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ON INTEGRABLE THREE-BODY PROBLEMS ON THE LINE

A.Ya. Vus

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The natural systems of three pair-interacting particles on the line are investigated. The properties of interactive potentials are considered under assumption that the given system has the first integral which is a polynomial of prescribed degree in the momenta. The functional equations for those potentials are obtained. All such potentials for special functional classes are described.

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Исследуются натуральные системы трех попарно взаимодействующих частиц на прямой. Рассмотрены свойства потенциалов взаимодействия в предположении, что данная система обладает первым интегралом, полиномиальным по импульсам. Получены функциональные уравнения на потенциалы и описаны все их решения для некоторых функциональных классов.

The dynamics of n equal pair-interactive particles on the line is described by the Hamiltonian system with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i<j} V(x_i - x_j), \tag{1}$$

where the x_i and $p_i, i = 1,\dots,n$, are the coordinates and momenta of the particles. We henceforth call the function V a potential. We say that a potential V admits an integral F if F is the first integral of the Hamiltonian system (1). We call the first integral F to be nontrivial if F is functionally independent with H. Such systems were considered in [1],[2] and the complete integrability for special cases of the Weierstrass $\wp$ function as the interaction potential was established. It is known ([3]), that the Hamiltonian system (1) is completely integrable for $V(x)$ being the Weierstrass $\wp$ function. A distinguishing feature of this problem is the polynomial character in the momenta of their additional integrals. It is therefore natural to obtain a description of Hamiltonians (1) which admits integrals that are polynomials in the momenta. In the paper this problem will be considered for $n = 3$ and $V(x)$ satisfying tne following conditions:

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1) V is meromorphic in the vicinity of zero,
2) $V(x) = V(-x)$.

The total momentum $P = \sum p_i$ is the first integral of the Hamiltonian system (1). Therefore, this system can be reduced to the system with two degrees of freedom and the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + W(x_1, x_2), \tag{2}$$

where

$$W = V(x_1) + V(-\frac{x_1}{2} + \frac{x_2\sqrt{3}}{2}) + V(-\frac{x_1}{2} - \frac{x_2\sqrt{3}}{2}). \tag{3}$$

Without loss of generality we can assume that the first integral of system (2) has the form

$$F = F_{2N} + F_{2N-2} + \cdots + F_0, \tag{4}$$

$F_k = \sum_{i=1}^{k} E^{k-i,i}(x_1, x_2) p_1^{k-i} p_2^i$. Then the equation $\{F, H\} = 0$ can be written in the form

$$\begin{gathered} 0 = p_1 \partial_{x_1} F_{2N} + p_2 \partial_{x_2} F_{2N}, \\ \partial_{p_1} F_{2N} \cdot W_1 + \partial p_2 F_{2N} \cdot W_2 = p_1 \partial_{x_1} F_{2N-2} + p_2 \partial_{x_2} F_{2N-2}, \\ \dots\dots\dots\dots\dots \\ \partial_{p_1} F_2 \cdot W_1 + \partial p_2 F_2 \cdot W_2 = p_1 \partial_{x_1} F_0 + p_2 \partial_{x_2} F_0, \end{gathered} \tag{5}$$

where $\partial_t = \frac{\partial}{\partial t}$, $W_i = \frac{\partial W}{\partial x_i}$.

Let $R_k(x_1, x_2, p_1, p_2) = \sum f_i(x_1, x_2) p_1^{k-i} p_2^i$ be a homogeneous polynomial in the momenta. We shall put

$$[R_k] = R_k(x_1, x_2, \partial_{x_2}, -\partial_{x_1}) = \sum \partial_{x_2}^{k-i} (-\partial_{x_1})^i f_i(x_1, x_2).$$

Then equations (5) yield

$$P_i(V) = 0, \tag{6}$$

where $P_i(V) = [\partial_{p_1}(F_{2N+2-2i}) W_1] + [\partial_{p_2}(F_{2N+2-2i}) W_2], i = 1, \ldots, N$.

Each nontrivial relation (6) is called the *addition theorem.*

The following result holds:

Lemma 1. *The relation (6) is nontrivial for $i = 1$ or $i = 2$.*

The proof of Lemma 1 is based on the consideration of relation (6) for $V(x) = x^{-1}$. It is easy to show that if $\partial_{x_1} E^{0,2N} \neq 0$, then $P_1(V) \neq 0$. Otherewise, $P_2(V) \neq 0$.

