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# NONSTANDARD STURM-LIOUVILLE DIFFERENCE OPERATOR

V.E. Lyantse, Yu.M. Yavorsky

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A nonstandard ordinary singular Sturm-Liouville operator $L$ in finite differences is considered. The expansion in principal functions of $L$ is described. The results are similar to the ones for the differential operator of Naimark, though $L$ is an operator in (hyper) finite space.

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Рассмотрен нестандартный обычный сингулярный конечно разностный оператор Штурма-Лиувилля. Описано разложение $L$ по основных функциях. Результаты аналогичны результатам для дифференциального оператора Неймарка, хотя $L$ — оператор в (гипер) конечном пространстве.

**1. Preliminaries.** Our point of view in this paper is that of Internal Set Theory ([9]). Thus we consider that the collections ${}^{st}\mathbb{N}$ and $\mathbb{N}\setminus{}^{st}\mathbb{N}$ of standard and nonstandard natural numbers respectively are not empty and each $m\in\mathbb{N}\setminus{}^{st}\mathbb{N}$ is infinite in the sense that $\forall p\in{}^{st}\mathbb{N}\ p<m$. Below we investigate an operator $L$ with $\operatorname{rank} L\in\mathbb{N}\setminus{}^{st}\mathbb{N}$. We find that majority of eigenvalues $\lambda$ of $L$ are infinitely close with each other. The standard version of this phenomenon is the fact that the corresponding standard analogue in $\ell_2(\mathbb{N})$ (or $L_2(\mathbb{R}_+)$) has a continuous spectrum. Moreover, eigenfunctions of the above mentioned eigenvalues are not square summable. (Recall that eigenfunctions of the continuous spectrum of a standard differential operator are not square integrable.) We give a condition for minority of eigenvalues to have an S-compact eigenprojector. The standard version here is the finite (in the standard sense) algebraic multiplicity of eigenvalues of the corresponding standard operator. The standard version of our function $e(\cdot)$ is the Jost function. Its roots which are infinitely close to continuous spectrum are said to be the *spectral singularities* of $L$. It turns out that similarly to the standard case the quantity of authentic eigenvalues and spectral singularities is finite (in the standard sense). We suspect that the eigenprojector of a spectral singularity has an infinite norm, but we cannot prove this.

Now let us pass to details.

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Let $L$ be the operator generated by the difference expression $\ell$:

$$\ell x(t) = -\frac{1}{2}\big[x(t-1) + x(t+1)\big] + a(t)x(t), \quad t \in \mathbb{Z}, \tag{1.1}$$

with the boundary conditions

$$x(0) = 0, \quad x(m+1) = 0; \tag{1.2}$$

$a \in \mathbb{C}^{\mathbb{Z}}$, $m \in \mathbb{N}$ are considered as given.

Let

$$T = T_m := \{t \in \mathbb{Z} : \ 1 \le t \le m\}. \tag{1.3}$$

By definition, $L$ is a map $\mathbb{C}^T \to \mathbb{C}^T$ such that $\forall x \in \mathbb{C}^T \ \ \forall t \in T$

$$Lx(t) = \begin{cases} -\frac{1}{2}x(2) + a(1)x(1) & \text{for } \ t = 1, \\ \ell x(t) & \text{for } \ 1 < t < m, \\ -\frac{1}{2}x(m-1) + a(m)x(m) & \text{for } \ t = m. \end{cases} \tag{1.4}$$

We exit outside the scope of ordinary linear algebra by the simple conjecture: $m \in \mathbb{N} \setminus {}^{st}\mathbb{N}$, that is the natural number $m$ is *nonstandard*. Therefore, the space $\mathbb{C}^T$ and the operator $L$ are nonstandard. Since "discrete" interval $T$ is unlimited, the operator $L$ should be considered as *singular*. It is also *nonselfadjoint*, because the values $a(t)$ of "potential" are complex numbers.

M.A. Naimark was the first who has constructed the expansion in eigenfunctions of a singular nonselfadjoint Sturm-Liouville operator. His famous paper [1] initiated many further investigations (for instance, [2–8]). The principal interest has been caused by so called spectral singularities (denomination of J. Schwartz) associated with the operator of Naimark. For spectral singularities this operator is not spectral in the sense of Dunford–Bade, and its spectral function is not a measure but some distribution (generalized function, see [2,3,5,6]). It will be seen that our operator $L$ and Naimark's operator posess many similar properties (see also [15]). But we hope that the spectral theory of $L$ is more elementary for $\operatorname{rang} L \in \mathbb{N}$.

