = f'$, we define a Z-embedding $\bar{f}\colon C \to X$ such that $\bar{f}|B = f|B$, $(\bar{f}, f) \prec \mathcal{U}$, and $\bar{f}^{-1}(\mathfrak{L}) = \mathfrak{C}$. Thus $X \cap \mathfrak{L}$ is a strongly $\vec{\mathcal{C}}$-universal system and the theorem is proven. □

5.2. Remark. If L is a locally convex space then every star-shaped set $X \subset L$ with dense kernel is an AR with LCAP. Indeed, since the closure $\bar{X}$ is convex, $\bar{X}$ is an AR with LCAP according to 2.12. Because $\operatorname{Ker}(X)$ is a convex dense set in $\bar{X}$ with $\operatorname{Ker}(X) \subset X \subset \bar{X}$, X is homotopy dense in $\bar{X}$, and thus, X is an AR with LCAP by 2.4.

6. Characterizing the strongly universal property in linear metric spaces.

6.1. Theorem. *Let L be a linear metric AR-space, $\mathfrak{L} = (L, L_\gamma)_{\gamma \in \Gamma}$ a linear Γ-system such that $\bigcap \mathfrak{L}$ is dense in L, and $\vec{\mathcal{C}}$ a class of Γ-systems. The system $\mathfrak{L}$ is strongly $\vec{\mathcal{C}}$-universal if and only if L contains a closed convex AR-set $X \subset L$ such that $X \cap (\bigcap \mathfrak{L})$ is dense in X and the system $X \cap \mathfrak{L}$ is strongly $\vec{\mathcal{C}}$-universal.*

Proof. The "only if" part is trivial (just let $X = L$). Assume that $X \subset L$ is a convex closed AR-set such that $X \cap (\bigcap \mathfrak{L})$ is dense in X and $X \cap \mathfrak{L}$ is strongly $\vec{\mathcal{C}}$-universal. Let $X_0 = X \cap (\bigcap \mathfrak{L})$.

Without loss of generality, L is infinite-dimensional, and consequently, L satisfies SDAP [DT]. By Proposition 2.7, L has LCAP. If X has infinite codimension in $\bigcap \mathfrak{L}$ then by Theorem 5.1(3), the system $\mathfrak{L}$ is strongly $\vec{\mathcal{C}}$-universal.

So further, we assume that X has finite codimension in $\bigcap \mathfrak{L}$. Without loss of generality, $0 \in X_0$. Let $L_0 = \mathrm{Cl}(\operatorname{span} X)$ and consider the quotient map $\pi\colon L \to L/L_0$. Since X has finite codimension in $\bigcap \mathfrak{L}$, $\pi(\bigcap \mathfrak{L})$ is a dense finite-dimensional linear subspace in L/L_0. This immediately yields that $\pi(\bigcap \mathfrak{L}) = L/L_0$ and the system $\mathfrak{L}$ is homeomorphic to $(L_0 \cap \mathfrak{L}) \times L/L_0$. According to Lemma 1.3, to prove the theorem, it suffices to show that the system $L_0 \cap \mathfrak{L}$ is strongly $\vec{\mathcal{C}}$-universal.

Let $\{x_n\}_{n=1}^\infty \subset \operatorname{span} X_0$ be a dense countable set in L_0. If the interior $\operatorname{Int} X$ of X in L_0 is not empty then, by Lemma 1.2(1), the system $\operatorname{Int} X \cap \mathfrak{L}$ is strongly $\vec{\mathcal{C}}$-universal, and consequently, $\mathcal{U} = \{\operatorname{Int} X + x_n \mid n \in \mathbb{N}\}$ is an open cover of L_0 such that each system $(\operatorname{Int} X + x_n) \cap \mathfrak{L}$, $n \in \mathbb{N}$, is strongly $\vec{\mathcal{C}}$-universal. By Lemma 1.4, the Γ-system $L_0 \cap \mathfrak{L}$ is strongly $\vec{\mathcal{C}}$-universal.

