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# STRONGLY SUMMABLE ULTRAFILTERS ON ABELIAN GROUPS

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*Strongly summable* ultrafilters on a commutative semigroup are those that are generated by sets of finite sums. We establish several facts about strongly summable ultrafilters on a countable abelian group $G$ that were previously known to hold only for the group $(\mathbb{Z},+)$ and for Boolean groups. It is shown that Martin's Axiom implies the existence of nonprincipal strongly summable ultrafilters, that their existence cannot be established in ZFC, and that, if $G$ is embeddable in the circle group, they satisfy strong algebraic properties regarding uniqueness of solutions to certain equations.

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*Сильно суммируемыми* называются ультрафильтры на коммутативных полугруппах, порожденные множествами конечных сумм. В работе установлены некоторые факты о сильно суммируемых ультрафильтрах на счетной абелевой группе $G$, ранее известные только для группы $(\mathbb{Z},+)$ и булевых групп. Показано, что из аксиомы Мартина следует существование неглавных сильно суммируемых ультрафильтров, что их существование невозможно установить в ZFC и что если $G$ вложима в циклическую группу, они удовлетворяют сильным алгебраическим свойствам, связанным с единственностью решения определенных уравнений.

## 1. Introduction.

We regard the points of the Stone-Čech compactification $\beta G$ of the discrete space $G$ as being ultrafilters on $G$, with the points of $G$ itself being identified with the principal ultrafilters. The topology of $\beta G$ can be defined by choosing the sets of the form $\bar{A} = \{x \in \beta G : A \in x\}$, where $A \subseteq G$, as a base for the open sets. Then $\bar{A}$ is a clopen subset of $\beta G$ and is, in fact, equal to $\mathrm{cl}_{\beta G}(A)$. We shall use $A^*$ to denote

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$\bar{A} \setminus A$. We shall use the fact that, for every $x \in \beta G$ and every neighbourhood $U$ of $x$ in $\beta G$, $G \cap U \in x$.

If $(G, +)$ is a semigroup, then the semigroup operation on $G$ can be extended in a natural way to $\beta G$ by putting $x + y = \lim_{s \to x} \lim_{t \to y}(s + t)$, where $x$ and $y$ denote elements of $\beta G$ and $s$ and $t$ denote elements of $G$. Although we use the symbol $+$ for the extended operation, it is usually very far from being commutative, even when $G$ is commutative. With this operation, $\beta G$ is a right topological semigroup. This means that, for every $x \in \beta G$, the map $\rho_x : \beta G \to \beta G$, defined by $\rho_x(y) = y + x$, is continuous. It is also true that the map $\lambda_x : \beta G \to \beta G$, defined by $\lambda_x(y) = x + y$, is continuous for every $x \in G$. We note that, for every $x, y \in \beta G$, $x + y$ is the ultrafilter which has as base the sets of the form $\bigcup_{s \in X}(s + Y_s)$, where $X \in x$ and $Y_s \in y$ for every $s \in X$. See [5] for an elementary derivation of these properties, as well as for other unfamiliar facts cited below.

There are significant algebraic implications which follow from the statement that a semigroup has a topology for which it is compact, Hausdorff and right topological. A simple and important example is the fact that it contains idempotents; i.e. elements $x$ for which $x + x = x$.

If $S$ is any set, $\mathcal{P}_f(S)$ will denote the set of finite non-empty subsets of $S$. If $G$ is a commutative semigroup, then for any non-empty $X \subseteq G$, $FS\langle X \rangle$ will denote $\{\sum_{x \in F} x : F \in \mathcal{P}_f(X)\}$. If $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $G$, $FS\langle x_n \rangle_{n=1}^{\infty}$ will denote $\{\sum_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$. (If $G$ is not commutative, one needs to specify the order in which the sums are taken. We shall not be concerned with this situation in this paper.)

It is well known that, if $G$ is a commutative semigroup and $q \in G^*$ is idempotent, then every member of $q$ contains a set of the form $FS\langle X \rangle$ for some infinite subset $X$ of $G$. However, we do not normally expect that $FS\langle X \rangle \in q$.

**1.1. Definition.** Let $G$ be a commutative semigroup. An ultrafilter $p \in \beta G$ which has a base of sets of the form $FS\langle X \rangle$ is called *strongly summable*.

Thus $p$ is strongly summable if and only if, for every $A \in p$, there exists $X \subseteq G$ such that $FS\langle X \rangle \in p$ and $FS\langle X \rangle \subseteq A$.

Throughout the rest of this paper, $(G, +)$ will denote a countable abelian group. The restriction to a group rather than an arbitrary semigroup is made for our convenience. We need the group properties for some of the proofs. Once this restriction is made, we lose nothing by adding the countability assumption. Indeed, any strongly summable ultrafilter on an abelian group has some countable member [8, Theorem 3].

The principal ultrafilter on $G$ which has $\{0\}$ as a member is a trivial example of a strongly summable ultrafilter. This is the only example of a strongly summable ultrafilter on $G$ whose existence can be established in ZFC. We shall show that Martin's Axiom implies that there are nonprincipal strongly summable ultrafilters on $G$, but that their existence cannot be demonstrated in ZFC.

If $G$ can be embedded in the unit circle, we shall show that a strongly summable ultrafilter $p$ on $G$ has the property that the equation $p + x = p$ has the unique solution $x = p$ in $G^*$, and so does the equation $x + p = p$. We shall also show that Martin's Axiom implies the existence of certain strongly summable ultrafilters $p$ on $G$ with the property that $x + y = p$, with $x, y \in G^*$, implies that $x$ and $y$ are both in $G + p$.

Our results generalise theorems already known for the case in which $G = \mathbb{Z}$ ([2] and [1]) and the case in which $G$ is Boolean [6].

We note in passing that strongly summable ultrafilters on $G$ give rise to interesting topologies. (See [5, Section 9.2].) Any strongly summable ultrafilter $p \in G^*$ defines an extremally disconnected regular left invariant topology on $G$ for which $\{\{0\} \cup A : A \in p\}$ is the filter of neighbourhoods of 0. This topology has the property of being maximal subject to having no isolated points. In the case in which $G$ is Boolean, $G$ is a topological group in this topology. It is not known whether every ultrafilter converging to 0 on a maximal topological group has to be strongly summable. It is also an open question whether the existence of extremally disconnected topological groups without isolated points can be demonstrated in ZFC.

If $q \in \beta G$ is a given idempotent and $B \in q$, we shall use $B^\star$ to denote $\{b \in B : b + q \in \overline{B}\}$. We shall use the fact that $B^\star \in q$ and that, for every $b \in B^\star$, $-b + B^\star \in q$ [5, Lemma 4.14].

We shall use $\mathbb{T}$ to denote the unit circle $\mathbb{R}/\mathbb{Z}$, and shall use the element $t \in (-\frac{1}{2}, \frac{1}{2}]$ to represent the element $t + \mathbb{Z}$ of $\mathbb{T}$. It is well known that any countable abelian group can be embedded in the direct sum $\bigoplus_{n\in\mathbb{N}} \mathbb{T}$ of countably many copies of $\mathbb{T}$ and so we shall assume that $G \subseteq \bigoplus_{n\in\mathbb{N}} \mathbb{T}$ and shall use $\pi_n$ for the projection map from $\bigoplus_{n\in\mathbb{N}} \mathbb{T}$ onto its $n$'th factor.

Of course, any ultrafilter $q \in \beta G$ converges to a point $\gamma(q) \in \times_{n\in\mathbb{N}} \mathbb{T}$ where $\times_{n\in\mathbb{N}} \mathbb{T}$ has the product topology. (By this we mean — slightly incorrectly — that every neighbourhood of $\gamma(q)$ contains a member of $q$). It is easy to prove that the mapping $\gamma\colon \beta G \to \times_{n\in\mathbb{N}} \mathbb{T}$ is a continuous homomorphism. In particular, $\gamma(q) = 0$ if $q$ is idempotent. We shall prove that any strongly summable ultrafilter $p$ on $G$ is idempotent. However, prior to proving this, we can conclude that $\gamma(p) = 0$, because every member of $p$ contains three points of the form $a$, $b$ and $a + b$.