In what follows we use Nelson's IST (see, for example, one of the papers [9–14]). We explain that for $\tau \in \mathbb{R}$ $\ (\tau \approx 0) \Leftrightarrow (\forall^{st} n \in \mathbb{N})(|\tau| < \frac{1}{n})$, $(|\tau| \ll \infty) \Leftrightarrow (\exists^{st} n \in \mathbb{N})(|\tau| < n)$, $(\tau \gg 0) \Leftrightarrow (\exists^{st} n \in \mathbb{N})(\tau > \frac{1}{n})$, where "$\exists^{st}$" means "there exists a standard". Let $X$, $Y$ be (internal) sets. Then "$f \in Y^X$" denotes "$f$ is an (internal) function such that $\operatorname{dom} f = X$, $im\, f \subseteq Y$". For any (internal or external) set $E$, ${}^{st}E := \{x \in E : \ x \text{ is standard}\}$.

Let $s(\cdot, \lambda)$ be a solution of the equation $ls = \lambda s$ with initial values

$$s(0, \lambda) = 0, \quad s(1, \lambda) = 1. \tag{1.5}$$

Obviously, eigenvalues of $L$ are $\lambda$–roots of the equation $s(m+1, \lambda) = 0$. To describe them we use three complex planes: $\lambda$–plain, $\varrho$–plain, and $\widehat{\tau}$–plain, related with each other by

$$\varrho = e^{i\widehat{\tau}}, \quad \lambda = -\frac{1}{2}(\varrho + \varrho^{-1}) = -\cos\widehat{\tau}. \tag{1.6}$$

We note that values $a(t)$, which correspond to $t \in \mathbb{Z} \setminus T$, are indifferent for $L$. For definiteness we assume that

$$\forall t \in \mathbb{Z} \setminus T \quad a(t) = 0. \tag{1.7}$$

**2. Potential-zero**. At beginning we assume that $\forall t \in \mathbb{Z}\ \ a(t) = 0$. In this case we write $\ell_0$ and $L_0$ respectively in place of $\ell$ and $L$. On account of (1.6) the equation $\ell_0 x = \lambda x$ takes the form

$$x(t-1) + x(t+1) = (\varrho + \varrho^{-1})x(t). \tag{2.1}$$

If $\varrho \neq \pm 1$, the general solution of (2.1) is as follows

$$x(t) = C_- \varrho^{-t} + C_+ \varrho^{+t}, \tag{2.2}$$

and for $\varrho = 1$ ($\varrho = -1$)

$$x(t) = C_0 + C_1 t \qquad (x(t) = (-1)^t (C_0 + C_1 t), \tag{2.3}$$

where $C_\pm$, $C_0$, $C_1$ are arbitrary complex constants. In particular, the solution $s_0$ defined by initial conditions (1.5), is (for $\varrho \neq \pm 1$, that is for $\widehat{\tau} \neq 0, \pi$, $\lambda \neq \pm 1$)

$$s_0(t, \lambda) = \frac{\varrho^t - \varrho^{-t}}{\varrho - \varrho^{-1}} = \frac{\sin \widehat{\tau} t}{\sin \widehat{\tau}}, \tag{2.4}$$

and

$$s_0(t, -1) = t, \quad s_0(t, +1) = (-1)^{t-1} t. \tag{2.5}$$

Since $s_0(m+1, \pm 1) \neq 0$, the eigenvalues of $L_0$ in $\varrho$-plain coincide with the roots of the following equation

$$\varrho^{2m+2} = 1, \quad \varrho \neq \pm 1. \tag{2.6}$$

This implies the following. Put

$$\widehat{h} := \frac{\pi}{m+1}, \quad \widehat{T} := \{\widehat{h}, 2\widehat{h}, \ldots, \pi - \widehat{h}\}; \tag{2.7}$$

note that $\pi - \widehat{h} = m\widehat{h}$. The "interval" $\widehat{T}$ may be considered as dual to $T = \{1, \ldots, m\}$ with respect to the operator $L_0$. Indeed, eigenvalues of $L_0$ in $\widehat{\tau}$–plane, $\varrho$-plain, and $\lambda$-plain are respectively

$$\hat{t} \in \widehat{T}, \quad \varrho_{\hat{t}} = e^{i\hat{t}}, \quad \lambda_{\hat{t}} = -\cos \hat{t}. \tag{2.8}$$

To the eigenvalue $\lambda_{\hat{t}}$ there corresponds the eigenfunction $s_0(\cdot, \lambda_{\hat{t}})$ (cf. (2.4)). Put

$$\forall \hat{t} \in \widehat{T}, \forall t \in T \ \ S_0(t, \hat{t}) := \sqrt{\frac{2}{m+1}} \sin \hat{t} t. \tag{2.9}$$

Let $H$ be the Hilbert space of functions $x \in \mathbb{C}^T$ with the inner product

$$\forall x, y \in \mathbb{C}^T \quad (x|y) = \sum_{t \in T} x(t)\overline{y(t)}. \tag{2.10}$$

