УДК 517.95

ON THE BEHAVIOUR OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR ELLIPTIC NONDIVERGENCE SECOND ORDER EQUATIONS NEAR THE CONICAL BOUNDARY POINT

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M. Borsuk. *On the behaviour of solutions of the Dirichlet problem for elliptic nondivergence second order equations near the conical boundary point*, Matematychni Studii, **10**(1998) 163–172.

We investigate regularity of the solutions of the Dirichlet problem for second order linear and quasilinear uniformly elliptic nondivergence equations in a neighbourhood of a conical boundary point.

М. Борсук *О поведении решений задачи Дирихле для недивергентного эллиптического уравнения второго порядка около конической граничной точки*// Математичні Студії. – 1998. – Т.10, № 2. – С.163–172.

Исследована регулярность решений задачи Дирихле для линейных и квазилинейных однородных эллиптических недивергентных уравнений второго порядка в окрестности конической граничной точки.

1. Introduction

In this work we examine the question of a smoothness rise of the solutions of the Dirichlet problem for elliptic second order nondivergence linear and quasilinear equations near the conical boundary point of a bounded domain. This problem was investigated earlier in [2,3] for linear equation (see below remark 1). The case of nonlinear equation is considered here for the first time.

Let $G \subset \mathbb{R}^n$, $n > 2$, be a bounded domain with the boundary ∂G which is assumed to be a smooth surface everywhere, exepting the point $O \in \partial G$, an origin of rectangular coordinates, and near the point O it is a convex conic surface with the vertex at O. Let $x = (x_1, \dots, x_n)$ be a point in $\mathbb{R}^n$; let (r, ω) be its spherical coordinates. We shall assume: there are a domain Ω on unit sphere S^{n-1} with the smooth boundary $\partial\Omega$ and a positive number $d > 0$ such that $G \cap \{x : |x| \leq d\} = \{(r,\omega)| 0 < r \leq d, \quad \omega \in \Omega\}$ is a convex cone. Let $G_a^b = G \cap \{(r,\omega)|0 \leq a < r < b \leq d\}$ be a layer in $\mathbb{R}^n$; $\Gamma_a^b = \partial G \cap \{(r,\omega)|0 \leq a < r < b \leq d\}$ a lateral surface of the layer G_a^b; $G_\varepsilon = G/G_0^\varepsilon \ \forall\ \varepsilon > 0$; $\Omega_\rho = \Gamma \cap \{x : |x| = \rho\}$, $0 < r < d$.

We shall use the following functional spaces: $C^l(\overline{G})$ is the space of functions having continious derivatives in $\overline{G}$ up to the order $l \geq 0$, if l is an integer, and up to the $[l]$ (the integral part of l), otherwise, which satisfy the Hölder condition with the exponent $l - [l]$; we denote the norm in $C^l(\overline{G})$ by $|u|_{l;G}$, the norm of an element

1991 *Mathematics Subject Classification.* 35J25, 35J65.

in the Banach space $L_q(G)$ by $||u||_{q;G}$ $(q \geq 1)$, the norm of an element in the Sobolev space $W^{m,q}(G)$ by $||u||_{m,q;G}$, $m \geq 1$ is integer. We define the weight space $V^m_{q,\alpha}(G)$ as the set of functions with the norm

$$||u||_{V^m_{q,\alpha}(G)} = \left(\iint_G \sum_{|\beta|=0}^{m} r^{q(|\beta|-m+\alpha/q)} \left| D^\beta u \right|^q dx \right)^{\frac{1}{q}},$$

where $q \geq 1$, $m \geq 0$ is an integer, α is any real number. The spaces $W^{m-1/q,q}(\partial G)$ and $V^{m-1/q}_{q,\alpha}(\partial G)$ are the spaces of traces of functions from the spaces $W^{m,q}(G)$ and $V^m_{q,\alpha}(G)$ respectively [4].

At last, $X_{loc}(\overline{G}\backslash O)$ is extended to the mean space of functions belonging to $X(G')$ for any compactum $G' \subset \overline{G}\backslash O$.

In the domain G we consider the Dirichlet problem

$$\begin{cases} a^{ij}(x)u_{x_ix_j} + a^i(x)u_{x_i} + a(x)u = f(x), & x \in G, \\ u(x) = \varphi(x), & x \in \partial G \end{cases} \tag{L}$$

or

$$\begin{cases} a^{ij}(x,u,u_x)u_{x_ix_j} + a(x,u,u_x)u = 0, & x \in G, \\ u(x) = \varphi(x), & x \in \partial G. \end{cases} \tag{QL}$$

In the work [5] we have stated a *minimal* condition on the smoothness of data in the problem (L) and (QL) under which there is a solution of (L) and (QL) from $W^{2,q}(G) \cap C^{1+\gamma}(\overline{G})$, $q \geq n$, $\gamma \in (0,1)$. Here we shall investigate a smoothness rise for exactly these solutions.

As an auxiliary problem we consider the eigenvalue problem

$$\begin{cases} \nabla_\omega \psi + \lambda(\lambda + n - 2)\psi = 0, & \omega \in \Omega, \\ \psi = 0, & \omega \in \partial\Omega, \end{cases} \tag{EV}$$

where ∇_ω is the Laplace-Beltrami operator on the unit sphere. We shall denote by $\lambda = \lambda(\Omega)$ the least positive eigenvalue of the problem (EV). It is known [6] that $\lambda > 1$ if Ω is a domain on S^{n-1} being cut out by the convex cone.

Finally, we denote by $\delta^j_i = \begin{cases} 1, \text{ if } i=j, \\ 0, \text{ if } i \neq j, \end{cases}$ the Kronecker symbol and by C various constants.