We note that, if $f$ is any function from $G$ to a set $S$ and if $q \in \beta G$, then $\{T \subseteq S : f^{-1}[T] \in q\}$ is an ultrafilter on $S$. We shall use $f(q)$ to denote this ultrafilter. (So $f$ also denotes also the continuous extension of $f$ mapping $\beta G$ to $\beta S$.)

## 2. Existence.

We show in this section that Martin's Axiom implies the existence of nonprincipal strongly summable ultrafilters on $G$.

**2.1. Lemma.** *Let $p \in \beta G$. Suppose that, for every $A \in p$, there exists a sequence $\langle x_n\rangle_{n=1}^\infty$ in $G$ such that $FS\langle x_n\rangle_{n=1}^\infty \subseteq A$ and $x_1 + FS\langle x_n\rangle_{n=2}^\infty \in p$. Then $-p+p = p$.*

*Proof.* Let $B \in p$ and suppose that $\{x \in G : x + B \in p\} \notin p$. Let $A = B \setminus \{x \in G : x + B \in p\}$. Pick a sequence $\langle x_n\rangle_{n=1}^\infty$ as guaranteed for $A$. Now $x_1 \in A$ so $x_1 + B \notin p$. But $x_1 + FS\langle x_n\rangle_{n=2}^\infty \in p$ and $x_1 + FS\langle x_n\rangle_{n=2}^\infty \subseteq x_1 + A \subseteq x_1 + B$, a contradiction. □

**2.2. Lemma.** *Let $p$ be a strongly summable ultrafilter on $G$ and let $B = \{b \in G : \pi_i(b) \in \{0, \frac{1}{2}\}$ for every $i \in \mathbb{N}\}$. If $B \notin p$, then $-p + p \neq p$.*

*Proof.* Suppose that $B \notin p$. We consider two cases.

Case (i). Suppose that there exists $i \in \mathbb{N}$ such that $\{b \in G : \pi_i(b) \notin \{0, \frac{1}{2}\}\} \in p$. Let $P = \{b \in G \setminus B : \pi_i(b) \in (0, \frac{1}{2})\}$ and $Q = \{b \in \setminus B : \pi_i(b) \in (-\frac{1}{2}, 0)\}$. If $b \in Q$, the fact that $p$ converges to 0 implies that $b + p$ converges to $b$ and hence that $b+p \in \overline{Q}$. Thus $\overline{Q}+p \subseteq \overline{Q}$. So, if $P \in p$, we have $Q \in -p$ and $-p+p \subseteq \overline{Q}+p \subseteq \overline{Q}$. Similarly, if $Q \in p$, we have $-p + p \in \overline{P}$.

Case (ii). Now suppose that, for every $i \in \mathbb{N}$, $\{b \in G : \pi_i(b) \in \{0, \frac{1}{2}\}\} \in p$. Since $\pi_i(p)$ converges to 0, this implies that $\{b \in G : \pi_i(b) = 0\} \in p$. For each $b \in G \setminus B$, let $m(b) = \min\{i \in \mathbb{N} : \pi_i(b) \notin \{0, \frac{1}{2}\}\}$. We now put $P = \{b \in G \setminus B : \pi_{m(b)}(b) \in (0, \frac{1}{2})\}$ and $Q = \{b \in G \setminus B : \pi_{m(b)}(b)(-\frac{1}{2}, 0)\}$. Let $b \in Q$. If $X = \{x \in G : \pi_i(x) = 0$ for every $i \leq m(b)\}$, then $X \in p$. Since $b + X \subseteq Q$,

$b+p \in \overline{Q}$. So, if $P \in p$, we have $Q \in -p$ and $-p+p \in \overline{Q}+p \subseteq \overline{Q}$. Similarly, if $Q \in p$, we have $-p+p \in \overline{P}$. □

**2.3. Theorem.** *Let $p$ be a strongly summable ultrafilter on $G$. Then $p$ is an idempotent.*

*Proof.* Notice that 0 is an idempotent, so we may presume that $p \in G^*$. Assume first that the hypotheses of Lemma 2.1 do not hold and pick $A \in p$ such that there is no sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $G$ with $FS\langle x_n \rangle_{n=1}^{\infty} \subseteq A$ and $x_1 + FS\langle x_n \rangle_{n=2}^{\infty} \in p$.

Let $B \in p$ and suppose that $B \notin p+p$ so that $\{x \in G : -x+B \in p\} \notin p$. Then

$$\big((A \cap B) \setminus \{x \in G : -x+B \in p\}\big) \in p$$

so pick $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS\langle x_n \rangle_{n=1}^{\infty} \subseteq (A \cap B) \setminus \{x \in G : -x+B \in p\}$ and $FS\langle x_n \rangle_{n=1}^{\infty} \in p$. Notice that

$$FS\langle x_n \rangle_{n=1}^{\infty} = FS\langle x_n \rangle_{n=2}^{\infty} \cup \{x_1\} \cup (x_1 + FS\langle x_n \rangle_{n=2}^{\infty}) .$$

Now $p$ is nonprincipal and by assumption $x_1 + FS\langle x_n \rangle_{n=2}^{\infty} \notin p$ so $FS\langle x_n \rangle_{n=2}^{\infty} \in p$. Also $FS\langle x_n \rangle_{n=2}^{\infty} \subseteq -x_1 + B$ and so $-x_1 + B \in p$, a contradiction.

We may therefore suppose that the hypotheses of Lemma 2.1 are satisfied and hence that $-p+p=p$. It then follows from Lemma 2.2 that $B = \{b \in G : \pi_i(b) \in \{0, \frac{1}{2}\}$ for every $i \in \mathbb{N}\} \in p$. However, this implies that $-p=p$. So we again have $p+p=p$. □

**2.4. Definition.** Let $p \in \beta G$. We shall say that $p$ is *a sparse strongly summable ultrafilter* if and only if for every $A \in p$, there exists a set $X \subseteq G$ and a set $Y \subseteq X$ such that $X \setminus Y$ is infinite, $FS\langle Y \rangle \in p$ and $FS\langle X \rangle \subseteq A$.

We shall show that Martin's Axiom implies that nonprincipal strongly summable ultrafilters exist on $G$. Indeed, we shall show that Martin's Axiom implies that any family of subsets of $G$ which is contained in an idempotent and has cardinality less than $\mathfrak{c}$, is contained in a sparse strongly summable idempotent.

We remind the reader of the version of Martin's Axiom which we shall use. A partially ordered set $Q$ is said to satisfy the countable chain condition if every antichain in $Q$ is countable. A subset $D$ is said to be dense if, for every $a \in Q$, there exists $d \in D$ such that $d \leq a$. A non-empty subset $\Phi$ of $Q$ is called a filter if it satisfies the two following conditions:

(i) for every $a \in \Phi$ and $b \in Q$, $a \leq b$ implies that $b \in \Phi$ and
(ii) for every $a, b \in \Phi$, there exists $c \in \Phi$ such that $c \leq a$ and $c \leq b$.

Then Martin's Axiom asserts that, if $Q$ satisfies the countable chain condition and if $\mathcal{F}$ is a family of dense subsets of $Q$ for which $|\mathcal{F}| < \mathfrak{c}$, then there is a filter in $Q$ which meets every set in $\mathcal{F}$.

**2.5. Definition.** We now assume that the elements of $G$ have been arranged as a sequence, and write $s < t$ if $s$ occurs before $t$ in this sequence. Then every infinite subset $X$ of $G$ defines a unique sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $G$ with the property that $x_n < x_{n+1}$ for every $n$ and $X = \{x_n : n \in \mathbb{N}\}$. We put $FS_m\langle X \rangle = FS(\langle x_n \rangle_{n=m}^{\infty})$ for each $m \in \mathbb{N}$ and $FS_\infty\langle X \rangle = \bigcap_{m \in \mathbb{N}} \mathrm{cl}_{\beta G}(FS_m\langle X \rangle)$.

**2.6. Lemma.** *Let $\mathcal{F}$ denote a family of subsets of $G$ with the finite intersection property. Suppose that $B \in \mathcal{F}$ and that $\overline{B}$ contains an idempotent $q \in G^*$. Suppose also that $B^\star = \{b \in B : B \in b+q\} \in \mathcal{F}$ and that $-b+B^\star \in \mathcal{F}$ for every $b \in B^\star$. Then, if $|\mathcal{F}| < \mathfrak{c}$, it follows from Martin's Axiom that there exists a set $X \subseteq G$ such that $FS\langle X \rangle \subseteq B$ and $X \cap A \neq \varnothing$ for every $A \in \mathcal{F}$.*

*Proof.* We may suppose that $\mathcal{F}$ is closed under finite intersections.

