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PARABOLIC EQUATION ON THE RIEMANN MANIFOLD

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The article describes an approach to solution of partial derivative parabolic equation that generalizes the known parametrix method. The iteration technique proposed exhibits faster convergence than the classical approach.

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В статье описан подход к решению уравнения в частных производных параболического типа, который обобщает известный параметрический метод. Предложенная итерационная техника дает более быструю сходимость чем классический подход.

1. Problem Statement

Many problems relative to the control theory require solving of Cauchy problem for n-dimensional parabolic type equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} g^{jk}(x) \frac{\partial^2 u}{\partial x^j \partial x^k} + b^k(x) \frac{\partial u}{\partial x^k} \tag{1}$$

with initial condition $u(0,x) = \varphi(x)$. Here $\mathbf{x} = [x^1, \dots, x^n]^T \in \mathbb{R}^n$, $g^{jk}(x)$ and $b^k(x)$ are coefficients of diffusion and transfer, respectively. Right hand side of (1) contains summing operation on indexes j and k (symbol of sum is skipped as it is usually done in tensor algebra). Generally, the solution of Cauchy problem has the form

$$u(t,x) = \int_{\mathbb{R}^n} \varphi(y) p(t,x,y)\, dy,$$

where $p(t,x,y)$ is a fundamental solution (here the transfer kernel) for equation (1).

The standard analytic method to find a fundamental solution for (1) is the so called iteration parametrix method [1], where initial approximation is chosen in a form of the function

$$(2nt)^{-\frac{n}{2}} \exp\Big\{ -\frac{1}{2t} g_{jk}(x)(y^j - x^j)(y_l - x_l) \Big\}, \tag{2}$$

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and g_{jk} is the inverse diffusion matrix. The drawback of this traditional method is that discrepancy of equation (1) with usage of function (2) represents special case with respect to variable t. Due to this fact the rate of convergence is very slow.

A different initial approximation is proposed in this work for the parametrix method. Due to this approach, the equation discrepancy does not contain any special case. The respective known result [2] can be used for studying asymptotic behavior of the fundamental solution

$$\lim_{t\to 0} t\ln p(t,x,y) = -\frac{\varrho^2(x,y)}{2}, \tag{3}$$

where ϱ is a Riemann distance in $\mathbb{R}^n$ determined by

$$\varrho^2(x,y) = \min \int_0^t g_{jk}(s)\dot\gamma^j(s)\dot\gamma^k(s)\,ds. \tag{4}$$

Here minimum is found with respect to all possible curves connecting the points $x, y \in \mathbb{R}^n$. In other words it is natural to consider equation (1) on Riemann manifold M (in this case M is the initial space $\mathbb{R}^n$ with the new metric (4)).

2. Notations and conditions

Let M be a Riemann manifold. The geodesic connecting points x and y on M (i.e. solution of variation problem (4)) will be denoted as $\gamma(s)$, where s is a natural parameter; $\gamma(0) = y$, $\gamma(\varrho(x,y)) = x$. The matrix $g_{jk}(x)$ is the inverse to diffusion matrix and generates the metrics tensor of manifold that is used to determine:

— connection coefficients

$$\Gamma^k_{ij} = \frac{1}{2}g^{kr}\Big(\frac{\partial g_{jr}}{\partial x^i} + \frac{\partial g_{ir}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^r}\Big),$$

— curvature tensor

$$R_{ijkl} = \frac{1}{2}\Big(\frac{\partial^2 g_{jk}}{\partial x^i\partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i\partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j\partial x^i} + \frac{\partial^2 g_{il}}{\partial x^j\partial x^k}\Big) + g_{pq}\Big(\Gamma^p_{jk}\Gamma^q_{jl} - \Gamma^p_{ik}\Gamma^q_{jl}\Big).$$

The sectional curvature of manifold in direction u, v is defined as $r = -R_{ijkl}u^iv^ju^kv^l$ up to positive multiplier.

It is supposed that the following condition is fulfilled everywhere.