2. The rise of the solution smoothness for the problem (L)

Conjectures:

(a) the condition of uniform ellipticity $\nu\xi^2 \leq a^{ij}(x)\xi_i\xi_j \leq \mu\xi^2$, $\forall x \in \overline{G}$, $\forall \xi \in \mathbb{R}^n$, *where* $\nu, \mu = \text{const} > 0$, *holds;* $a^{ij}(0) = \delta^j_i$, $i,j = 1,\dots,n$;

(aa) $a^{ij}(x) \in C^0(\overline{G})$, $i,j = 1,\dots,n$, $a^i(x) \in L_n(G)$, $i = 1,\dots,n$, $a(x) \in L_p(G)$, $p \geq n$; *the following inequality holds*

$$\left(\sum_{i,j=1}^{n} |a^{ij}(x) - a^{ij}(0)|^2 \right)^{\frac{1}{2}} + |x| \left(\sum_{i=1}^{n} |a^i(x)|^2 \right)^{\frac{1}{2}} + |x|^2 |a(x)| \leq A(|x|), \quad x \in G,$$

where $A(t)$ *being determined under* $t \geq 0$ *is nonnegative monotony increasing Dini-continuous at* 0 *function and* $A(0) = 0$;

(b) $a(x) \leq 0$ $\forall x \in G$;

(c) $f(x) \in V^m_{p,0}(G) \cap V^0_{2,4-n}(G)$, $\varphi(x) \in V^{m+2-1/p}_{p,0}(\partial G) \cap V^{3/2}_{2,4-n}(\partial G)$, *for some integer* $m \geq 0$ *and real* $p \geq n$; *there exists nonnegative numbers* k, k_m *and* $s > \lambda$ *such that for* $\rho \in (0,d)$

$$||f||_{V^0_{2,4-n}(G^\rho_0)} + ||\varphi||_{V^{3/2}_{2,4-n}(\Gamma^\rho_0)} \leq k\rho^s,$$

$$||f||_{V^m_{p,0}(G^{2\rho}_{\rho/4})} + ||\varphi||_{V^{m+2-1/p}_{p,0}(\Gamma^{2\rho}_{\rho/4})} \leq k_m \rho^{\lambda-m-2+n/p}.$$

Let us now turn to the consideration of the rise of the problem (L) solution smoothness. The smoothness of the solution depends on the value λ defining a solid angle of the cone in a neighbourhood of the point O.

Theorem 1. *Let be λ, p and m such, that the following inequalities*

$$p \geq n, \quad m \geq 1, \quad \lambda > m + 2 - n/p \tag{1}$$

hold. Let assumptions (a)–(d) be fulfilled and also:
(d) there exist generalized derivatives of functions $a(x)$, $a^{ij}(x)$, $a^i(x)$, $i, j = 1, \dots, n$ up to order m and numbers $\mu_m \geq 0$ such that $\forall x \in \overline{G}$:

$$|x|^m \Big(\sum_{i,j=1}^{n} |\nabla^m a^{ij}(x)|^2 \Big)^{1/2} + |x|^{m+1} \Big(\sum_{i=1}^{n} |\nabla^m a^i(x)|^2 \Big)^{1/2} + |x|^{m+2} |\nabla^m a(x)|^2 \leq \mu_m.$$

Then any solution u of problem (L) belongs to $V^{m+2}_{p,0}(G) \subset C^{m+2-n/p}(\overline{G})$ and there exist $\tilde{d} \in (0, d)$ and $C_m \geq 0$ such that

$$||u||_{V^{m+2}_{p,0}(G^\rho_0)} \leq C_m \rho^{\lambda - 2 - m + n/p} \quad \forall \rho \in (0, \tilde{d}), \tag{2}$$

where the constant C_m is determined only by the quantities n, p, ν, μ, m, μ_m, k, k_m, s, λ, $\max\{A(|x|) : x \in \overline{G}\}$, $M_0 = \max\{|u(x)| : x \in \overline{G}\}$, G and $\int_0^d t^{-1} A(t)\, dt$.

Proof. It is proved (see assertion 3 of Theorem 3 [5]) that $u(x) \in V^2_{p,0}(G)$. Let us consider the two sets $G^\rho_{\rho/2}$ and $G^{2\rho}_{\rho/4} \supset G^\rho_{\rho/2}$, $\rho > 0$. Let us perform the transformation $x = \rho x'$, $v(x') = \rho^{-\lambda} u(\rho x')$. The function $v(x')$ is the solution of the problem

$$\begin{cases} a^{ij}(\rho x') v_{x'_i x'_j} + \rho a^i(\rho x') v_{x'_i} + \rho^2 a(\rho x') v = \rho^{2-\lambda} f(\rho x'), & x' \in G^2_{1/4}, \\ v(x') = \rho^{-\lambda} \varphi(\rho x'), & x' \in \Gamma^2_{1/4}. \end{cases} \tag{L'}$$