Let $Q = \{F \in \mathcal{P}_f(G) : FS\langle F\rangle \subseteq B^\star\}$. We define a partial order on $Q$ by stating that $F' \leq F$ if $F \subseteq F'$. Since $Q$ is countable, it is trivial that it satisfies the countable chain condition.

For each $A \in \mathcal{F}$, let $D(A) = \{F \in Q : F \cap A \neq \varnothing\}$. To see that $D(A)$ is dense in $Q$, let $F \in Q$. We can choose $a \in A \cap B^\star \cap \bigcap_{b \in FS\langle F\rangle}(-b + B^\star)$. Then $F \cup \{a\} \in Q$, $F \cup \{a\} \leq F$ and $F \cup \{a\} \in D(A)$.

Thus it follows from Martin's Axiom that there is a filter $\Phi \subseteq Q$ such that $\Phi \cap D(A) \neq \varnothing$ for every $A \in \mathcal{F}$. Let $X = \bigcup \Phi$.

If $H$ is any finite subset of $X$, $H \subseteq F$ for some $F \in Q$ and so $FS\langle H\rangle \subseteq B$. Thus $FS\langle X\rangle \subseteq B$. Furthermore, for any $A \in \mathcal{F}$, there exists $F \in \Phi \cap D(A)$ and so $X \cap A \neq \varnothing$. □

**2.7. Lemma.** *Let $\mathcal{F}$ be a family of subsets of $G$ contained in an idempotent $q \in G^*$. If $|\mathcal{F}| < \mathfrak{c}$, it follows from Martin's Axiom that there exists an infinite subset $X$ of $G$ such that $FS_\infty\langle X\rangle \subseteq \bigcap_{A \in \mathcal{F}} \bar{A}$.*

*Proof.* Let $\overline{\mathcal{F}}$ denote the family of sets which are finite intersections of sets in

$$\mathcal{F} \cup \{B^\star : B \in \mathcal{F}\} \cup \{-b + B^\star : B \in \mathcal{F}, b \in B^\star\} \cup \{G \setminus F : F \in \mathcal{P}_f(G)\}.$$

We note that $\overline{\mathcal{F}} \subseteq q$. Let $\mathcal{F}$ be well ordered as $\langle A_\lambda\rangle_{\lambda \leq \kappa}$. By Lemma 2.6, there exists a subset $X_0$ of $G$ for which $FS\langle X_0\rangle \subseteq A_0$ and $X_0 \cap A \neq \varnothing$ for every $A \in \overline{\mathcal{F}}$.

We then make the inductive assumption that $0 < \beta \leq \kappa$ and that we have defined $X_\alpha \subseteq G$ for every $\alpha < \beta$ so that the following conditions are satisfied:

(a) $FS\langle X_\alpha\rangle \subseteq A_\alpha$ and $X_\alpha \cap A \neq \varnothing$ for every $A \in \overline{\mathcal{F}}$ and
(b) if $\alpha' < \alpha$, then $X_\alpha^* \subseteq X_{\alpha'}^*$.

We apply Lemma 2.6, with $\overline{\mathcal{F}} \cup \{X_\alpha : \alpha < \beta\}$ in place of $\mathcal{F}$ and $A_\beta$ in place of $B$. By this lemma, there exists a set $W_\beta \subseteq G$ such that $FS\langle W_\beta\rangle \subseteq A_\beta$ and $W_\beta \cap A \cap X_\alpha \neq \varnothing$ for every $A \in \overline{\mathcal{F}}$ and every $\alpha < \beta$. By [5, Corollary 12.12], there exists an infinite subset $X_\beta$ of $G$ such that $X_\beta^* \subseteq \overline{W_\beta \cap A \cap X_\alpha}$ for every $A \in \overline{\mathcal{F}}$ and every $\alpha < \beta$. We may suppose that $X_\beta \subseteq W_\beta$.

It is clear that conditions (a) and (b) are satisfied with $\beta$ in place of $\alpha$. We can therefore define $X_\alpha$ for every $\alpha \leq \kappa$ so that these conditions hold.

We put $X = X_\kappa$. If $\alpha \leq \kappa$, $X^* \subseteq X_\alpha^*$ and so $X \setminus X_\alpha$ is finite. Thus, for every $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ for which $FS_n\langle X\rangle \subseteq FS_m\langle X_\alpha\rangle$. So $FS_\infty\langle X\rangle \subseteq \overline{FS_m\langle X_\alpha\rangle}$ and therefore $FS_\infty\langle X\rangle \subseteq FS_\infty\langle X_\alpha\rangle \subseteq \overline{A_\alpha}$. □

**2.8. Theorem.** *Let $\mathcal{F}$ be a family of subsets of $G$ contained in an idempotent $q \in S^*$. If $|\mathcal{F}| < \mathfrak{c}$, Martin's Axiom implies that there is a sparse strongly summable ultrafilter $p$ on $G$ for which $\mathcal{F} \subseteq p$.*

*Proof.* We assume Martin's Axiom.

Let $\langle S_\alpha\rangle_{\alpha<\mathfrak{c}}$ be an enumeration of $\mathcal{P}(G)$. We can choose $Z_0 \in \{S_0, G \setminus S_0\}$ such that $Z_0 \in q$. By Lemma 2.7, we can choose an infinite subset $X_0$ of $S$ such that $FS_\infty\langle X_0\rangle \subseteq \overline{A} \cap \overline{Z_0 \setminus \{0\}}$ for every $A \in \mathcal{F}$. We can then choose an infinite subset $Y_0$ of $X_0$ for which $X_0 \setminus Y_0$ is infinite. We now make the inductive assumption that $0 < \beta < \mathfrak{c}$ and that $Y_\alpha \subseteq X_\alpha \subseteq G$ have been defined for every $\alpha < \beta$ so that the following conditions hold:

(a) $FS_\infty\langle X_\alpha\rangle \subseteq \overline{S_\alpha}$ or $FS_\infty\langle X_\alpha\rangle \subseteq \overline{G \setminus S_\alpha}$;
(b) if $\alpha' < \alpha$, then $FS_\infty\langle X_\alpha\rangle \subseteq FS_\infty\langle Y_{\alpha'}\rangle$; and
(c) $X_\alpha \setminus Y_\alpha$ is infinite.

By [5, Lemma 5.11], $\bigcap_{\alpha<\beta} FS_\infty\langle Y_\alpha\rangle$ is a compact subsemigroup of $\beta G$ and therefore contains an idempotent $r \in \beta G$. Since $0 \notin \bigcap_{\alpha<\beta} FS_\infty\langle Y_\alpha\rangle$, $r \in G^*$. We can choose

$Z_\beta \in \{S_\beta, G \setminus S_\beta\}$ satisfying $Z_\beta \in r$. By Lemma 2.7 (applied to $\{FS_m\langle Y_\alpha\rangle : \alpha < \beta \text{ and } m \in \mathbb{N}\} \cup \{Z_\beta\}$ in place of $\mathcal{F}$) we can choose $X_\beta \subseteq S$ such that $FS_\infty\langle X_\beta\rangle \subseteq \overline{Z_\beta}$ and $FS_\infty\langle X_\beta\rangle \subseteq \overline{FS_m\langle Y_\alpha\rangle}$ for every $\alpha < \beta$ and every $m \in \mathbb{N}$. Thus $FS_\infty\langle X_\beta\rangle \subseteq FS_\infty\langle Y_\alpha\rangle$ for every $\alpha < \beta$. We choose $Y_\beta$ to be an infinite subset of $X_\beta$ for which $X_\beta \setminus Y_\beta$ is infinite. Then conditions (a)–(c) are satisfied with $\beta$ in place of $\alpha$.