**2.1. Proposition.** *The eigenfunctions $S_0(\cdot,\hat t)$, $\hat t \in \widehat T$ form some orthonormal basis of the space $H$. Therefore, for the "discrete"* sin*-transformation $x \mapsto \hat x$:*

$$\forall x \in H \ \forall \hat t \in \widehat T \ \hat x(\hat t) := \sqrt{\frac{2}{m+1}} \sum_{t\in T} x(t) \sin \hat t t \tag{2.11}$$

*we have the inverse–transform formula*

$$\forall t \in T \ x(t) = \sqrt{\frac{2}{m+1}} \sum_{\hat t \in \widehat T} \hat x(\hat t) \sin \hat t t. \tag{2.12}$$

*Proof.* The matrix of $L_0$ with respest to the natural basis of $H$ is Hermitian. So $L_0$ is a selfadjoint operator $H \to H$. On account of $\dim H = \operatorname{card} T = \operatorname{card} \widehat T = m \in \mathbb{N}$, $\big(S_0(\cdot,\hat t)\big)_{\hat t\in\widehat T}$ is an orthonormal basis of $H$. We have $\sum_{t\in T} \sin^2 \hat t t = \frac14(2m+1) - \frac14 e^{-2i\hat t m}(e^{2i\hat t}-1)^{-1}(e^{2(2m+1)\hat t}-1)$. Since $(2m+2)\hat t = 0 \ (\operatorname{mod} 2\pi)$, the latter summand is equal to $\frac14(2m+1) - \frac14 e^{2i\hat t}(e^{2i\hat t}-1)^{-1}(e^{-2i\hat t}-1)$, hence

$$\forall \hat t \in \widehat T \ \sum_{t\in T} \sin^2 \hat t t = \frac12 (m+1). \tag{2.13}$$

Therefore, $\|S_0(\cdot,\hat t)\| = 1$, where $\|\cdot\|$ is the norm corresponding to inner product (2.10). □

Where is here displayed the nearstandardness of the operator $L_0$? (Note that at this moment we are staying in the framework of the ordinary linear algebra.)

*2.2. Remark.* The eigenvalues $\lambda_{\hat t}$ in (2.8) are increasing:

$$-1 \approx \lambda_{\widehat h} < \lambda_{2\widehat h} < \cdots < \lambda_{(m-1)\widehat h} < \lambda_{m\widehat h} \approx +1. \tag{2.14}$$

But $\lambda_{\hat t+\widehat h} - \lambda_{\hat t} = 2\sin(\hat t + \frac{\widehat h}{2}) \sin\frac{\widehat h}{2} \le \widehat h \approx 0$, therefore,

$$\forall n \in \mathbb{N} \ \forall \hat t \in \widehat T \ (n \ll \infty \implies \lambda_{\hat t + n\widehat h} \approx \lambda_{\hat t}). \tag{2.15}$$

We formulate (2.15) by the phrase "The spectrum of $L_0$ is purely continuous on $[-1,+1]$". Note that in the $\varrho$-plain the values $\varrho_{\hat t}$ are uniformly distributed on the circle $|\varrho| = 1$, where $\varrho$ and $\varrho^{-1}$ are regarded as the same point.

**2.3. Definition.** A function $f \in \mathbb{C}^T$ is said to be *square–integrable* iff
1° $\|f\|^2 := \sum_{t\in T} |f(t)|^2 \ll \infty$, and
2° $(\forall t \approx \infty)\ \sum_{u>t} |f(u)|^2 \approx 0$.

There is one more argument for the assertion that the spectrum of $L_0$ is "continuous".

**2.4. Proposition.** *Normed eigenfunctions $S_0(\cdot,\hat t)$, $\hat t \in \widehat T$, are not square–integrable.*

*Proof.* We have $\sum_{u=t+1}^m |S_0(u,\hat t)|^2 = 1 - 2/(m+1)\sum_{u=1}^t \sin^2 \hat t t \ge 1 - (2t)/(m+1)$. This is $\gg 0$ for $t = [\frac14(m+1)] \approx \infty$. □

**3. Integrable potential**. We consider the operator $L_0$ as unperturbated and assume that a perturbation by the potential $a$ is nice in some sense.

**3.1. Conjecture.** *A function $a$ is integrable on $T$, that is*
1° $\|a\|_1 := \sum_{t\in T} |a(t)| \ll \infty$, *and*
2° $(\forall t \approx \infty)\ \sum_{u>t} |a(u)| \approx 0$.