If the interior of X in L_0 is empty, then by [Ba$_2$], X is a Z-set in L_0. Inductively, we shall construct sequences $\{y_n\}_{n=0}^\infty \subset (-\infty, 0] \cdot X_0$ and $\{r_n\}_{n=0}^\infty \subset \mathbb{N}$ such that for every $n \in \mathbb{N}$ the following conditions are satisfied:

$$x_n \in y_n + r_n X_0 \quad \text{and} \quad y_{n-1} + (r_{n-1} + 1) X_0 \subset y_n + r_n X_0 \tag{$*_n$}$$

Let $y_0 = 0$ and $r_0 = 1$. Assume that the point $y_n \in (-\infty, 0] \cdot X_0$ and the number $r_n \in \mathbb{N}$ satisfying $(*_n)$ have been constructed. Since $X_0 \ni 0$ is a convex set, $x_{n+1} = tx - t'x'$ for some $t, t' \geq 0$ and $x, x' \in X_0$. Let $y_{n+1} = y_n - t'x'$. Since $y_n \in (-\infty, 0] \cdot D'$, we have $y_n = -t''x''$ for certain $t'' \geq 0$ and $x'' \in X_0$. Choose

$r_{n+1} \in \mathbb{N}$ to satisfy $r_{n+1} > r_n + t + t' + t'' + 1$. Obviously, y_{n+1} and r_{n+1} satisfy $(*_{n+1})$, and the inductive step is over.

For every $n \in \mathbb{N}$ let $Y_n = y_n + r_n X_0$. Since the couple (L_0, X_n) is affinely homeomorphic to (L_0, X), we get that X_n is a convex Z-set in L_0 and X_n is an AR. Since $y_n \in \operatorname{span} X_0 \subset \bigcap L$ we have also $X_n \cap \mathfrak{L} \cong X \cap \mathfrak{L}$ is a strongly $\vec{\mathcal{C}}$-universal system. Since $\bigcup_{n=1}^{\infty} X_n \supset \{x_n\}_{n=1}^{\infty}$ is dense in L_0, by Proposition 3.3, the tower $(X_n)_{n\in\mathbb{N}}$ has MAPFDC in L_0, and by Proposition 3.2, it has the strong flattening property in L_0. Now Theorem 4.1 implies that the system $L_0 \cap \mathfrak{L}$ is strongly $\vec{\mathcal{C}}$-universal. □

Recall that $Q = [-1, 1]^\omega$ is the Hilbert cube, $s = (-1, 1)^\omega$ is its pseudo-interior, and $\Sigma = \{(t_n)_{n\in\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |t_n| < 1\}$ is its radial interior. By $\mathcal{M}_0$, $\mathcal{M}_1$, and $\mathcal{M}_2$ we denote the Borel classes consisiting of metrizable compacta, Polish spaces, and absolute $F_{\sigma\delta}$-spaces respectively; σ-$\mathcal{M}_1$ denotes the class of σ-complete spaces, i.e., spaces which are countable unions of closed complete-metrizable sets. A Z_σ-space is a space that is a countable union of Z-sets.

We derive from Theorem 6.1 the following Proposition which will be applied in [BDP].

6.2. Proposition. *Let $\tilde{L}$ be a separable Fréchet space, $L \subset \tilde{L}$, $L \neq \tilde{L}$, a dense linear subspace, and $C \subset \tilde{L}$ a closed convex set. The pair $(\tilde{L}, L)$ is homeomorphic to*