This shows that we can define $X_\alpha$ and $Y_\alpha$ for every $\alpha < \mathfrak{c}$ so that conditions (a)–(c) are satisfied. We put $p = \{B \subseteq G : FS_\infty\langle Y_\alpha\rangle \subseteq \overline{B} \text{ for some } \alpha < \mathfrak{c}\}$. It is clear that $p$ is a filter. For every $S \subseteq G$, $S \in p$ or $G \setminus S \in p$, and so $p$ is an ultrafilter. It is evident that $p$ is a sparse strongly summable ultrafilter and that $\mathcal{F} \subseteq p$. □

## 3. Independence.

We now set out to show that the existence of a nonprincipal strongly summable ultrafilter on $G$ cannot be demonstrated in ZFC. We shall do this by showing that the existence of an ultrafilter of this kind implies the existence of a P-point in $\mathbb{N}^*$. It is well known that this cannot be proved in ZFC [10, VI §4].

**3.1. Definition.** For each $x \in G \setminus \{0\}$, we put $\min(x) = \min\{n \in \mathbb{N} : \pi_n(x) \neq 0\}$ and $\max(x) = \max\{n \in \mathbb{N} : \pi_n(x) \neq 0\}$.

We omit the easy proof of the following lemma.

**3.2. Lemma.** *Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $(0, \frac{1}{2})$ with the property that $x_n > 4x_{n+1}$ for every $n$. Then $x_n > \sum_{i=n+1}^\infty 3x_i$ for every $n$. Furthermore, if $\sum_{n=1}^\infty a_n x_n = \sum_{n=1}^\infty b_n x_n$, where each $a_n$ and $b_n$ is 0, 1 or 2, then $a_n = b_n$ for every $n$.*

**3.3. Lemma.** *Suppose that $p$ is a nonprincipal strongly summable ultrafilter on $G$. If $\{x \in G \setminus \{0\} : \pi_{\min(x)}(x) = \frac{1}{2}\} \notin p$, then there is a P-point in $\mathbb{N}^*$.*

*Proof.* We may suppose without loss of generality that $\{x \in G \setminus \{0\} : \pi_{\min(x)}(x) \in (0, \frac{1}{2}]\} \in p$.

For each $i \in \{0, 1, 2\}$, we put $X_i = \bigcup_{m=0}^\infty [\frac{1}{2^{3m+i+2}}, \frac{1}{2^{3m+i+1}})$. We choose $j \in \{0, 1, 2\}$ such that $X = \{x \in G \setminus \{0\} : \pi_{\min(x)}(x) \in X_j\} \in p$. Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $G$ for which $FS\langle x_n\rangle_{n=1}^\infty \in p$ and $FS\langle x_n\rangle_{n=1}^\infty \subseteq X$.

Given $i \in \mathbb{N}$, let $M_i = \{n \in \mathbb{N} : \min(x_n) = i\}$. If $n$ and $n'$ are distinct elements of $M_i$, then $\min(x_n + x_{n'}) = i$, because $\pi_i(x_n + x_{n'}) \neq 0$ since $0 < \pi_i(x_n) < \frac{1}{2}$ and $0 < \pi_i(x_{n'}) < \frac{1}{2}$. It follows that $\pi_i(x_n)$ and $\pi_i(x_{n'})$ cannot be in the same interval of the form $[\frac{1}{2^{m+1}}, \frac{1}{2^m})$. So $\pi_i(x_n) < \pi_i(x_{n'})$ implies that $4\pi_i(x_n) < \pi_i(x_{n'})$. Consequently, if $F \in \mathcal{P}_f(M_i)$, then $\min(\sum_{n\in F} x_n) = i$.

Let $x \in G \setminus \{0\}$. Suppose that $\min(x) = i$ and that $x = \sum_{n\in F} a_n x_n$, where $F \in \mathcal{P}_f(\mathbb{N})$ and each $a_n \in \{1, 2\}$. We claim that $\min(x_n) \geq i$ for every $n \in F$. To see this, let $m = \min\{\min(x_n) : n \in F\}$. Let $H = \{n \in F \cap M_m : a_n = 2\}$. Then $\pi_m(\sum_{n\in F\cap M_m} a_n x_n) \neq 0$, because both $\pi_m(\sum_{n\in F\cap M_m} x_n)$ and $\pi_m(\sum_{n\in H} x_n)$ are in $(0, \frac{1}{2})$. It follows that $\min(x) = m$ and hence that $m = i$.

Suppose that $F, H \in \mathcal{P}_f(\mathbb{N})$ and that $x = \sum_{n\in F} a_n x_n = \sum_{n\in H} b_n x_n$, where each $a_n$ and $b_n$ is 1 or 2. We claim that $F = H$ and that $a_n = b_n$ for every $n \in F$. To see this, suppose that $x \in M_i$. Then $\pi_i(x) = \sum_{n\in F\cap M_i} a_n \pi_i(x_n) = \sum_{n\in H\cap M_i} b_n \pi_i(x_n)$. It follows from Lemma 3.2 that $\pi_i[F \cap M_i] = \pi_i[H \cap M_i]$ and that $a_n = b_n$ for every $n \in F \cap M_i$. We have observed that, if $n \neq n'$ in $M_i$, then $\pi_i(x_n) \neq \pi_i(x_{n'})$. So $F \cap M_i = H \cap M_i$ and $a_n = b_n$ for every $n \in F \cap M_i$. The terms $a_n x_n$ for which $n \in F \cap M_i$ can then be cancelled from the equation

$\sum_{n\in F} a_n x_n = \sum_{n\in H} b_n x_n$ and the argument repeated. Thus $F = H$ and $a_n = b_n$ for every $n \in F$.

In consequence, for any $F, H \in \mathcal{P}_f(\mathbb{N})$, $\sum_{n\in F} x_n + \sum_{n\in H} x_n \in FS\langle x_n\rangle_{n=1}^{\infty}$ implies that $F \cap H = \varnothing$. It also follows that, for every $x \in FS\langle x_n\rangle_{n=1}^{\infty}$, there is a unique set $H_x \in \mathcal{P}_f(\mathbb{N})$ for which $x = \sum_{n\in H_x} x_n$.

We now claim that for each $\ell \in \mathbb{N}$, $FS\langle x_n\rangle_{n=\ell+1}^{\infty} \in p$. Otherwise since

$$FS\langle x_n\rangle_{n=1}^{\infty} = FS\langle x_n\rangle_{n=\ell+1}^{\infty} \cup FS\langle x_n\rangle_{n=1}^{\ell} \cup \bigcup\{a + FS\langle x_n\rangle_{n=\ell+1}^{\infty} : a \in FS\langle x_n\rangle_{n=1}^{\ell}\}$$

and $FS\langle x_n\rangle_{n=1}^{\ell}$ is finite, there is some $a \in FS\langle x_n\rangle_{n=1}^{\ell}$ such that $a + FS\langle x_n\rangle_{n=\ell+1}^{\infty} \in p$. Pick $\langle y_n\rangle_{n=1}^{\infty}$ such that $FS\langle y_n\rangle_{n=1}^{\infty} \subseteq a + FS\langle x_n\rangle_{n=\ell+1}^{\infty}$. Then $y_1 = \sum_{n\in F} x_n$, $y_2 = \sum_{n\in H} x_n$, $y_1 + y_2 \in FS\langle x_n\rangle_{n=1}^{\infty}$, and $F \cap H \neq \varnothing$, a contradiction.

We define $h\colon FS\langle x_n\rangle_{n=1}^{\infty} \to \mathbb{N}$ by $h(x) = \max(H_x)$. We shall show that $h(p)$ is a P-point in $\mathbb{N}^*$. To see this, we choose any function $f\colon \mathbb{N} \to \mathbb{N}$ and show that there is a set in $h(p)$ on which $f$ is bounded or a set in $h(p)$ on which $f$ has finite preimages.

Let $P = \{x \in FS\langle x_n\rangle_{n=1}^{\infty} : \min(H_x) \geq f\big(\max(H_x)\big)\}$. Suppose that $P \in p$. Then $P^\star = \{x \in P : x + p \in \overline{P}\} \in p$ and, for every $x \in P^\star$, $-x + P^\star \in p$. Let $x \in P^\star$ and let $\ell = \max(H_x)$. Suppose that $y \in (-x + P^\star) \cap FS\langle x_n\rangle_{n=\ell+1}^{\infty}$. Then $\min(H_x) = \min(H_{x+y}) \geq f\big(\max(H_{x+y})\big) = f\big(\max(H_y)\big)$, and so $f$ is bounded on a set in $h(p)$.