We will show that the eigenvalues of $L_0$ change only a little if $\|a\|_1$ is sufficiently small. Let

$$\forall t \in T \;\; Z_0(t,\varrho) = \varrho^t - \varrho^{-t}; \tag{3.1}$$

$Z_0$ is a solution of the equation $(\ell_0 - \lambda)x = 0$ with $x(0) = 0$. Let $f \in \mathbb{C}^T$. By the variation of arbitrary constants we obtain a solution $x$ of $(\ell_0 - \lambda)x = f$ with $x(0) = x(1) = 0$ in the form

$$x(t) = -\frac{2}{\varrho - \varrho^{-1}} \sum_{u=1}^{t-1} f(u) Z_0(t-u,\varrho); \tag{3.2}$$

here and in what follows we admit that any $f \in \mathbb{C}^T$ is extended by zero to $\mathbb{Z}\backslash T$. This implies that a solution $Z(\cdot,\varrho)$ of the equation $(\ell - \lambda)Z = 0$ with initial conditions

$$Z(0,\varrho) = 0, \quad Z(1,\varrho) = \varrho - \varrho^{-1} \tag{3.3}$$

(the same as for $Z_0(\cdot,\varrho)$) satisfies the "integral" equation

$$\forall t \in T \;\; Z(t,\varrho) = \frac{2}{\varrho - \varrho^{-1}} \sum_{u=1}^{t-1} a(u) Z(u,\varrho) Z_0(t-u,\varrho) + Z_0(t,\varrho). \tag{3.4}$$

In order to prove it rewrite $(\ell - \lambda)Z = 0$ as $(\ell_0 - \lambda)Z = -a(\cdot)Z$ and change $f(u)$ in (3.2) by $-a(u)Z(u,\varrho)$.

**3.2. Proposition.** *Let* $|\varrho| \ge 1$, *then*

$$\forall t \in T \quad |Z(t,\varrho)| \le C_{\widehat{\tau}} |\varrho|^t, \tag{3.5}$$

*where*

$$C_{\widehat{\tau}} := 2 \exp \frac{2\|a\|_1}{\sin \widehat{\tau}}, \quad \varrho = e^{i\widehat{\tau}}. \tag{3.6}$$

*Proof.* Since (3.1) implies that for $|\varrho| \ge 1$

$$\forall t \in T \quad |Z_0(t,\varrho)| \le 2|\varrho|^t, \tag{3.7}$$

we can conclude from (3.4) that

$$|Z(t,\varrho)| \le \frac{1}{\sin \widehat{\tau}} \sum_{u=1}^{t-1} |a(u)|\, |Z(u,\varrho)|\, 2|\varrho|^{t-u} + 2|\varrho|^t.$$

Define $S(t)$ by

$$\forall t \in T \quad S(t) = \frac{2}{\sin \widehat{\tau}} \sum_{u=1}^{t-1} |a(u)| S(u) + 1. \tag{3.8}$$

By the induction we get

$$\forall t \in T \quad |Z(t,\varrho)| \le 2|\varrho|^t S(t). \tag{3.9}$$

But $S(t+1) - S(t) = \frac{2}{\sin \widehat{\tau}} |a(t)| S(t)$. Therefore, $S(t+1) \le S(t) \exp \frac{2a(t)}{\sin \widehat{\tau}}$. From (3.8) we see that $S(1) = 1$, hence $S(t) \le \exp \frac{2}{\sin \widehat{\tau}} \sum_{u=1}^{t-1} |a(u)|$. Now, by virtue of (3.9), we have

$$\forall t \in T \;\; |Z(t,\varrho)| \le 2|\varrho|^t \exp \frac{2}{\sin \widehat{\tau}} \sum_{u=1}^{t-1} |a(u)| \tag{3.10}$$

□

**3.3. Corollary.** *Let* $|\varrho| \geq 1$*, then* $\forall t \in T$

$$|Z(t,\varrho) - Z_0(t,\varrho)| \leq \frac{2C_{\widehat{\tau}}|\varrho|^t}{\sin\widehat{\tau}} \sum_{u=1}^{t-1} |a(u)| \leq \frac{2C_{\widehat{\tau}}\|a\|_1}{\sin\widehat{\tau}}|\varrho|^t. \tag{3.11}$$

This follows from (3.4), (3.5), and (3.7).

*3.4. Remark.* We will obtain valid inequalities if we change $|\varrho|^t$ in (3.5), (3.7), and (3.11) by $|\varrho|^{-t}$. Indeed, obviously, $Z(t,\varrho^{-1}) = Z(t,\varrho)$.

**3.5. Theorem.** *Let* $\hat{t} \in \widehat{T}$ *be such that* $0 \ll \hat{t} \ll \pi$*. Assume that*

$$\|a\|_1 \leq 0,1 \sin\hat{t}. \tag{3.12}$$

*Then there exists exactly one eigenvalue* $\lambda_{\widehat{\tau}_{\hat{t}}} = -\cos\widehat{\tau}$ *of the operator* $L$ *such that* $\widehat{\tau}_{\hat{t}}$ *is contained in the infinitely small rectangle* $\Gamma_{\hat{t}}$ *(with center* $\hat{t}$*), formed by the lines*

$$\operatorname{Re}\widehat{\tau} = \hat{t} \pm \frac{\pi}{2(m+1)}, \quad \operatorname{Im}\widehat{\tau} = \pm\frac{0,9}{m+1}. \tag{3.13}$$

In particular, this is so for all $\hat{t} \in \widehat{T}$, $0 \leq \hat{t} \ll \pi$, if $\|a\|_1 \approx 0$.