By Theorem 15.3 [1], whose conditions are satisfied due to the above hypotheses, the following estimate is true:

$$||v||_{m+2,p;G^1_{1/2}} \leq C_m \left(||v||_{p;G^2_{1/2}} + \rho^{2-\lambda} ||f||_{m,p;G^2_{1/4}} + \rho^{-\lambda} ||\varphi||_{m+2-1/p,p;\Gamma^2_{1/4}} \right), \tag{3}$$

where C_m is independent of v and is determined only by the quantities n, p, ν, μ, m, μ_m, $\max\{A(|x|) : x \in \overline{G}\}$ and by the domain G. Returning to the variables x and $u(x)$ we obtain from (3):

$$||u||_{V^{m+2}_{p,0}(G^\rho_{\rho/2})} \leq C_m \Bigg\{ \rho^{\lambda-2-m+n/p} \Bigg(\iint\limits_{G^{2\rho}_{\rho/4}} r^{-n-\lambda p} |u|^p dx \Bigg)^{1/p} + \\ + ||f||_{V^m_{p,0}(G^{2\rho}_{\rho/4})} + ||\varphi||_{V^{m+2-1/p}_{p,0}(\Gamma^{2\rho}_{\rho/4})} \Bigg\}. \tag{4}$$

By Theorem 3 [5] we have

$$|u(x)| \leq C|x|^\lambda \tag{5}$$

with C independent of $u(x)$ and being determined only by the quantities ν, p, n, μ, s, λ, k, M_0 (from hypothesis (c)), $\int_0^d t^{-1} A(t) dt$ and by the domain G. From (5) and assumption (c) from (4) we now obtain

$$||u||_{V^{m+2}_{p,0}(G^\rho_{\rho/2})} \leq \hat{C}_m \rho^{\lambda-2-m+n/p}, \quad \forall \rho \in (0, d). \tag{6}$$

Examining inequality (6) with ρ having being replaced by $2^{-k}\rho$, k is an integer, and summing the inequalities obtained over all k we have

$$||u||_{V_{p,0}^{m+2}(G_0^\rho)} \le \hat{C_m}\sum_{k=0}^{\infty} 2^{-k(\lambda-2-m+n/p)}\rho^{\lambda-2-m+n/p} \tag{7}$$

The numerical series (7) by inequality (1) converges and inequality (2) is, therefore, proved.

Theorem 2. *Let all assumptions of Theorem 1 excepting (1) be satisfied. Suppose that the inequalities*

$$p > n, \quad m \ge 0, \quad m+1 < \lambda \le m+2-n/p \tag{8}$$

hold. Then $u \in C^\lambda(\overline{G})$ *and there exist constants* C'_k, $k = 0, 1 \dots m+1$, *independent of* $u(x)$ *and such that*

$$|\nabla^k u(x)| \le C'_k |x|^{\lambda-k}, \quad x \in \overline{G}_0^d. \tag{9}$$

If

$$\lambda = m+1, \quad p \ge n, \quad m \ge 1$$

then $u \in C^{\lambda-\varepsilon}(\overline{G})$ $\forall\varepsilon > 0$.

Proof. Let us consider in the domain $G^2_{1/4}$ a function $v(x')$ that is the solution of problem (L'). By Sobolev's imbedding theorem (see, e.g., Th.1.4.5. [8]) $W^{m+2,p}(G) \subset C^{m+1+\gamma}(\overline{G})$, $0 < \gamma \le 1-n/p$, $p > n$ and thus:

$$\sup_{x' \in G^2_{1/4}} |v(x')| + \sup_{x' \in G^2_{1/4}} |\nabla' v(x') + \dots + \sup_{x' \in G^2_{1/4}} |\nabla'^{m+1} v(x')| + \\ + \sup_{\substack{x',y' \in G^2_{1/4} \\ x' \ne y'}} \frac{|\nabla'^{m+1} v(x') - \nabla'^{m+1} v(y')|}{|x'-y'|^{1-n/p}} \le C||v||_{m+2,p;G^2_{1/4}}. \tag{10}$$

with a constant C independent of $u(x)$ and being determined only by n, p and by the domain G. Rewriting (10) in variables x, $u(x)$ we have:

$$|u(x)| \le C\rho^{m+2-n/p}||u||_{V_{p,0}^{m+2}(G^{2\rho}_{\rho/4})}, \quad x \in G^{2\rho}_{\rho/4}$$

$$|\nabla u(x)| \le C\rho^{m+1-n/p}||u||_{V_{p,0}^{m+2}(G^{2\rho}_{\rho/4})}, \quad x \in G^{2\rho}_{\rho/4}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$|\nabla^{m+1} u(x)| \le C\rho^{1-n/p}||u||_{V_{p,0}^{m+2}(G^{2\rho}_{\rho/4})}, \quad x \in G^{2\rho}_{\rho/4}$$

$$\sup_{\substack{x,y \in G^2_{1/4} \\ x \ne y}} \frac{|\nabla^{m+1} u(x) - \nabla^{m+1} u(y)|}{|x-y|^{1-n/p}} \le C||u||_{V_{p,0}^{m+2}(G^2_{1/4})}.$$

According to Theorem 1 inequality (6) holds due to which the inequalities obtained above acquire the form:

$$|u(x)| < c\rho^\lambda, |\nabla u(x)| < c\rho^{\lambda-1}, \dots, |\nabla^{m+1} u(x)| < c\rho^{\lambda-m-1}, \quad x \in G^{2\rho}_{\rho/4} \tag{11}$$

$$\sup_{\substack{x,y \in G^2_{1/4} \\ x \ne y}} \frac{|\nabla^{m+1} u(x) - \nabla^{m+1} u(y)|}{|x-y|^{1-n/p}} \le c\rho^{\lambda-2-m+n/p}. \tag{12}$$

Taking in (11) $|x| = \rho$ we obtain desired inequality (9). Now from (12) under $\tau = \lambda - 2 - m + n/p \leq 0$ we have:

$$|\nabla^{m+1}u(x) - \nabla^{m+1}u(y)| \leq c\rho^{\tau}|x-y|^{\lambda-1-m-\tau}, \quad \forall x, y \in G^{2\rho}_{\rho/4}. \tag{13}$$

Thus, if $\tau \leq 0$ we have $|x-y|^{\tau} \geq (4\rho)^{\tau}$ $\forall x, y \in G^{2\rho}_{\rho/4}$ and so we have from (13):

$$|\nabla^{m+1}u(x) - \nabla^{m+1}u(y)| \leq c4^{-\tau}|x-y|^{\lambda-1-m}, \quad \forall x, y \in G^{2\rho}_{\rho/4}. \tag{14}$$

Inequality (14) with (9) under $k = m+1$ leads us to the assertion $u(x) \in C^{\lambda}(\overline{G^d_0})$ for λ and p, that satisfy (8).