We may therefore suppose that $Q = \{x \in FS\langle x_n\rangle_{n=1}^{\infty} : \min(H_x) < f\big(\max(H_x)\big)\} \in p$. Let $\langle y_n\rangle_{n=1}^{\infty}$ be a sequence in $G$ for which $FS\langle y_n\rangle_{n=1}^{\infty} \in p$ and $FS\langle y_n\rangle_{n=1}^{\infty} \subseteq Q \cap FS\langle x_n\rangle_{n=1}^{\infty}$. We note that, for any $n \neq n'$ in $\mathbb{N}$, the fact that $y_n + y_{n'} \in FS\langle x_n\rangle_{n=1}^{\infty}$ implies that $H_y \cap H_{y'} = \varnothing$.

We shall show that $f$ has finite preimages on $h[FS\langle y_n\rangle_{n=1}^{\infty}]$. To see this, suppose on the contrary that, for some $k \in \mathbb{N}$, $f$ assumes the value $k$ infinitely often on this set.

Choose any $z_1 \in FS\langle y_n\rangle_{n=1}^{\infty}$ with $f\big(\max(H_{z_1})\big) = k$. Suppose that $z_1 = \sum_{n\in F_1} y_n$, where $F_1 \in \mathcal{P}_f(\mathbb{N})$. This implies that $H_{z_1} = \bigcup_{n\in F_1} H_{y_n}$. We can choose $z \in FS\langle y_n\rangle_{n=1}^{\infty}$ such that $f\big(\max(H_z)\big) = k$ and $\max(H_z) > \max(H_{y_n})$ for every $n \in F_1$. Suppose that $z = \sum_{n\in H} y_n$, where $H \in \mathcal{P}_f(\mathbb{N})$. We put $F_2 = H \setminus F_1$ and $z_2 = \sum_{n\in F_2} y_n$. We observe that $\max(H_{z_2}) = \max(H_z)$ and so $f\big(\max(H_{z_2})\big) = k$. In this way, we can construct a sequence $\langle z_n\rangle_{n=1}^{\infty}$ in $G$ and a pairwise disjoint sequence $\langle F_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that $f\big(\max(H_{z_n})\big) = k$ and $z_n = \sum_{i\in F_n} y_i$ for every $n$.

We have $\min(H_{z_n}) < f\big(\max(H_{z_n})\big) = k$ for every $n$. So there exists $n \neq n'$ in $\mathbb{N}$ for which $\min(H_{z_n}) = \min(H_{z_{n'}})$. This is a contradiction, because $H_{z_n} \cap H_{z_{n'}} = \varnothing$. □

**3.4. Lemma.** *Let $p$ be a nonprincipal strongly summable ultrafilter on $G$ such that, for every $n \in \mathbb{N}$, $\{x \in G \setminus \{0\} : \pi_i(x) = 0 (\forall i \leq n)\} \in p$. Let $X \in p$. Suppose that* $\max(p)$ *is not a P-point in* $\mathbb{N}^*$. *Then there exist a function $f\colon \mathbb{N} \to \mathbb{N}$, a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $G$, a pairwise disjoint sequence $\langle F_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ and an integer $k \in \mathbb{N}$ such that $FS\langle x_n\rangle_{n=1}^{\infty} \subseteq X \cap \{x \in G\setminus\{0\} : \min(x) < f\big(\max(x)\big)\}$, $\min(\sum_{i\in F_n} x_i) < f(\max \sum_{i\in F_n} x_i) = k$ for every $n \in \mathbb{N}$, and $\max \sum_{i\in F_n} x_i < \max \sum_{i\in F_{n+1}} x_i$ for every $n \in \mathbb{N}$.*

*Proof.* There is a function $f : \mathbb{N} \to \mathbb{N}$ with the property that, for every $P \in p$, $f$ is unbounded on $\max[P]$ and does not have finite preimages on this set. Let $P = \{x \in G \setminus \{0\} : \min(x) \geq f\big(\max(x)\big)\}$. Suppose that $P \in p$. Then $P^\star = \{y \in$

$P : y + p \in \overline{P}\} \in p$ and, for every $y \in P^\star$, $-y + P^\star \in p$. Pick any $y \in P^\star$ and let $A = (-y + P^\star) \cap \{z \in G \setminus \{0\} : \pi_i(x) = 0(\forall i \leq \max(y))\}$. Then $A \in p$ and if $z \in A$, then $\min(y) = \min(y + z) \geq f(\min(y + z)) = f(\max(z))$ so $f$ is bounded on $\max[A]$.

We may therefore suppose that $Q = \{x \in G \setminus \{0\} : \min(x) < f(\max(x))\} \in p$. We can choose a sequence $\langle x_n \rangle_{n+1}^\infty$ in $G$ for which $FS\langle x_n \rangle_{n=1}^\infty \in p$ and $FS\langle x_n \rangle_{n=1}^\infty \subseteq X \cap Q$. Since $f$ does not have finite preimages on $\max[FS\langle x_n \rangle_{n=1}^\infty]$, there exists $k \in \mathbb{N}$ for which there are an infinite number of values of $t$ in $\max[FS\langle x_n \rangle_{n=1}^\infty]$ satisfying $f(t) = k$.

Choose any $y_1 \in FS\langle x_n \rangle_{n=1}^\infty$ with $f(\max(y_1)) = k$. Suppose that $y_1 = \sum_{n \in F_1} x_n$, where $F_1 \in \mathcal{P}_f(\mathbb{N})$. We can choose $w \in FS\langle x_n \rangle_{n=1}^\infty$ such that $f(\max(w)) = k$ and $\max(w) > \max(x_n)$ for every $n \in F_1$. Suppose that $w = \sum_{n \in H} x_n$, where $H \in \mathcal{P}_f(\mathbb{N})$. We put $F_2 = H \setminus F_1$ and $y_2 = \sum_{n \in F_2} x_n$, noting that $\max(y_2) = \max(w)$. In this way, we can construct a sequence $\langle y_n \rangle_{n=1}^\infty$ in $G$ and a pairwise disjoint sequence $\langle F_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $f(\max(y_n)) = k$ and $y_n = \sum_{i \in F_n} x_i$ for every $n$.

For every $n \in \mathbb{N}$, we have $\min(y_n) < f(\max(y_n)) = k$. □

**3.5. Lemma.** *Let $p$ be a nonprincipal strongly summable ultrafilter on $G$ with the property that, for every $n \in \mathbb{N}$, $\{x \in G : \pi_i(x) = 0(\forall i \leq n)\} \in p$. Suppose that $X = \{x \in G \setminus \{0\} : \pi_{\min(x)}(x) = \frac{1}{2}\} \in p$. Then $\max(p)$ is a P-point in $\mathbb{N}^*$.*

*Proof.* Suppose that $\max(p)$ is not a P-point in $\mathbb{N}^*$. Let $\langle x_n \rangle_{n=1}^\infty$, $\langle F_n \rangle_{n=1}^\infty$ and $k$ have the properties guaranteed by Lemma 3.3. For each $n \in \mathbb{N}$, let $y_n = \sum_{i \in F_n} x_i$. We may suppose that there exists $m \in \mathbb{N}$ such that $\min(y_n) = m$ for every $n$, because this could be achieved by replacing $\langle F_n \rangle_{n=1}^\infty$ by a subsequence. We observe that $\min(\sum_{i \in H} y_i) < k$ for every $H \in \mathcal{P}_f(\mathbb{N})$, because $\sum_{i \in H} y_i \in FS\langle x_n \rangle_{n=1}^\infty$ and $\max(\sum_{i \in H} y_i) = \max(y_t)$ where $t = \max H$.