*Proof.* For $Z(0,\varrho) = 0$, the eigenvalues of $L$ in $\varrho$-plain are the $\varrho$–roots of $Z(m+1,\varrho) = 0$. If $\widehat{\tau} \in \Gamma_{\hat{t}}$, then the point $z_{\widehat{\tau}} := (m+1)(\widehat{\tau} - \hat{t})$ belongs to the rectangle $\Gamma$ formed by lines $\operatorname{Re} z = \pm\frac{\pi}{2}$, $\operatorname{Im} z = \pm 0,9$. Since $|\sin z|^2 = \sinh^2(\operatorname{Im} z) + \sin^2(\operatorname{Re} z)$, $\sinh 0,9 > 1$, and $\sin\frac{\pi}{2} = 1$, we have

$$\forall z \in \Gamma \quad |\sin z| \geq 1 \tag{3.14}$$

Let $\widehat{\tau} \in \Gamma_{\hat{t}}$ and $\varrho_{\widehat{\tau}} = e^{i\widehat{\tau}}$. Then (see (3.1)) $Z_0(m+1,\varrho_{\widehat{\tau}}) = \varrho_{\widehat{\tau}}^{m+1} - \varrho_{\widehat{\tau}}^{-m-1} = (-1)^t 2i\sin z$, where $t := \widehat{\tau}/\widehat{h} \in T \subset \mathbb{N}$ and $z = (m+1)(\widehat{\tau} - \hat{t}) \in \Gamma$. Therefore, by (3.14),

$$\forall\widehat{\tau} \in \Gamma_{\hat{t}} \quad |Z_0(m+1,\varrho_{\widehat{\tau}})| \geq 2. \tag{3.15}$$

On the other hand, by inequality (3.11) and remark 3.4, $\forall\widehat{\tau} \in \Gamma_{\hat{t}}$

$$|Z(m+1,\varrho_{\widehat{\tau}}) - Z_0(m+1,\varrho_{\widehat{\tau}})| \leq \frac{2C_{\widehat{\tau}}\|a\|_1}{\sin\widehat{\tau}}|\varrho_{\widehat{\tau}}|^{\pm(m+1)}.$$

But on $\Gamma_{\hat{t}}$ $|\varrho_{\widehat{\tau}}|^{\pm(m+1)} = e^{\mp(m+1)\operatorname{Im}\widehat{\tau}} = e^{0,9}$. For $\forall\widehat{\tau} \in \Gamma_{\hat{t}}$ $\sin\widehat{\tau} \approx \sin\hat{t}$, by (3.6) and (3.12), we have $|\sin\widehat{\tau}|^{-1} C_{\widehat{\tau}}\|a\|_1 e^{0,9} < 1$. Therefore, $\forall\widehat{\tau} \in \Gamma_{\hat{t}}$ $|Z(m+1,\varrho_{\widehat{\tau}}) - Z_0(m+1,\varrho_{\widehat{\tau}})| < Z_0(m+1,\varrho_{\widehat{\tau}})$. Thus, by the Rouché theorem, the equation $Z(m+1,\varrho) = 0$ (as well as the equation $Z_0(m+1,\varrho) = 0$) has a unique solution in $\Gamma_{\hat{t}}$. □

**4. Relative standardness.** This notion depends on the choice of topology. For instance, it is a natural interrelation between $H = \mathbb{C}^T$ (see (2.10)) and the standard Hilbert space $\mathbf{H} = \ell_2(\mathbb{N})$ endowed with the inner product

$$\forall\xi,\eta \in \mathbf{H} \quad (\xi|\eta) := \sum_{t\in\mathbb{N}} \xi(t)\overline{\eta(t)}. \tag{4.1}$$

**4.1. Definition.** Let $x \in H$ and $\|x\| \ll \infty$ (we denote the norms generated by (2.10) and (4.1) by the same symbol $\|\cdot\|$). Let $\xi := (\xi(t))_{t\in\mathbb{N}}$ be the *standard extension* of $({}^\circ[x(t)])_{t\in{}^{st}\mathbb{N}}$. The function $x$ is said to be $\|\cdot\|$-standard and we write $x \in {}^{st}H$, whenever $\forall t \in T \;\; x(t) = \xi(t)$. Such $\xi$ is called the *standardized image* of $x$ and is denoted by ${}^\bullet x$.