Now, let $\lambda = m+1$, $p = n$. Then by Theorem 1.4.5 (d) [8] we have:

$$\sup_{\substack{x', y' \in G^2_{1/4} \\ x' \neq y'}} \frac{|\nabla'^m v(x') - \nabla'^m v(y')|}{|x'-y'|^{\alpha}} \leq c||v||_{m+2,n;G^2_{1/4}}, \quad \forall \alpha \in (0,1), \quad m \geq 0.$$

In variables x, $u(x)$ this inequality acquires the form ($\forall \alpha \in (0,1)$, $m \geq 0$):

$$\sup_{\substack{x, y \in G^{2\rho}_{\rho/4} \\ x \neq y}} \frac{|\nabla^m u(x) - \nabla^m u(y)|}{|x-y|^{\alpha}} \leq c\rho^{2-\alpha-n/p}||u||_{V^{m+2}_{n,0}(G^{2\rho}_{\rho/4})} \leq \hat{c}\rho^{\lambda-m-\alpha} = \hat{c}\rho^{1-\alpha},.$$

Taking $\alpha = 1 - \varepsilon$ $\forall \varepsilon > 0$ from that we have:

$$\sup_{\substack{x, y \in G^{2\rho}_{\rho/4} \\ x \neq y}} \frac{|\nabla^m u(x) - \nabla^m u(y)|}{|x-y|^{1-\varepsilon}} \leq \hat{c}\rho^{\varepsilon}, \quad \forall \varepsilon > 0. \tag{15}$$

Besides that, inequalities (9) remain valid for $k = 0, \ldots, m$. Together with (15) they are extended to the mean $u(x) \in C^{\lambda-\varepsilon}(\overline{G^d_0})$ $\forall \varepsilon > 0$. So, the proof of Theorem 2 is completed.

Remark 1. A. Azzam has proved in [2,3] the following: let $G \subset \mathbb{R}^2$ be a domain with the angular boundary point $\mathcal{O}$, ω_0 the opening of angle at the vertex $\mathcal{O}$; when $\pi/\omega_0 > k + 2 + \alpha$, $\alpha \in (0,1)$, the coefficients and the right-hand side of (L) belong to $C^{k+\alpha}(\overline{G})$, $\varphi \in C^0(\overline{G}) \cap C^{k+2+\alpha}(\overline{G} \setminus \mathcal{O})$ then any bounded in $\overline{G}$ solution of (L) $u(x) \in C^{k+2+\alpha}(\overline{G})$. It is easily seen that our assertions of Theorems 1, 2 are more precise and our requirements on the smoothness of problems data are *minimal* (see with this reference to [4, 5]).

3. The rise of solution smoothness for the problem (QL)

Let us consider the question of rise of solution (from $W^{2,q}(G) \cap C^{1+\gamma}(\overline{G})$) smoothness of (QL). As well as in linear case the solution smoothness depends on the quantity λ determining value of the solid angle of the cone in a neigbourhood of the point 0.

Let us define the set $\mathfrak{M}_{M_0,M_1} = \{(x,u,z) | x \in \overline{G}, u \in \mathbb{R}, |u| \leq M_0, z \in \mathbb{R}^n, |z| \leq M_1\}$.

As for the equation of the problem (QL) we assume that the following conditions are satisfied on the set $\mathfrak{M}_{M_0,M_1}$:

(A) *the uniform ellipticity: there exist positive constants* ν, μ *such that for* $\forall(x,y,z) \in \mathfrak{M}_{M_0,M_1}$, $\forall \xi \in \mathbb{R}^n$ $\nu\xi^2 \le a_{ij}(x,u,z)\xi_i\xi_j < \mu\xi^2$; $a_{ij}(0,0,0) = \delta_i^j$, $i,j = 1,\dots,n$;

(B) $a_{ij}(x,u,z) \in C^m(\mathfrak{M}_{M_0,M_1})$, $i,j = 1,\dots,n$, *for some integer* $m \ge 0$ *and the partial derivatives of functions* $a_{ij}(x,u,z)$ *over all their arguments up to order* m *are bounded on* $\mathfrak{M}_{M_0,M_1}$;

(C) *there exist generalized partial derivatives of function* $a(x,u,z)$ *over all their arguments up to order* $m \ge 1$, *functions* $f_l(x)$ *and numbers* $\tilde{\mu}_l$, $\tilde{k}_l$ $(l = 1,\dots,m)$ *such that the following inequalities*

$$|D_u^{l_1} D_x^{l_2} a(x,u,z)| \le \mu_1 |z|^2 + f_{l_2}(x); 1 \le l_1 + l_2 \le m; \tag{16}$$

$$\begin{gathered} |D_u^{l_1} D_x^{l_2} D_z a(x,u,z)| \le \mu_1 |z| + f_{l_2}(x); \quad 0 \le l_1 + l_2 \le m-1; \\ |D_u^{l_1} D_x^{l_2} D_z^{l_3} a(x,u,z)| \le \tilde{\mu}_{l_3}, \quad 2 \le l_1 + l_2 + l_3 \le m, \end{gathered} \tag{17}$$

$$\textit{where} \quad f_l(x) \le \tilde{k}_l |x|^{\lambda-2-l}, \quad f_0(x) \le \tilde{k}_0 |x|^{\beta}, \quad \beta > \lambda - 2, \tag{18}$$

are fulfilled.