Let $\ell$ denote the largest positive integer for which there exists an infinite pairwise disjoint sequence $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $\min(\sum_{i \in H_n} y_i) = \ell$ for every $n$. Let $K_n = H_{2n-1} \cup H_{2n}$. Then $\ell < \min(\sum_{i \in K_n} y_i) < k$. So there exists $\ell' > \ell$ and an infinite subsequence $\langle K_{n_r} \rangle_{r=1}^\infty$ of $\langle K_n \rangle_{n=1}^\infty$ such that $\min(\sum_{i \in K_{n_r}} x_i) = \ell'$ for every $r$, contradicting our choice of $\ell$. □

**3.6. Theorem.** *The existence of a nonprincipal strongly summable ultrafilter on $G$ implies the existence of a P-point in $\mathbb{N}^*$.*

*Proof.* Let $p$ be a nonprincipal strongly summable ultrafilter on $G$. If $G \subseteq \mathbb{T}$, it follows from Lemma 3.3 that the existence of a nonprincipal strongly summable ultrafilter on $G$ implies the existence of a P-point in $\mathbb{N}^*$. We observe that, for each $i \in \mathbb{N}$, $\pi_i(p)$ is a strongly summable ultrafilter on $\pi_i[G]$. If $\pi_i(p)$ were a nonprincipal ultrafilter, the existence of a P-point in $\mathbb{N}^*$ would follow. So we may assume that $\pi_i(p)$ is the principal ultrafilter which has $\{0\}$ as a member. This implies that, for any $n \in \mathbb{N}$, $\{x \in G : \pi_i(x) = 0(\forall i \leq n)\} \in p$. The conclusion then follows immediately from Lemmas 3.3 and 3.5. □

## 4. Solving the Equation $x + y = p$.

We see in this section that if $G \subseteq \mathbb{T}$ and $p$ is a strongly summable ultrafilter on $G$, then there is only one solution to the equations $p + x = p$ and $x + p = p$. Moreover, if $p$ is a sparse strongly summable ultrafilter on $G$, then there are only the trivial solutions to the equation $x + y = p$.

**4.1. Theorem.** *Suppose that $G \subseteq \mathbb{T}$ and that $p$ is a nonprincipal strongly summable ultrafilter on $G$. Then the equation $p + x = p$ has the unique solution $x = p$ in $G^*$.*

*Proof.* We may suppose that $(0, \frac{1}{2}) \in p$. For each $i \in \{0, 1, 2\}$, we put

$$X_i = \bigcup_{m=0}^{\infty} \Big[\frac{1}{2^{3m+i+2}}, \frac{1}{2^{3m+i+1}}\Big).$$

We choose $j \in \{0, 1, 2\}$ such that $X_j \in p$. Assume that $p + x = p$ for some $x \in G^*$ with $x \neq p$. Pick $P \in p$ and $Q \in x$ such that $P \cap Q = \varnothing$. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $G$ for which $FS\langle x_n \rangle_{n=1}^{\infty} \in p$ and $FS\langle x_n \rangle_{n=1}^{\infty} \subseteq P \cap X_j$.

Exactly as in the proof of Lemma 3.3, we can conclude that, for any $n \neq n'$ in $\mathbb{N}$, $x_n \neq x_{n'}$. Furthermore, $x_n < x_{n'}$ implies that $4x_n < x_{n'}$. It follows from Lemma 3.2 that, if $x_{n_1} > x_{n_2} > \cdots > x_{n_k}$, then $x_{n_1} > 3 \sum_{i=2}^{k} x_{n_i}$.

Consider the equation

$$x_{n_1} + x_{n_2} + \cdots + x_{n_k} + t = x_{m_1} + x_{m_2} + \cdots + x_{m_\ell},$$

where $k$ and $\ell$ are in $\mathbb{N}$, $x_{n_1} > x_{n_2} > \cdots > x_{n_k}$, $x_{m_1} > x_{m_2} > \cdots > x_{m_\ell}$ and $t \in \mathbb{T}$ satisfies $-x_{n_k} < 2t < x_{n_k}$.

We claim that this implies that $t \in FS\langle x_n \rangle_{n=1}^{\infty}$. To see this, note that we cannot have $x_{n_1} > x_{m_1}$, because otherwise we should have $x_{n_1} + t > \frac{1}{2}x_{n_1} > x_{m_1} + x_{m_2} + \cdots + x_{m_\ell}$. We also cannot have $x_{n_1} < x_{m_1}$, because otherwise we should have $x_{m_1} > x_{n_1} + x_{n_2} + \cdots + x_{n_{k-1}} + 2x_{n_k} > x_{n_1} + x_{n_2} + \cdots + x_{n_k} + t$. So $x_{n_1} = x_{m_1}$. This term can be cancelled from the equation and the argument can be repeated if $k > 1$. We shall eventually have $t \in FS\langle x_n \rangle_{n=1}^{\infty}$.

We note that the equation $p + x = p$ implies that $x$ converges to 0. Let $Q \in x$. Let $Y$ denote the set of elements of the form $y + t$, where $y = \sum_{n \in F} x_n$ for some $F \in \mathcal{P}_f(\mathbb{N})$, $t \in Q$ and $-\min\{x_n : n \in F\} < 2t < \min\{x_n : n \in F\}$. Then $Y$ is a member of $p + x$. So there is an element $y + t$ of this form in $FS\langle x_n \rangle_{n=1}^{\infty}$. We have seen that this implies that $t \in FS\langle x_n \rangle_{n=1}^{\infty} \subseteq P$. So $P \cap Q \neq \varnothing$, a contradiction. □

**4.2. Lemma.** *Suppose that $G \subseteq \mathbb{T}$ and that $p$ is a nonprincipal strongly summable ultrafilter on $G$, with $(0, \frac{1}{2}) \in p$. Suppose that $x + y = p$, where $x, y \in G^*$, and that $y$ converges to 0. For each $i \in \{0, 1, 2\}$, let $X_i = \bigcup_{m=0}^{\infty} [\frac{1}{2^{3m+i+2}}, \frac{1}{2^{3m+i+1}})$. Let $j \in \{0, 1, 2\}$ be such that $X_j \in p$. Suppose that $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $G$ for which $FS\langle x_n \rangle_{n=1}^{\infty} \in p$ and $FS\langle x_n \rangle_{n=1}^{\infty} \subseteq X_j$. Let $B = \{\sum_{n=1}^{\infty} a_n x_n : \text{each } a_n \in \{0, 1\}\}$. Then $B \in x$.*

*Proof.* We note that $(0, \frac{1}{2}) \in x$, because $a < 0$ implies that $a + y \in \overline{(-\frac{1}{2}, 0)}$ and hence, if $(-\frac{1}{2}, 0) \in x$, then $(-\frac{1}{2}, 0) \in x + y$, contradicting the assumption that $x + y = p$.

We may assume that $FS\langle x_n \rangle_{n=1}^{\infty} \notin x$ because $FS\langle x_n \rangle_{n=1}^{\infty} \subseteq B$.

We first note that, if $n$ and $n'$ are distinct elements of $\mathbb{N}$, then $x_n$ and $x_{n'}$ cannot be in the same interval of the form $[\frac{1}{2^{m+1}}, \frac{1}{2^m})$. So $x_n < x_{n'}$ implies that $4x_n < x_{n'}$.

Let $A = \{a \in (0, \frac{1}{2}) \setminus FS\langle x_n \rangle_{n=1}^{\infty} : a + y \in \overline{FS\langle x_n \rangle_{n=1}^{\infty}}\}$. Then $A \in x$. Let $a \in A$. Then, if $Y_a = \{b \in G \cap (-\frac{a}{4}, \frac{a}{4}) : a + b \in FS\langle x_n \rangle_{n=1}^{\infty}\}$, we have $Y_a \in y$. Choose any $b \in Y_a$, and choose $n_1, n_2, \ldots, n_k \in \mathbb{N}$ with $x_{n_1} > x_{n_2} > \cdots > x_{n_k}$ such that $a + b = x_{n_1} + x_{n_2} + \cdots + x_{n_k}$. We note that $x_{n_1} \leq x_{n_1} + x_{n_2} + \ldots x_{n_k} < \frac{4}{3}x_{n_1}$ and that $\frac{3a}{4} < a + b < \frac{5a}{4}$. Hence $\frac{9a}{16} < x_{n_1} < \frac{5a}{4}$. Now $n_1$ is the unique positive integer for which $x_{n_1} \in (\frac{9a}{16}, \frac{5a}{4})$, because $n \neq n_1$ implies that $x_n > 4x_{n_1}$ or $4x_n < x_{n_1}$.

We define $f\colon A \to \{x_n : n \in \mathbb{N}\}$ by putting $f(a) = x_{n_1}$.

We claim that, for every $a \in A$, $a - f(a) \in A$. To see that $a > f(a)$, we note that, for every $b \in Y_a$, we have an equation of the form $a+b = x_{n_1}+x_{n_2}+\cdots+x_{n_k}$, where $x_{n_1} = f(a)$. This implies that $f(a) \le a+b$ and hence that $f(a) \le a$. The possibility that $a = f(a)$ is ruled out by the assumption that $a \notin FS\langle x_n\rangle_{n=1}^{\infty}$. Thus we have shown that $\frac{9a}{16} < f(a) < a$.