We shall consider only potentials satisfying the following conditions:

a) zero is a pole of order 2 for V;
b) V is holomorphic in $\mathbb{R}\backslash\{0\}$; (7)
c) $\lim\limits_{x \to \infty} V(x) = 0$.

The description of integrable potentials $V(x)$ in the class of functions under consideration is given in Theorems 1,2.

Theorem 1. *Let the leading homogeneous component of F has constant coefficients. Then $V(x)$ is one of the Weierstrass $\wp$ function degenerated cases x^{-2}, $\sinh^{-2} kx$.*

The proof of Theorem 1 is based on three following lemmas.

Lemma 2. *The addition theorem is $P_2(V)=0$ and $P_2(V)=[L]\varrho(V)$, where L is a homogeneous polynomial in p_1,p_2 of degree $2N-3$ and*

$$\begin{aligned}\varrho(V) = V(x_1)V'\Big(-\frac{x_1}{2}+\frac{x_2\sqrt{3}}{2}\Big) - V'(x_1)V\Big(-\frac{x_1}{2}+\frac{x_2\sqrt{3}}{2}\Big)+\\ +V\Big(-\frac{x_1}{2}+\frac{x_2\sqrt{3}}{2}\Big)V'\Big(-\frac{x_1}{2}-\frac{x_2\sqrt{3}}{2}\Big) - V'\Big(-\frac{x_1}{2}+\frac{x_2\sqrt{3}}{2}\Big)V\Big(-\frac{x_1}{2}-\frac{x_2\sqrt{3}}{2}\Big)+\\ +V\Big(-\frac{x_1}{2}-\frac{x_2\sqrt{3}}{2}\Big)V'(x_1) - V'\Big(-\frac{x_1}{2}-\frac{x_2\sqrt{3}}{2}\Big)V(x_1).\end{aligned}$$

Proof. Consider the expression F_{2N}. Without loss of generality we can assume that the integral F is invariant with respect to the canonical transformation

$$x_i \to \widetilde{x}_i, \quad p_i \to \widetilde{p}_i,$$

where

$$\begin{aligned}\widetilde{x}_1 &= -x_1/2 + x_2\sqrt{3}/2, \quad \widetilde{x}_2 = -x_1\sqrt{3}/2 - x_2/2,\\ \widetilde{p}_1 &= -p_1/2 + p_2\sqrt{3}/2, \quad \widetilde{p}_2 = -p_1\sqrt{3}/2 - p_2/2.\end{aligned}$$

Thus we can represent F_{2N} in the form

$$F_{2N} = (p_1\widetilde{p}_1\widetilde{\widetilde{p}}_1)^2 \cdot G(T,J),$$

where $T = p_1^2 + p_2^2$, $J = p_2\widetilde{p}_2\widetilde{\widetilde{p}}_2$ and G is a polynomial in its variables. Then $P_1(V) \equiv 0$ and

$$F_{2N-2} = (p_1)^{-1}\frac{\partial F_{2N}}{\partial p_1}\cdot V(x_1) + (\widetilde{p}_1)^{-1}\frac{\partial F_{2N}}{\partial \widetilde{p}_1}\cdot V(\widetilde{x}_1) + (\widetilde{\widetilde{p}}_1)^{-1}\frac{\partial F_{2N}}{\partial \widetilde{\widetilde{p}}_1}\cdot V(\widetilde{\widetilde{x}}_1) + \omega, \quad (8)$$

where $[\omega]=0$.

Then we get

$$P_2(V) = \Big[(3JG - (p_1\widetilde{p}_1\widetilde{\widetilde{p}}_1)^2\frac{\partial G}{\partial J}p_1\widetilde{p}_1\widetilde{\widetilde{p}}_1\Big]\varrho(V).$$

The expression $L = \big(3JG - (p_1\widetilde{p}_1\widetilde{\widetilde{p}}_1)^2\partial G/\partial J\big)p_1\widetilde{p}_1\widetilde{\widetilde{p}}_1$ is a homogeneous polynomial in p_1,p_2.