Theorem 3. *Let* λ, p, m *be numbers such that*

$$p > n, \quad m \ge 1, \quad \lambda > m + 2 - n/p. \tag{19}$$

Let assumptions (A)–(C) be satisfied and let a function $u(x)$ *be a solution of the problem (QL), if* $M_0 = \max\{|u(x)| : x \in \overline{G}\}$. $M_1 = \max\{|\nabla u(x)| : x \in \overline{G}\}$. *Moreover, let* $\varphi(x) \in V_{p,0}^{m+2-1/p}(\partial G) \cap V_{2,4-n}^{3/2}(\partial G)$ *and there exist nonnegative numbers* $\tilde{k}'_0, \tilde{k}'_1, \dots, \tilde{k}'_m$ *and* $s > l$ *such that the inequalities*

$$||\varphi||_{V_{2,4-n}^{3/2}(\Gamma_0^\rho)} \le \tilde{k}'_0 \rho^s, \quad ||\varphi||_{V_{p,0}^{2-1/p+m}(\Gamma_{\rho/2}^\rho)} \le \tilde{k}'_m \rho^{\lambda-2-m+n/p}, \quad \rho \in (0,d) \tag{20}$$

hold. Then $u(x) \in V_{p,0}^{m+2}(G)$ *and there exist numbers* $\tilde{d} \in (0,d)$ *and* $C_m > 0$ *such that*

$$||u||_{V_{p,0}^{m+2}(G_0^\rho)} \le C_m \rho^{\lambda-2-m+n/p}, \quad \rho \in (0,\bar{d}), \tag{21}$$

where C_m *is determined only by quantities taking part in assumptions of the theorem and by* G.

Proof. We apply usual iteration procedure over m. Let $m = 1$. Let us consider the equation of (QL) in the domain $G_{\rho/2}^\rho$, $\rho \in (0,d)$. The lateral surface $\Gamma_{\rho/2}^\rho$ of $G_{\rho/2}^\rho$ is unboundedly smooth, because G_0^d is a convex cone. By definition of smooth domains (see, e.g., p.6.2 [2]) for every point $x_0 \in \Gamma_{\rho/2}^\rho$ there exists a neighbourhood $\Gamma \subset \Gamma_{\rho/2}^\rho$ of this point and a diffeomorphism χ from C^{2+m} rectifying boundary in Γ. Let $\mathfrak{D} \subset G_{\rho/2}^\rho$ be such that $\Gamma \subset \mathfrak{D}$. Let us perform transformation $y = \chi(x) = (\chi_1(x), \dots, \chi(x))$ and let $\chi(\mathfrak{D}) = \mathfrak{D}'$, $\chi(\Gamma) = \Gamma' \subset \partial\mathfrak{D}'$ (Γ' is a plane portion of the boundary $\mathfrak{D}'$), $v(y) = u(\chi^{-1}(y))$. In this case χ, $\chi^{-1} \in C^{2+m}$ and Jacobi's determinant $|\nabla\chi| \ne 0$. Besides, one can suppose the norms in C^{2+m} of transformations χ determining local representation of the boundary $\Gamma_{\rho/2}^\rho$ to be uniformly bounded with respect to $x_0 \in \Gamma^{\rho}_{\rho/2}$. In new variables the equation of (QL) takes the form:

$$A_{ij}(y,v,v_y)v_{y_iy_j} + A(y,v,v_y) = 0, \quad y \in \mathfrak{D}', \tag{22}$$

where $A(y,v,v_y) = a(x,u,u_x) + a_{ij}(x,u,u_x)v_{y_k}\frac{\partial^2\chi_k}{\partial x_i \partial x_j}$,

$$A_{ij}(y,v,v_y) = a_{kl}(x,u,u_x)\frac{\partial \chi_i}{\partial x_k}\frac{\partial \chi_j}{\partial x_l}. \tag{23}$$

Let us notice that in $\mathfrak{D}'$ by condition (A)

$$\varkappa_1 \nu \xi^2 \le A_{ij}\xi_i\xi_j \le \varkappa_2 \mu \xi^2, \tag{24}$$

where $\varkappa_1 = \inf\{|\nabla\chi(x)|^2 > 0 : x \in \mathfrak{D}'\}$; $\varkappa_2 = \sup\{|\nabla\chi(x)|^2 > 0 : x \in \mathfrak{D}'\}$. The coordinate system can be taken so that the axis y_n would be directed like the normal toward Γ' and the axes $y_1, \ldots, y_{n-1}$ like the rays at plane Γ'. Let $\mathbf{e}_k$ be fixed coordinate vectors ($k = 1, \ldots, n-1$). We define difference quotients $v_k(y;h) = \frac{1}{h}\{v(y) - v(y_1,\ldots,y_{k-1},y_k - h, y_{k+1},\ldots,y_n)\}$, $k = 1,\ldots,n-1$; $|h|$ is small enough. We set: $y^t = ty + (1-t)(y - h\mathbf{e}_k)$; $v^t(y) = tv(y) + (1-t)v(y - h\mathbf{e}_k)$. Then the function $w(y) \equiv v_k(y,h)$ satisfies the linear equation