To see that $a - f(a) \notin FS\langle x_n\rangle_{n=1}^{\infty}$, suppose instead that $a - f(a) = x_{r_1} + x_{r_2} + \cdots + x_{r_\ell}$ with $x_{r_1} > x_{r_2} > \cdots > x_{r_\ell}$. This implies that $f(a) + x_{r_1} \le a$ and hence that $f(a) > x_{r_1}$, because otherwise we should have $f(a) + x_{r_1} \ge 2f(a) > a$. So $a \in FS\langle x_n\rangle_{n=1}^{\infty}$, a contradiction.

To see that $\big(a - f(a)\big) + y \in \overline{FS\langle x_n\rangle_{n=1}^{\infty}}$, we note that, for every $b \in Y_a$, we have an equation of the form $a + b = x_{n_1} + x_{n_2} + \cdots + x_{n_k}$, with $x_{n_1} = f(a)$ and $x_{n_1} > x_{n_2} > \cdots > x_{n_k}$. Thus $\big(a - f(a)\big) + b \in FS\langle x_n\rangle_{n=1}^{\infty}$, and so $\big(a - f(a)\big) + y \in \overline{FS\langle x_n\rangle_{n=1}^{\infty}}$. Furthermore, if $b \in Y_a \cap (-\frac{a-f(a)}{4}, \frac{a-f(a)}{4})$, then this equation implies that $x_{n_2} = f\big(a - f(a)\big)$. Thus $f\big(a - f(a)\big) < f(a)$.

We now define a sequence $\langle x_{n_i}\rangle_{i=1}^{\infty}$ by putting $x_{n_1} = f(a)$ and $x_{n_i} = f(a - \sum_{m=1}^{i-1} x_{n_m})$ if $i > 1$. By an immediate inductive argument, we have $a - \sum_{m=1}^{i} x_{n_m} \in A$ for every $i \in \mathbb{N}$. To see that $\langle x_{n_i}\rangle_{i=1}^{\infty}$ is decreasing, choose $i > 1$ and put $c = a - \sum_{m=1}^{i-1} x_{n_m}$. Then $f\big(c - f(c)\big) = x_{n_{i+1}} < f(c) = x_{n_i}$. We have observed that $f(a) < a < \frac{16}{9} f(a)$ for every $a \in A$, and so $0 < a - \sum_{m=1}^{i} x_{n_m} < \frac{16}{9} x_{n_{i+1}}$.

Thus $a = \sum_{m=1}^{\infty} x_{n_m}$. □

**4.3. Theorem.** *Suppose that $G \subseteq \mathbb{T}$ and that $p$ is a nonprincipal strongly summable ultrafilter on $G$. Then the equation $x + p = p$ has the unique solution $x = p$ in $G^*$.*

*Proof.* We may suppose that $(0, \frac{1}{2}) \in p$. Suppose that $x + p = p$, where $x \in G^*$ and $x \ne p$. Let $X_j \in p$ be defined as in Lemma 4.2. We can choose a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $G$ for which $FS\langle x_n\rangle_{n=1}^{\infty} \in p$, $FS\langle x_n\rangle_{n=1}^{\infty} \notin x$ and $FS\langle x_n\rangle_{n=1}^{\infty} \subseteq X_j$. Let $B = \{\sum_{i=1}^{\infty} x_{n_i} : \langle n_i\rangle_{i=1}^{\infty}$ is an infinite injective sequence in $\mathbb{N}\}$. By Lemma 4.2, $B \in x$. So $B + FS\langle x_n\rangle_{n=1}^{\infty} \in x + p$. By Lemma 3.2, this set is disjoint from $FS\langle x_n\rangle_{n=1}^{\infty}$, which is a member of $p$, contradicting the assumption that $x + p = p$. □

*Remark.* It is possible to prove in ZFC that there are idempotents $p \in \mathbb{Z}^*$ for which the equation $x + p = p$ has the unique solution $x = p$ in $\mathbb{Z}^*$ [5, Theorem 9.10]. We do not know of any ZFC proof that there are idempotents $p \in \mathbb{Z}^*$ for which the equation $p + x = p$ has the unique solution $x = p$ in $\mathbb{Z}^*$. Indeed, we do not know of any ZFC proof that there are idempotents in $\mathbb{Z}^*$ which are maximal for the relation $\le_L$. (This is the relation defined on idempotents by putting $p \le_L q$ if $p + q = p$.)

We now show that, if $G \subseteq \mathbb{T}$, sparse strongly summable ultrafilters defined on $G$ have remarkable algebraic properties.

**4.4. Lemma.** *Suppose that $G \subseteq \mathbb{T}$ and that $p$ is a sparse strongly summable ultrafilter on $G$. Let $x, y \in G^*$ satisfy $x + y = p$. If $y$ converges to 0, then $x = y = p$.*

*Proof.* We may suppose that $(0, \frac{1}{2}) \in p$ and, by Theorem 4.1, that $x \ne p$. Let $X_j$ be defined as in the statement of Lemma 4.2. Suppose that $\langle x_n\rangle_{n=1}^{\infty}$ is a sequence in $G$ for which $FS\langle x_n\rangle_{n=1}^{\infty} \in p$, $FS\langle x_n\rangle_{n=1}^{\infty} \notin x$ and $FS\langle x_n\rangle_{n=1}^{\infty} \subseteq X_j$.

Let $A = \{a \in (0, \frac{1}{2}) \setminus FS\langle x_n\rangle_{n=1}^{\infty} : a + y \in \overline{FS\langle x_n\rangle_{n=1}^{\infty}}\}$ and let $B = \{\sum_{i=1}^{\infty} x_{n_i} : \langle n_i\rangle_{i=1}^{\infty}$ is an injective sequence in $\mathbb{N}\}$. By Lemma 4.2, $B \in x$. Choose $a \in A \cap B$ and choose an injective sequence $\langle n_i\rangle_{i=1}^{\infty}$ in $\mathbb{N}$ for which $a = \sum_{i=1}^{\infty} x_{n_i}$.

Let $a'$ be any other element of $A \cap B$. There is a sequence of distinct positive integers $\langle n_i'\rangle_{i=1}^{\infty}$ for which $a' = \sum_{i=1}^{\infty} x_{n_i'}$. We can choose $b \in G$ such that $a + b$

and $a' + b$ are both in $FS\langle x_n\rangle_{n=1}^{\infty}$. So $a + \sum_{n\in F} x_n = a' + \sum_{n\in F'} x_n$ for some $F, F' \in \mathcal{P}_f(\mathbb{N})$. By Lemma 3.2, this implies that the terms in the sequences $\langle n_i\rangle_{i=1}^{\infty}$ and $\langle n'_i\rangle_{i=1}^{\infty}$ are eventually the same.

We claim that $FS\langle x_{n_i}\rangle_{i=1}^{\infty} \in p$. To see this, suppose the contrary. Then we can choose a sequence $\langle y_n\rangle_{n=1}^{\infty}$ in $G$ for which $FS\langle y_n\rangle_{n=1}^{\infty} \in p$, $FS\langle y_n\rangle_{n=1}^{\infty} \cap FS\langle x_{n_i}\rangle_{i=1}^{\infty} = \varnothing$, and $FS\langle y_n\rangle_{n=1}^{\infty} \subseteq FS\langle x_n\rangle_{n=1}^{\infty}$. We note that it follows from Lemma 3.2 that, for each $n \in \mathbb{N}$, there is a unique set $H_n \in \mathcal{P}_f(\mathbb{N})$ for which $y_n = \sum_{i\in H_n} x_i$. Furthermore, $H_n \nsubseteq \{n_i : i \in \mathbb{N}\}$ and $H_n \cap H_{n'} = \varnothing$ if $n \neq n'$. By Lemma 4.2, with $\langle y_n\rangle_{n=1}^{\infty}$ in place of $\langle x_n\rangle_{n=1}^{\infty}$, we can choose $a' \in A \cap B$ such that $a' = \sum_{i=1}^{\infty} y_{r_i}$ for some infinite injective sequence $\langle r_i\rangle_{i=1}^{\infty}$ in $\mathbb{N}$. This is a contradiction, because we then have $a' = \sum_{i=1}^{\infty} x_{n'_i}$, where $\langle n'_i\rangle$ is an injective sequence in $\mathbb{N}$ which contains infinitely many terms which are not in $\{n_i : i \in \mathbb{N}\}$.