Condition. Let the manifold be complete and simple connected, and its sectional curvature is nonpositive at each point and decreases fast enough with $||x|| \to \infty$ (see details in [4]).

If the condition holds, then for each pair of points $x, y \in M$ a nonnegative function $a(x,y)$ can be introduced that is defined in terms of Jacobi fields [4] and is dependent on the manifold curvature. Let

$$\varphi(x,y) = \int_0^{\varrho(x,y)} \frac{a(\gamma(s),y)}{s}\,ds.$$

3. Results

Consider again equation (1) supposing that the transfer coefficients are defined in a special way as follows:

$$b^k(x) = \frac{1}{2}\Gamma^j_{jr} g^{rk}(x) + \frac{1}{2}\frac{\partial g^{jk}}{\partial x^j}.$$

Then equation (1) takes the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u, \tag{5}$$

where Δ is the Laplace-Beltrami operator; in this case $p(t,x,y) = p(t,y,x)$.

Introduce the function

$$m(t,x,y) = (2\pi t)^{-\frac{n}{2}} \exp\Big\{-\frac{\varrho^2(x,y)}{2} - \frac{\varphi(x,y)}{2}\Big\}.$$

Lemma. *If the condition given above is fulfilled for innovation*

$$h(t,x,y) = \frac{1}{2}\Delta m - \frac{\partial m}{\partial t},$$

then the following inequality holds:

$$|h(t,x,y)| < cm(t,x,y), \quad t > 0, \quad x,y \in M.$$

The proof is indirect computing of innovation taking into consideration that derivatives of the function $\varphi(x,y)$ are defined in rather sophisticated way via Jacobi fields.

Write a fundamental solution of (5) as

$$p(t,x,y) = m(t,x,y) + \int_0^t d\tau \int_M m(t-\tau,x,z) r(\tau,z,y)\, dz, \tag{6}$$

where the required function r satisfies the integral Volterra equation

$$r(t,x,y) = h(t,x,y) + \int_0^t d\tau \int_M h(t-\tau,x,z) r(\tau,z,y)\, dz. \tag{7}$$

Theorem. *In conditions of the Lemma equation (6) has a unique solution satisfying the estimate*

$$|r(t,x,y)| < c \exp ct \exp\Big\{-\frac{\varrho^2(x,y)}{2t}\Big\},$$

where c is a constant.

Proof. Using ordinary iteration technique to solve (7) we get

$$r(t,x,y) = h(t,x,y) + \sum_{n=1}^{\infty} r_n(t,x,y), \tag{8}$$

where

$$r_n(t,x,y) = \int_0^t \int_M h(t-\tau,x,z) r_{n-1}(\tau,z,y)\,dz.$$

It is easy to establish the estimate

$$|r_n(t,x,y)| < \frac{c^n t^n}{n!} \exp\Big\{-\frac{\varrho^2(x,y)}{2t}\Big\}$$

from which the theorem follows.

Note. According to the classic parametrix method for r_n the asymptotic behavior with respect to n of the form $t^{\frac{n}{2}}(\Gamma(\frac{n}{2}))^{-1}$ holds, where $\gamma(x)$ is the gamma-function, i.e. the series (8) exhibits slow convergence proposed in this work is provided via approprite selection of initial approach $m(t,x,y)$ and small (not containing singularities with respect to t) innovation $h(t,x,y)$.

Consequence. *Representation (6) for fundamental solution takes place together with*

$$\Big|\int_0^t \int_M h(t-\tau,x,z) r(\tau,z,y)\,dz\Big| < ct\exp(ct)\exp\Big\{-\frac{\varrho^2(x,y)}{2t}\Big\}.$$

Example. Consider the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\sigma(x)\sigma'(x)\frac{\partial u}{\partial x}.$$

Its fundamental solution is

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} \exp\Big\{-\frac{1}{2t}\Big(\int_x^y \frac{dz}{\sigma(z)}\Big)^2\Big\}.$$

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