$$a^{ij}(y)w_{y_iy_j} + a^i(y)w_{y_i} + a(y)w = f(y), \quad y \in \mathfrak{D}', \tag{L$'$}$$

where $a^{ij}(y) = A_{ij}(y, v(y), v_y(y))$;

$$a^i(y) = v_{y_py_l}(y-h)\int_0^1 \frac{A_{pl}(y^t,v^t,v^t_y)}{\partial v^t_{y_i}}dt + \int_0^1 \frac{A(y^t,v^t,v^t_y)}{\partial v^t_{y_i}}dt,$$
$$a(y) = v_{y_py_l}(y-h)\int_0^1 \frac{A_{pl}(y^t,v^t,v^t_y)}{\partial v^t}dt + \int_0^1 \frac{A(y^t,v^t,v^t_y)}{\partial v^t}dt,$$
$$-f(y) = v_{y_py_l}(y-h)\int_0^1 \frac{A_{pl}(y^t,v^t,v^t_y)}{\partial y^t_k}dt + \int_0^1 \frac{A(y^t,v^t,v^t_y)}{\partial y^t_k}dt,$$

$k = 1,\ldots,n-1$. Since the directions $\mathbf{e}_k$ ($k = 1,\ldots,n-1$) are parallel to the tangent plane to Γ, we have $w\big|_{\Gamma'} = \psi_k(y,h)$, $y \in \Gamma'$, $\psi(y) = \varphi(\chi^{-1}y)$. Let us apply to the solution $w(y)$ local the Schauder L_p-estimate near smooth boundary portion (Theorem 15.3 [1]; see also remark to Theorem 9.13 at the end of p.9.5 [2]). Let us verify fulfillment of all conditions of the above estimate. (24) implies fulfillment of uniform ellipticity condition for equation (L)$'$. Since our solution $u(x) \in C^{1+\gamma}(\bar{G})$, the hypothesis (B) guarantees continuity of the coefficients $a^{ij}(y)$ in $\mathfrak{D}'$. Further, it is proved in Theorems 5.1, 6.1 [5] that $u_{xx} \in L_p(\mathfrak{D})$, $p > n$; then by assertions 1 and 3 of Theorem 5.1 [5] in view of assumptions (B),(C) we have:

$$\left|\left|\Big(\sum_{i=1}^{n}|a^i(y)|^2\Big)^{\frac{1}{2}}\right|\right|_{n,\mathfrak{D}'} \le C|\chi|_{2,\mathfrak{D}'}\Big(\bar{\mu}_1|u_{xx}|_{n,G_0^{3\rho/2}} + ||\mu + (\bar{\mu}_1 + \tilde{\mu}_1)|\nabla u| + f_0(x)||_{n,G^{\rho}_{\rho/2}}\Big) \le$$
$$\le C\left(|\chi|_{2,\mathfrak{D}'}, n, p, \tilde{k_0}, \mu, \overline{\mu}_1, \tilde{\mu}_1\right)\left(\rho + \rho^{\lambda} + \rho^{\lambda-2+n/p} + \rho^{\lambda-1}\right) \le \text{const}$$

by inequality (19). The same way:

$$||a||_{p,\mathfrak{D}'} \le C\left(|\chi|_{2,\mathfrak{D}'}\right)\left(\left|\left|\mu_1|\nabla u|^2 + \bar\mu_1|\nabla u| + f_0(x)\right|\right|_{p,G^{\rho}_{\rho/2}} + \bar\mu_1||u_{xx}||_{p,G_0^{3\rho/2}}\right) \le$$
$$\le C\left(|\chi|_{2,\mathfrak{D}'}, n, p, \tilde k_0, \mu_1, \bar\mu_1\right)\left(\rho^{2(\lambda-1)+n/p} + \rho^{\lambda-2+n/p} + \rho^{\lambda-1+n/p}\right) \le \text{const}\,.$$

So the Shauder local L_p-estimate ([7]) for the solutions of $(L)'$ gives us the inequality:

$$||\omega||_{2,p;\mathfrak{D}''} \le \text{const}\left(||\omega||_{p,\mathfrak{D}'} + ||f||_{p,\mathfrak{D}'} + ||\varphi_k(y,h)||_{2-1/p,p;\Gamma'}\right), \quad \forall \mathfrak{D}'' \Subset \mathfrak{D}' \cup \Gamma', \tag{25}$$

where const is independent of ω, f, φ_k, h and is determined only by k, p, ν, μ, $\varkappa_1$, $\varkappa_2$ and by modules of continuity of $a^{ij}(y)$ in $\mathfrak{D}'$; the latter are estimated in the following way:

$$|a^{ij}(y_1) - a^{ij}(y_2)|_{\mathfrak{D}'} = |A_{ij}(y_1, v(y_1), v_y(y_1)) - A_{ij}(y_2, v(y_2), v_y(y_2))| =$$
$$= \left|a_{kl}(x_1, u(x_1), u_x(x_1))\frac{\partial\chi_i(x_1)}{\partial x_k}\frac{\partial\chi_j(x_1)}{\partial x_l} - a_{kl}(x_2, u(x_2), u_x(x_2))\frac{\partial\chi_i(x_2)}{\partial x_k}\frac{\partial\chi_j(x_2)}{\partial x_l}\right| \le$$
$$\le |(a_{kl}(x_1, u(x_1), u_x(x_1)) - a_{kl}(x_2, u(x_2), u_x(x_2))| \cdot |\nabla\chi|^2 +$$
$$+\mu\left|\frac{\partial\chi_i(x_1)}{\partial x_k}\frac{\partial\chi_j(x_1)}{\partial x_l} - \frac{\partial\chi_i(x_2)}{\partial x_k}\frac{\partial\chi_j(x_2)}{\partial x_l}\right| \le \varkappa_2^{1/2}|\chi|_{2,\mathfrak{D}}|x_1 - x_2| +$$
$$+\varkappa_2\bar\mu_1\left(|x_1 - x_2| + |u(x_1) - u(x_2)| + |\nabla u(x_1) - \nabla u(x_2)|\right) \le$$
$$\le 2\rho\left(\varkappa_2\bar\mu_1 + \mu_1\varkappa_2^{1/2}|\chi|_{2,\mathfrak{D}} + \bar c_1\rho^{\gamma}\right) + C(2\rho)^{\gamma}$$