By the definition of a sparse strongly summable ultrafilter, we can now choose a sequence $\langle u_n\rangle_{n=1}^{\infty}$ in $G$ and an infinite subsequence $\langle v_n\rangle_{n=1}^{\infty}$ of $\langle u_n\rangle_{n=1}^{\infty}$ such that $FS\langle v_n\rangle_{n=1}^{\infty} \in p$, $FS\langle u_n\rangle_{n=1}^{\infty} \subseteq FS\langle x_{n_i}\rangle_{i=1}^{\infty}$ and $M = \big\{n \in \mathbb{N}\colon u_n \notin \{v_r : r \in \mathbb{N}\}\big\}$ is infinite. We apply an argument similar to the one used in the last paragraph. For each $n \in \mathbb{N}$, there is a unique set $K_n \in \mathcal{P}_f(\mathbb{N})$ such that $u_n = \sum_{i\in K_n} x_{n_i}$ and $K_n \cap K_{n'} = \varnothing$ if $n \neq n'$. We can choose $a' \in A \cap B$ such that $a' = \sum_{i=1}^{\infty} v_{r_i}$ for some infinite injective sequence $\langle r_i\rangle_{i=1}^{\infty}$ in $\mathbb{N}$. This is a contradiction because we then have $a' = \sum_{i=1}^{\infty} x_{n'_i}$, where $\langle n'_i\rangle$ is an injective sequence in $\mathbb{N}$ disjoint from $\{n_i : i \in \bigcup_{n\in M} K_n\}$. □

*Remark.* The conclusion of the following theorem is valid in the case in which $G$ is a Boolean group and $p$ is any strongly summable ultrafilter on $G$ [9, Corollary 4.4]. Notice also that as a consequence of Theorem 2.6, if $p$ is a strongly summable ultrafilter on $G \subseteq \mathbb{T}$, then the maximal group with $p$ as identity is just a copy of $G$. This is known to hold for any strongly summable ultrafilter on $\mathbb{Z}$ by [3, Corollary 3.2].

**4.5. Theorem.** *Suppose that $G \subseteq \mathbb{T}$ and that $p$ is a sparse strongly summable ultrafilter on $G$. Let $x, y \in G^*$ satisfy $x + y = p$. Then $x, y \in G + p$.*

*Proof.* Suppose that $y$ converges to $c \in \mathbb{T}$. Let $H$ denote the subgroup of $\mathbb{T}$ generated by $G \cup \{c\}$. By Lemma 4.4, with $H$ in place of $G$, we have $-c + y = c + x = p$. This implies that $c \in G$, because otherwise $c + G$ and $G$ would be disjoint and would be members of $c + x$ and $p$ respectively. □

**4.6. Theorem.** *Suppose that $G \subseteq \mathbb{T}$ and that $p \in G^*$ is a srongly summable ultrafilter on $G$. Let $x, y \in G^*$ satisfy the equation $x + y = y + x = p$. Then $x$ and $y$ are in $G + p$.*

*Proof.* We assume that $(0, \frac{1}{2}) \in p$.

We first consider the case in which $x$ and $y$ converge to $0$.

Let $P \in p$. For each $i \in \{0, 1, 2\}$, let $X_i$ be defined as in the statement of Lemma 4.2, and let $j \in \{0, 1, 2\}$ be such that $X_j \in p$. We can choose $\langle x_n\rangle_{n=1}^{\infty} \subseteq G$ such that $FS\langle x_n\rangle_{n=1}^{\infty} \subseteq P \cap X_j$ and $FS\langle x_n\rangle_{n=1}^{\infty} \in p$.

If $B = \big\{\sum_{n=1}^{\infty} a_n x_n : \text{each } a_n \in \{0, 1\}\big\}$, then, by Lemma 4.2, $B \in x$ and $B \in y$. If $X \in x$ and $Y \in y$, we can choose $a \in X \cap B$ and $b \in Y \cap B$ such that $a + b \in FS\langle x_n\rangle_{n=1}^{\infty}$. By Lemma 3.2, this implies that $a, b \in FS\langle x_n\rangle_{n=1}^{\infty}$ and hence that $X \cap P \neq \varnothing$ and $Y \cap P \neq \varnothing$. So $x = y = p$.

In the general case, in which $x$ and $y$ do not necessarily converge to $0$, we can prove that $x, y \in G + p$ exactly as in Theorem 4.5. □

*Remark.* The results in this paper were heavily dependent on the groups considered being abelian. However, they have implications about the existence of idempotents with remarkable algebraic properties in many non-commutative groups. Suppose that $G$ is a countable group which can be algebraically embedded in a compact topological group $C$, and that $V$ denotes the subgroup of $\beta G$ which contains all the ultrafilters converging to the identity in $C$. There is then a bijection $\psi: \mathbb{N} \to G$ with the property that its continuous extension $\widetilde{\psi}: \beta\mathbb{N} \to \beta G$ defines an isomorphism from $\bigcap_{n\in\mathbb{N}} \overline{2^n\mathbb{N}}$ onto $V$ [5, Theorem 7.28]. It thus follows easily from the results in this paper that Martin's Axiom implies the following statement: any family of subsets of $G$ which has cardinality less than $\mathfrak{c}$ and is contained in an idempotent in $\beta G$, is also contained in an idempotent $p \in \beta G$ with the property that the equation $xy = p$ has only trivial solutions in $\beta G$. By this we mean that $xy = p$ implies that there exists $a \in G$ such that $x = pa^{-1}$ and $y = ap$. Thus the maximal group in $\beta G$ which contains $p$ is a copy of the subgroup $H = \{g \in G : gp = pg\}$ of $G$.

In the case in which $G$ is the free group on two generators, $a$ and $b$, Martin's Axiom implies that there is an idempotent in $\beta G$ whose maximal group is a singleton. This follows from the fact that there is a $G_\delta$ subset of $G^*$ which contains an idempotent and has the property that none of its elements commute with any element of $G$, except the identity. We shall give an outline of the proof that a set of this kind exists.

Let $S \subseteq G$ denote the free semigroup with generators $a$ and $b$, and let $L = \bigcap_{n\in\mathbb{N}} S^*a^n$ and $R = \bigcap_{n\in\mathbb{N}} b^nS^*$. Then $L$ is a left ideal in $S^*$ and $R$ is a right ideal in $S^*$, and so $L \cap R$ contains an idempotent in $S^*$ (by [5, Theorem 1.64]). We note that $L \cap R$ is a $G_\delta$ subset of $G^*$. It is not hard to verify that, for any $x \in L \cap R$ and any $g \in G$, $gx = xg$ implies that $g$ is the identity.

## REFERENCES


[1] A. Blass and N. Hindman, *On strongly summable and union ultrafilters*, Trans. Amer. Math. Soc. **304** (1987), 83–99.

[2] N. Hindman, *Summable ultrafilters and finite sums*, Logic and Combinatorics (S. Simpson, ed.), vol. 65, Contemporary Math., 1987, pp. 263–274.

[3] N. Hindman, *Strongly summable ultrafilters on* $\mathbb{N}$ *and small maximal subgroups of* $\beta\mathbb{N}$, Semigroup Forum **42** (1991), 63–75.

[4] N. Hindman and D. Strauss, *Nearly prime subsemigroups of* $\beta\mathbb{N}$, Semigroup Forum **51** (1995), 379–397.

[5] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification – theory and applications*, de Gruyter, Berlin, 1998.

[6] V. Malykhin, *Extremally disconnected and similar groups*, Soviet Math. Dokl. **16** (1975), 21–25.

[7] P. Matet, *Some filters of partitions*, J. Symbolic Logic **53** (1988), 540–553.

[8] I. Protasov, *Ultrafilters on abelian groups close to being Ramsey ultrafilters*, Matem. Studii **7** (1997), 133–138.

[9] I. Protasov, *Finite groups in* $\beta G$, Algebra and Discrete Math (to appear).

[10] S. Shelah, *Proper forcing*, Springer-Verlag, Berlin, 1982.



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