*4.2. Remark.* Let $x \in {}^{st}H$, then
1° $\forall t \in {}^{st}\mathbb{N} \quad x(t) \in {}^{st}\mathbb{C}$,
2° $\|x\| \approx 0 \implies x = 0$,
3° ${}^\bullet x \in {}^{st}\mathbf{H}$ and $\|{}^\bullet x\| =^\circ \|x\|$.
Proof is easy.

**4.3. Definition.** A function $x \in H$ is said to be $\|\cdot\|$-*nearstandard* and we write $x \in {}^{nst}H$ iff $\exists y \in {}^{st}H \;\; \|x-y\| \approx 0$. Such $y$ (it is unique, for 4.2.2°) is said to be the *shadow* of $x$ and is denoted by ${}^\circ x$. The standardized image of $y =^\circ x$ is denoted by ${}^\bullet x$ and is called the *shadow* of $x$ on $\mathbf{H}$.

*4.4. Remark.* 1° ${}^{st}H \subset {}^{nst}H$;
2° $(\forall x \in H)(x \in {}^{nst}H \Leftrightarrow (\|x\| \ll \infty$ and $\forall t \approx \infty \;\; \sum_{u>t}^m |x(u)|^2 \approx 0)$;
3° $\forall x, y \in {}^{nst}H \;\; {}^\circ(x|y) = ({}^\bullet x|{}^\bullet y)$.
Proof is evident.

**4.5. Definition.** 1° Let $x \in H$. An extension of $x$ to $\mathbb{N}$ such that $\forall t \in \mathbb{N}\backslash T \; x(t) = 0$ is denoted by $Qx$. Obviously, $Q$ is an injection $H \to \mathbf{H}$; 2° Any $\xi \in \mathbf{H}$ *induces* its restriction $\Pi\xi$ on $T$. The map $\Pi$ is said to be the *inductor* $\mathbf{H} \to H$; 3° $\Pi$ denotes the orthoprojector $\mathbf{H} \to QH$.

**4.6. Proposition.** 1° *The injector $Q$ is isometric: $\forall x \in H \;\; \|Qx\| = \|x\|$;*
2° *The inductor $\Pi$ does not extend: $\forall \xi \in \mathbf{H} \; \|\Pi\xi\| \le \|\xi\|$;*
3° *We have $Q^* = \Pi$, $\Pi^* = Q$, $\Pi Q = 1_H$, $Q\Pi = P$.*

**4.7. Proposition.** 1° *The inductor $\Pi$ is exact in the following sense: $(\forall \xi \in {}^{st}\mathbf{H})$ $(\Pi\xi \approx 0 \implies \xi = 0)$; 2° The projector $P$ is a quasiunity in the following sense: $(\forall \xi \in {}^{nst}\mathbf{H})$ $(\|P\xi - \xi\| \approx 0)$; 3° Let $x \in H$, then $x \in {}^{st}H \Leftrightarrow (\exists \xi \in {}^{st}\mathbf{H})(x = \Pi\xi)$. In this case $\xi =^\bullet x$; 4° Let $x \in H$, then $x \in {}^{nst}H \Leftrightarrow (\exists \xi \in {}^{st}\mathbf{H})(\|x - \Pi\xi\| \approx 0)$ and in this case $\Pi\xi =^\circ x$.*

Proofs are omitted as they are easy.
Consider some examples.

**4.8. Proposition.** *Let $\varrho \in \mathbb{C}\backslash\{0\}$ and $\forall t \in T \; y(t) := \varrho^{-t}$. Set $x := \|y\|^{-1}y$. Then*

$$x \in {}^{nst}H \Longleftrightarrow |\varrho| \gg 1. \tag{4.2}$$

*If $1 \ll |\varrho| \ll \infty$, then*

$${}^\bullet x(t) = \sqrt{{}^\circ|\varrho|^2 - 1}\,({}^\circ\varrho)^{-t}, \;\; t \in \mathbb{N}. \tag{4.3}$$

*If $\varrho \approx \infty$, then*

$${}^\bullet x(t) = \begin{cases} {}^\circ(|\varrho|\varrho^{-1}) & \text{for } \; t = 1, \\ 0 & \text{for } \; t > 1. \end{cases} \tag{4.4}$$

*Proof.* Put

$$\mathfrak{X}_t := |x(t+1)|^2 + \cdots + |x(m)|^2. \tag{4.5}$$

Note that $x \in {}^{nst}H$ iff $\|x\| \ll \infty$ and $\forall t \approx \infty \; \mathfrak{X}_t \approx 0$. At the beginning, let $|\varrho| < 1$. Then $\mathfrak{X}_t = 1 - (|y(1)|^2 + \cdots + |y(t)|^2)(|y(1)|^2 + \cdots + |y(m)|^2)^{-1} = 1 - |\varrho|^{2(m-t)}(1-$