Lemma 3. (The uniqueness theorem). *Let* $[L]f = 0$, *where*

$$L = ap_1^2 + bp_1p_2 + cp_2^2, \quad a^2 + b^2 + c^2 \neq 0,$$

f *is a holomorphic function in* $\mathbb{R}^2$ *and* $\lim\limits_{x^2+y^2\to\infty} f(x,y) = 0$. *Then* $f \equiv 0$ *in* $\mathbb{R}^2$.

If $b^2 - 4ac < 0$, then this lemma is a generalization of the Liouville uniqueness theorem for the harmonic functions in $\mathbb{R}^2$.

If $b^2 - 4ac \geq 0$, then L can be decomposed into the product of two monomials. The equation $a\partial_x f + b\partial_y f = 0$ yields $f(x,y) = g(bx - ay)$. Using the asymptotical condition we get $f \equiv 0$.

Since every homogeneous polynomial L in p_1, p_2 can be represented as a product of polynomials of degree 1 or 2, we receive from Lemmas 1,2 the equation $\varrho(V) = 0$.

Lemma 4. *The identity* $\varrho(V) = 0$ *is the addition theorem for the integral of degree 3 and only the Weierstrass* $\wp$ *function satisfies it.*

Proof. Consider the Laurent expansion at $x_1 = 0$ of the expression $\varrho(V)$. Its first nontrivial coefficient can be written in the form

$$V'''(t) - 12V(t)V'(t) = 0, \tag{9}$$

where $t = x_2\sqrt{3}/2$. It is known that the solution of equation (9) is either the Weierstrass $\wp$ function or one of its degenerated cases.

Theorem 2. *Let the leading homogeneous component of* F *has nonconstant coefficients and* $\partial_{x_1} E^{0,2N} \neq 0$. *Then* $V(x) = x^{-2}$.

Proof. It is easy to show that under conditions of Theorem 2 the relation $P_1(V) = 0$ is nontrivial. Equations (5) yield

$$F_{2N-2} = (p_1)^{-1}\frac{\partial F_{2N}}{\partial p_1}\cdot V(x_1) + (p_1)^{-2}\frac{\partial F_{2N}}{\partial x_1}\cdot U(x_1) + (\widetilde{p}_1)^{-1}\frac{\partial F_{2N}}{\partial \widetilde{p}_1}\cdot V(\widetilde{x}_1) +$$
$$+ (\widetilde{p}_1)^{-2}\frac{\partial F_{2N}}{\partial \widetilde{x}_1}\cdot U(\widetilde{x}_1) + (\widetilde{\widetilde{p}}_1)^{-1}\frac{\partial F_{2N}}{\partial \widetilde{\widetilde{p}}_1}\cdot V(\widetilde{\widetilde{x}}_1) + (\widetilde{\widetilde{p}}_1)^{-2}\frac{\partial F_{2N}}{\partial \widetilde{\widetilde{x}}_1}\cdot U(\widetilde{\widetilde{x}}_1) + \omega,$$

where $V(t) = \partial_t U(t)$, $[\omega] = 0$.

Then the solution of the addition theorem $P_1 = 0$ can be written in the form

$$V(x) = \partial_x\big(Q_{2N+4}(x)/(\partial_x E^{0,2N}(x))\big), \tag{10}$$

where Q_{2N+4} is a polynomial of degree $2N+4$ in x.

Lemma 5. *Consider potential (10) as* $V(z), z \in \mathbb{C}$. *If* $a \neq 0$ *is a pole for* $V(z)$, *then* $2a$ *is also a pole.*

The proof of Lemma 5 is based on consideration of the Laurent expansion (at $x_1 = a$) of the coefficients $E^{k-i,i}$. Then one can show that $z = a$ is a pole of order 2 for $V(z)$. Let

$$B(x_2) = V\Big(\frac{x_2\sqrt{3}}{2} + \frac{a}{2}\Big) - V\Big(\frac{x_2\sqrt{3}}{2} - \frac{a}{2}\Big).$$

Consider the order of the pole at $x_1 = a$ of the coefficients $E^{2N-4-i,i}$. The expression $B(x_2)$ can be represented as $B(x_2) = g(x_2)^{-3/(2m)}$, where $g(x_2)$ is a polynomial of degree $k \leq 2m$.