by $u(x) \in C^{1+\gamma}(\overline{G})$. Further, we have by definition of $w(y)$:

$$||\omega||_{p,\mathfrak{D}'} = \left|\left|\frac{v(y) - v(y - he_k)}{h}\right|\right|_{p,\mathfrak{D}'} \le C(|\chi^{-1}|_1)\,||\nabla u(x)||_{p,G^{\rho}_{\rho/2}} \tag{26}$$

Analogously we obtain:

$$||\varphi_k(y,h)||_{2-1/p,p;\Gamma'} \le C(|\chi^{-1}|_1)||\varphi(x)||_{3-1/p,p;\Gamma^{\rho}_{\rho/2}}, \tag{27}$$

and finally

$$||f||_{p,\mathfrak{D}'} \le C(|\chi|_{2,\mathfrak{D}'})\left(\left|\left|\mu_1|\nabla u|^2 + \bar\mu_1|\nabla u| + f_1(x)\right|\right|_{p,G^{\rho}_{\rho/2}} + \bar\mu_1||u_{xx}||_{p,G_0^{3\rho/2}}\right) \tag{28}$$

Now from (25)–(28) we obtain the inequality:

$$\left|\left|\frac{v(y) - v(y - he_k)}{h}\right|\right|_{2,p;\mathfrak{D}'} \le \text{const}\left(||\varphi(x)||_{3-1/p,p;\Gamma^{\rho}_{\rho/2}} +\right.$$
$$\left. + \left|\left||\nabla u(x)|^2\right|\right|_{p,G^{\rho}_{\rho/2}} + ||\nabla u(x)||_{p,G^{\rho}_{\rho/2}} + + ||f_1(x)||_{p,G^{\rho}_{\rho/2}} + ||u_x x||_{p,G_0^{3\rho/2}}\right),$$

where const on the right is independent of h. This fact allows us to conclude on the basis of Fatou's theorem that there exists $v_{y_k} \in W^{2,p}(\mathfrak{D}'')$ and perform passage to the limit $h \to 0$:

$$\left|\left|\frac{\partial v}{\partial y_k}\right|\right|_{2,p;\mathfrak{D}'} \le \text{const}\Big(||\varphi(x)||_{3-1/p,p;\Gamma^\rho_{\rho/2}} + ||u_{xx}||_{p,G_0^{3\rho/2}} + \\ + \big|\big||\nabla u(x)|^2\big|\big|_{p,G^\rho_{\rho/2}} + ||\nabla u(x)||_{p,G^\rho_{\rho/2}} + ||f_1(x)||_{p,G^\rho_{\rho/2}}\Big),\ k = 1,\dots,n-1 \quad (29)$$

We consider again equation (22) and differentiate it over y_n:

$$A_{nn}(y,v,v_y)v_{y_ny_ny_n} = -\Big\{\sum_{k=1}^{n-1} A_{kn}v_{y_ky_ny_n} + \sum_{i,j=1}^{n-1} A_{ij}v_{y_iy_jy_n} + \\ + \frac{\partial A_{ij}}{\partial u_{y_l}}v_{y_iy_j}v_{y_ly_n} + \frac{\partial A_{ij}}{\partial u}v_{y_iy_j}v_{y_n} + \frac{\partial A_{ij}}{\partial y_n}v_{y_iy_j} + \frac{\partial A_{ij}}{\partial u_{y_l}}v_{y_ly_n} + \frac{\partial A}{\partial y_n} + \frac{\partial A}{\partial u}v_{y_n}\Big\}, \quad (30)$$

the $A_{nn} = a_{kl}(y,v,v_y)\frac{\partial \chi_n}{\partial x_k}\frac{\partial \chi_n}{\partial x_l} \ge \nu|\nabla\chi_n(x)|^2 \ge \varkappa_1\nu$. Since $u(x) \in W^{2,p}(G^\rho_{\rho/2})$, $v_{y_k} \in W^{2,p}(\mathfrak{D}")$, $1 \le k \le n-1$ then from (30) we obtain $v(y) \in W^{3,p/2}(\mathfrak{D}")$ by assumptions (B), (C). Then by Sobolev's imbedding theorem (see p. 1.4.5 (d) and Rem.2 to p. 1.4.5 [8]) we can write :

1. if $p > 2n$ then $v(y) \in C^2(\mathfrak{D}'')$ and in this case $|v|_{2;\mathfrak{D}''} \le c||v||_{3,p/2;\mathfrak{D}''}$.
2. if $n < p \le 2n$ then $v(y) \in W^{2,q_1}(\mathfrak{D}'')$ with $q_1 = \frac{np}{2n-p} > p$; in particular $v(y) \in W^{2,2p}(\mathfrak{D}'')$ for $p \le 3n/2$.