$|\varrho|^{2t})(1-|\varrho|^{2m})^{-1}$. If $|\varrho| \ll 1$, we take a $t \approx \infty$ such that $m-t \approx \infty$. Then we see that $\mathfrak{X}_t \approx 1$, hence $x \notin {}^{nst}H$. If $|\varrho| < 1$ and $|\varrho| \approx 1$, we find (by the Robinson lemma) a $t_0 \approx \infty$ such that $\forall t \le t_0$ $|\varrho|^t \approx 1$. Therefore, $\mathfrak{X} = 1-|\varrho|^{2(m-t)}(1+\cdots+|\varrho|^{2(t-1)})(1+\cdots+|\varrho|^{2(m-1)})^{-1} > 1-|\varrho|^{2(m-t)}t/(m|\varrho|^{2(m-1)}) = 1-t/(m|\varrho|^{2(t-1)})$. Hence if $t \approx \infty$ and $t < t_0$ but $2t > m$, we have $\mathfrak{X}_t \gg 0$ and once more $x \notin {}^{nst}H$. Let $|\varrho| = 1$, then $\mathfrak{X}_t = 1-t/m$, and as before $x \notin {}^{nst}H$. Let $|\varrho| > 1$, then

$$x(t) = C_\varrho \varrho^{-t}, \quad \text{where} \quad C_\varrho := (|\varrho|^2-1)^{1/2}(1-|\varrho|^{-2m})^{-1/2}. \tag{4.6}$$

If $1 \ll |\varrho| \ll \infty$, then $C_\varrho \ll \infty$. Since $\forall t \approx \infty$ $|\varrho|^{2(t+1)}+\cdots \approx 0$, we have $\forall t \approx \infty$ $\mathfrak{X}_t \approx 0$, hence $x \in {}^{nst}H$. Formula (4.3) is evident. If $|\varrho| \approx \infty$, then $(|\varrho|^2-1)^{-1/2}C_\varrho \approx 1$, $\varrho^{-t} \approx 0$, that implies (4.4).

At the end, let $|\varrho| > 1$ or $|\varrho| \approx 1$. Let $t_0 \approx 0$ be such that $t_0 \le m$ and $\forall t \le t_0$ $|\varrho|^{\pm t} \approx 1$. We have (see (4.5)) $\mathfrak{X}_t > 1-t|y(t)|^2[t_0|y(t_0)|^2]^{-1} = 1-tt_0^{-1}|\varrho|^{-2(t_0-1)}$. Hence $\mathfrak{X}_t \gg 0$, whenever $tt_0^{-1} \approx 0$. The latter is possible also for $t \approx \infty$, hence $x \notin {}^{nst}H$. □

Consider some case a little more complicated.

**4.9. Proposition.** *Let*

$$\forall t \in T \quad y(t) := C_+\varrho^t + C_-\varrho^{-t}, \quad x := \|y\|^{-1}y. \tag{4.7}$$

*Suppose that* $1 \le |\varrho| < \infty$ *and that*

$$|C_+| + |C_-| = 1. \tag{4.8}$$

*(We note that condition (4.8) has no influence on the value of* $\|x\|$*). It turns out that*

$$x \in {}^{nst}H \iff |\varrho| \gg 1 \quad \textit{and} \quad C_+[|\varrho|^2+\cdots+|\varrho|^{2m}] \approx 0. \tag{4.9}$$

*Proof.* Part ($\Leftarrow$) follows from 4.8. We note that $|\varrho| \gg 1$ and $C_+[|\varrho|^2+\cdots+|\varrho|^{2m}] \approx 0 \implies y \in {}^{nst}H$ and $(\forall t \in^{st} H)({}^\bullet y(t) = ({}^\circ C_-)({}^\circ\varrho)^{-t})$.

For ($\Rightarrow$) suppose that $|\varrho| \approx 1$. It is evident that $x \notin {}^{nst}H$, whenever $\varrho \approx \pm 1$. Admit that $\varrho \not\approx \pm 1$ and investigate the case $C_+ \ne 0$. In this case $y(t) = C_+\varrho^t[1-D\varrho^{-2t}]$, where $D := -C_-C_+^{-1}$. Put $S := \{t \in T : \ |1-De^{-2t}| \ge 1/2\}$. It is easy to see that

$$\nu := \operatorname{card} S \approx \infty. \tag{4.11}$$

Obviously,

$$\forall t \in T \ |y(t)| \le |\varrho|^t \ \text{ and } \ \forall t \in S \ |y(t)| \ge \frac{1}{2}|C_+||\varrho|^t. \tag{4.12}$$