Since $B(x_2)$ is a rational function, $B(x_2) = P_3^{-1}(x_2)$, where P_3 is a nontrivial polynomial of degree 3. Let $z = \pm a$ be the poles and $z = \pm 2a$ the regular points for $V(z)$. Then $B(x_2)$ has two poles of degree 3 at $x_2 = a\sqrt{3}$. But the expression $P_3^{-1}(x_2)$ cannot have more then 3 poles in $\mathbb{C}$. This contradiction proves Lemma 5.

It follows from the previous lemma that potential (10) cannot have a pole $a \neq 0$. Therefore, $V(x) = x^{-2} + Q(x)$, where $Q(x)$ is a polynomial in x. In the class of potentials under consideration, $V(x) = x^{-2}$.

Let the potential V satisfy only conditions (7a,7b). Then the following result is established:

Theorem 3. *Let the potential $V(x) = x^{-2} + Q(x)$ admit integral (4). Then* $\deg Q(x) \leq 4$.

Lemma 6. *Let the potential $V(x) = x^{-2} + Q(x)$ admit an integral F of degree $2N$ and $\deg Q(x) = k$. Then the potential x^k admits a nontrivial integral, which is polynomial of degree $d \leq 2N$.*

The proof of Lemma 6 is based on using of the canonical transformation

$$x_1 = u_1 \cdot \varepsilon,\ x_2 = u_2 \cdot \varepsilon, \quad p_1 = q_1 \cdot \varepsilon^{-1},\ p_2 = q_2 \cdot \varepsilon^{-1}.$$

Then H and F are holomorphic functions of the parameter ε. Let $H_0 = H|_{\varepsilon=0}$, $F_0 = F|_{\varepsilon=0}$. Then $\{H_0, F_0\} = 0$, where H_0 is a Hamiltonian for the system with $V(x) = x^k$ and F_0 is a nontrivial polynomial in p_1, p_2.

For proof of Theorem 3 we use Yoshida's theorem [4] on the nonintegrability of natural systems with the Hamiltonian (2), where W is the homogeneous polynomial in x_1, x_2 of degree k.

Following Yoshida's method, consider the system

$$\begin{cases} \partial_{x_1} W = x_1, \\ \partial_{x_2} W = x_2, \end{cases} \tag{11}$$

where

$$W = x_1^k + \widetilde{x}_1^k + \widetilde{\widetilde{x}}_1^k.$$

Its solution is

$$\begin{cases} x_1 = \big(2/(k(1+2^{k-1}))\big)^{1/(k-2)}, \\ x_2 = x_1\sqrt{3}. \end{cases}$$

For these x_1, x_2 the eigenvalues λ_1, λ_2 of the matrix $\left(\begin{smallmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{smallmatrix}\right)$ are

$$\lambda_1 = k - 1, \quad \lambda_2 = 3(k-1)/(1+2^{k-1}).$$

Calculating Kovalevska's indicators

$$\varrho_i = \sqrt{1 + 8k\lambda_i/(k-2)^2}$$

we get $\varrho_1 = (3k-2)/(k-2) \in \mathbb{Q}$ and

$$\varrho_2 = \sqrt{1 + \frac{24k(k-1)}{(k-2)^2(1+2^{k-1})}}.$$

If $\varrho_2 \notin \mathbb{Q}$, then it follows from Yoshida's theorem that the Hamiltonian system with $V(x) = x^k$ is nonintegrable.

Proposition. *If* (k, m) *is a solution of the Diophantine equation*

$$(k-2)^2(1+2^{k-1})^2 + 24k(k-1)(1+2^{k-1}) = m^2, \tag{12}$$

then $k \leq 4$.

Proof. For $k = 6, 8$ or 10 we can directly calculate m. Consider the case $k > 10$. Let (k, m) be a solution of (12). One can show that

$$m = 2(l-1)(1+2^{2l-1}) + 12s,$$

where $k = 2l$ and s is an integer number.Then s satisfies the Diophantine equation

$$(l(2l-1) - s(l-1))(1+2^{2l-1}) = 3s^2.$$

But for $l \geq 6$ its solution must satisfy the inequalities $2l+1 < s < 2l+2$.

The last proposition completes the proof of Theorem 3.

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Department of Mechanics and Mathematics, Lviv University

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