By the above statements and equation (30) we obtain $v(y) \in W^{3,p}(\mathfrak{D}'')$ and therefore $u(x) \in W^{3,p}(G^{7p/8}_{5p/8})$ if $p \ge 3n/2$. Now we need to examine only $p \in (n, 3n/2)$. From above we have $v(y) \in W^{2,q_1}(\mathfrak{D}'')$ and by (30) $v(y) \in W^{3,q_1/2}(\mathfrak{D}'')$. Let us use again imbedding $W^{3,q} \subset W^{2,q^*}$, $q^* = \frac{nq}{n-q}$, $(q < n)$; as a result $v(y) \in W^{2,q_2}(\mathfrak{D}'')$ $q_2 = nq_1/(2n-q_1) = np/(4n-3p)$ if $n < p < 4n/3$, and $v(y) \in C^2(\mathfrak{D}'')$ if $p \ge 4n/3$. Repeat that procedures times:

$$v(y) \in W^{3,q_s/2}(\mathfrak{D}'') \cap W^{2,q_s}(\mathfrak{D}''), \quad (31)$$

$q_s = \frac{np}{n2^s-(2^s-1)p}$, if $n < p < n/(1-2^{-s})$. We choose an integer number $s \ge 1$ so that $q_s \ge 2p$; solving that inequality we obtain $s = [\log_2((2p-n)/(p-n))]$. From (31) we find

$$u(x) \in W^{3,p}(G^{7\rho/8}_{5\rho/8}) \cap W^{2,2p}(G^{7\rho/8}_{5\rho/8}) \quad \forall\rho \in (0,d).$$

We proceed to derivation of estimate (21) under $m = 1$. From (30) by (29) we have

$$\Big(\iint\limits_{\mathfrak{D}''} |v_{y_ny_ny_n}|^p dy\Big)^{1/p} \le (\nu\chi_1)^{-1}\Big\{\mu\chi_2\sum_{k=1}^{n-1}\left|\left|\frac{\partial v}{\partial y_k}\right|\right|_{2,p;\mathfrak{D}''} + \\ +\Big(\mu + (\mu_1+\bar\mu_1)|\nabla_y v|_{\mathfrak{D}''} + |f_0(y)|_{\mathfrak{D}''}\Big)||v_{yy}||_{p,\mathfrak{D}''} + \\ + \Big|\Big|\mu_1|\nabla_y v|^3 + (\bar\mu_1+\mu_1)|\nabla_y v|^2 + (\bar\mu_1 + f_0(y))|\nabla_y v| + f_1(y)\Big|\Big|_{p,\mathfrak{D}''} + \\ + \bar\mu_1||v^2_{yy}||p,\mathfrak{D}'' + \bar\mu_1||v_{yy}||_{p,\mathfrak{D}''}(1 + |\nabla_y v|_{\mathfrak{D}''})\Big\}C(|\chi|_{2,\mathfrak{D}''}) \quad (32)$$

From (29), (32) in variables $x, u(x)$ taking into account hypothesis (C) we obtain

$$||u||_{3,p;G^{7\rho/8}_{5\rho/8}} \le C\left(|\chi|_{3,G^{\rho}_{\rho/2}}, \nu, \mu, n, p, \varkappa_1, \varkappa_2, \mu_1, \bar{\mu}_1\right)(||u_{xx}||p, G_0^{3p/2} + \\ + \left|\left||\nabla u|^3 + |\nabla u|^2 + |\nabla u|(1 + f_0(x)) + |f_1(x)| + u^2_{xx}\right|\right|_{p,G^{\rho}_{\rho/2}} + \\ + (1 + |f_0(x) + \nabla u|_{G^{\rho}_{\rho/2}})||u_{xx}||_{p,G^{\rho}_{\rho/2}} + ||\varphi||_{3-1/p,p,\Gamma^{\rho}_{\rho/2}}\Big).$$

From here basing on Theorem 5.1 [5] for $\rho \in (0, d)$

$$||u||_{3,p;G^{7\rho/8}_{5\rho/8}} \le C\left(|\chi|_{3,G^{\rho}_{\rho/2}}, \nu, \mu, p, \varkappa_1, \varkappa_2, \mu_1, \bar{\mu}_1, \lambda, \tilde{k}_0, \tilde{k}_1, d, \bar{c}_1, \bar{c}_3\right)\rho^{\lambda-3+n/p}. \quad (33)$$

Replacing in (33) ρ on $2^{-k}\rho$, summing inequalities obtained over all $k = 0, 1, 2, \ldots$ and taking into account (19) under $m = 1$ we come to sought for inequality (31).

Repeating such procedure by induction we conclude validity of the assertions of Theorem 3.

Theorem 4. *Let all assertions of Theorem 3 excepting of (19) be fulfilled. If $m \ge 0$ is integer and*

$$m + 1 < \lambda \le m + 2 - n/p, \ \ p > n, \ \ m \ge 0, \quad (34)$$

then $u(x) \in C^{\lambda}(\overline{G})$ and there exist constants $\tilde{c}_k (k = 0, \ldots, m+1)$ independent of $u(x)$ and such that

$$|\nabla^k u(x)| < \tilde{C}_k |x|^{\lambda-k}, \ \ x \in \overline{G^d_0}, \ \ k = 0, \ldots, m+1. \quad (35)$$

If $\lambda = m + 1,\ p \ge n$ then $u \in C^{\lambda-\varepsilon}(\overline{G})$ $\varepsilon > 0$.

Proof. Let function $v(x') = \rho^{-\lambda}u(\rho x')$ be a solution in a layer $G^1_{1/2}$ of $(QL)'$. Verbally repeating proof of Theorem 2 and using on Theorem 3 we obtain all assertions of Theorem 4.

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Received 14.03.1995
Revised 15.06.1998