1. *$(s \times s, \Sigma \times s)$ if L is an F_σ-set in $\tilde{L}$ and $C \subset L$ is non-locally compact;*
2. *$(s \times Q, \Sigma \times s)$ if $L \in \sigma$-$\mathcal{M}_1$ is contained in a σ-compact set in $\tilde{L}$ and $(C, C \cap L) \cong (Q, s)$;*
3. *$(s \times s \times Q, s \times \Sigma \times s)$ if $L \in \sigma$-$\mathcal{M}_1$ and $(C, \mathcal{C} \cap L) \cong (s \times Q, s \times s)$;*
4. *$(s \times Q^\omega, \Sigma \times \Sigma^\omega)$ if $L \in \mathcal{M}_2$ is contained in a σ-compact set in $\tilde{L}$ and $(C C \cap L) \cong (Q^\omega, \Sigma^\omega)$;*
5. *$(s^\omega, \Sigma^\omega)$ if $L \in \mathcal{M}_2$ is a Z_σ-space and $(C, C \cap L) \cong (s^\omega, \Sigma^\omega)$.*

Proof. We call a pair (X, Y) *$\vec{\mathcal{C}}$-absorbing*, where $\vec{\mathcal{C}}$ is a class of pairs, if it is strongly $\vec{\mathcal{C}}$-universal and there is a set $Z = \bigcup_{n\in\mathbb{N}} Z_n$ such that $X \subset Z \subset Y$ and for every $n \in \mathbb{N}$ Z_n is a Z-set in X with $(Z_n, Z_n \cap Y) \in \vec{\mathcal{C}}$.

Proofs of (1)—(5) rely on Theorem 6.1 and the following uniqueness

6.3. Theorem [Ca, 5.4]. *Any two $\vec{\mathcal{C}}$-absorbing pairs (s, X) and (s, X') are homeomorphic.*

Suppose $L \neq \tilde{L}$ is an F_σ-set in $\tilde{L}$ and $C \subset L$ is a (closed in $\tilde{L}$ convex) non-locally compact set. By [DT], C is homeomorphic to s, and hence, C is strongly $\mathcal{M}_1$-universal. Then the pair $(C, C) = (C, C \cap L)$ is strongly universal for the class $\vec{\mathcal{C}} = \{(M, M) \mid M \in \mathcal{M}_1\}$. By Theorem 6.1, the pair $(\tilde{L}, L)$ is strongly $\vec{\mathcal{C}}$-universal too. Since $L \neq \tilde{L}$ is a dense F_σ-set in $\tilde{L}$, it is actually a Z_σ-set in $\tilde{L}$ (see [BRZ, 4.2.8]). Thus $L = \bigcup_{n\in\mathbb{N}} Z_n$, where each Z_n is a Z-set in $\tilde{L}$. Noticing that $(Z_n, Z_n \cap L) = (Z_n, Z_n) \in \vec{\mathcal{C}}$, we see that $(\tilde{L}, L)$ is a $\vec{\mathcal{C}}$-absorbing pair. By similar arguments, it can be shown that the pair $(s \times s, s \times \Sigma)$ is $\vec{\mathcal{C}}$-absorbing too. Then by 6.3, $(s \times s, \Sigma \times s) \cong (\tilde{L}, L)$.

Proofs of (2)—(5) are analogous and use the fact that the pairs $(s \times Q, \Sigma \times s)$, $(s \times s \times Q, s \times \Sigma \times s)$, $(s \times Q^\omega, \Sigma \times \Sigma^\omega)$ and $(s^\omega, \Sigma^\omega)$ are absorbing for the classes $(\mathcal{M}_0, \mathcal{M}_1) = \{(K, C) \mid \mathcal{M}_1 \ni C \subset K \in \mathcal{M}_0\}$, $(\mathcal{M}_1, \mathcal{M}_1) = \{(K, C) \mid \mathcal{M}_1 \ni C \subset K \in \mathcal{M}_1\}$,

$(\mathcal{M}_0, \mathcal{M}_2) = \{(K, C) \mid \mathcal{M}_2 \ni C \subset K \in \mathcal{M}_0\}$, and $(\mathcal{M}_1, \mathcal{M}_2) = \{(K, C) \mid \mathcal{M}_2 \ni C \subset K \in \mathcal{M}_1\}$ respectively. □

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Received 1.09.1997
Revised 25.05.1998