Choose $t_0 \in T$ such that $t_0 \approx \infty$ and $\forall t \le t_0$ $|\varrho|^t \approx 1$. From (4.2) and (4.5) we see that $\mathfrak{X}_t \ge 1-(|\varrho|^2+\cdots+|\varrho|^{2t})\left(\sum_{t\in S}|y(t)|^2\right)^{-1}$. Hence if $t \le t_0$, we have $\mathfrak{X}_t \ge 1-2t(C_+\nu)^{-1}$. Therefore, we can find a $t \approx \infty$ such that $\mathfrak{X}_t \gg 0$ and so $x \notin {}^{nst}H$. Whenever $C_+ \approx 0$, we represent $y$ as $y(t) = C_-\varrho^{-t}(1-E\varrho^{2t})$. We note that $|C_-| \approx 1$ and $E := -C_+C_-^{-1} \approx 0$. Put $S := \{t \in T : \ |1-E\varrho^{2t}| \ge \frac{1}{2}\}$. As before, (4.11) is satisfied, and by the same reasoning $x \notin {}^{nst}H$. □

Let us introduce the relative standardness for operators. Let $X$ be a normed space and $\mathcal{B}(X)$ be the algebra of linear bounded operators $A \in X^X$. We claim that the inductor $\Pi$ is exact in the following sense:

**4.10. Proposition.** *Let* $\mathbf{A} \in {}^{st}\mathcal{B}(\mathbf{H})$*, then*

$$(\forall x \in {}^{st}H)\ \ (\Pi\mathbf{A}Qx \approx 0) \implies \mathbf{A} \approx 0. \tag{4.13}$$

*Proof.* Suppose that $\forall x \in^{st} H$ and $\Pi\mathbf{A}Qx \approx 0$. Let $\xi \in {}^{st}\mathbf{H}$ and $x := \Pi\xi$. Then $x \in {}^{st}H$, hence $\Pi\mathbf{A}Q\Pi\xi \approx 0$. But $Q\Pi\xi = P\xi \approx \xi$. Therefore, $\Pi\mathbf{A}\xi \approx 0$ and, by 4.7.1°, $\mathbf{A}\xi = 0$. Now, by transfer, $\mathbf{A} = 0$. □

**4.11. Definition.** 1° An operator $A \in \mathcal{B}(H)$ is said to be $\|\cdot\|$-*standard* and we write $A \in {}^{st}\mathcal{B}(H)$ iff $(\exists \mathbf{A} \in {}^{st}\mathcal{B}(\mathbf{H}))$ $(A = \Pi\mathbf{A}Q)$. Such $\mathbf{A}$ is called the *standardized image* of $A$ and is denoted by ${}^{\bullet}A$. (By 4.10, such $\mathbf{A}$ is unique.) 2° An operator $A \in \mathcal{B}(H)$ is said to be $\|\cdot\|$-*nearstandard* and we write $A \in {}^{nst}\mathcal{B}(H)$ iff $(\exists B \in {}^{st}\mathcal{B}(H))$ $(\|A - B\| \approx 0)$. Such $B$ is called the *shadow* of $A$ and is denoted by ${}^{\circ}A$. We denote the standardized image of ${}^{\circ}A$ by ${}^{\bullet}A$. Therefore, ${}^{\circ}A \in {}^{st}\mathcal{B}(H)$, ${}^{\bullet}A \in {}^{st}\mathcal{B}(\mathbf{H})$, $\|A -{}^{\circ} A\| \approx 0$ and ${}^{\circ}A = \Pi\mathbf{A}Q$.

*4.12. Remark.* A standard (in the usual sense) function is determined uniquely by its values at standard points. The same is true for an operator $A \in {}^{st}\mathcal{B}(H)$. Once more, let $A \in {}^{st}\mathcal{B}(H)$ and $(\forall x \in {}^{st}H)$ $(Ax \approx 0)$, then $A = 0$. Particularly, if $A \in {}^{st}\mathcal{B}(H)$ and $\|A\| \approx 0$, then $A = 0$. Therefore, the shadows ${}^{\bullet}A$ and ${}^{\circ}A$ for an $A \in {}^{nst}\mathcal{B}(H)$ are uniquely determined.

*Proof.* Put $\mathbf{A} := {}^{\bullet}A$ and $\xi := {}^{\bullet}x$. Then $Ax = \Pi\mathbf{A}Q\Pi\xi \approx \Pi\mathbf{A}\xi$. Hence if $Ax \approx 0$, then $\Pi\mathbf{A}\xi \approx 0$ and, by 4.7.1°, $A = 0$ □

*4.13. Remark.* Let $x \in {}^{nst}H$, $A \in {}^{nst}\mathcal{B}(H)$, then $Ax \in {}^{nst}H$ and ${}^{\bullet}(Ax) = ({}^{\bullet}A)({}^{\bullet}x)$.

*Proof.* Let $x \approx \Pi\xi$, $\xi \in {}^{st}\mathbf{H}$, $A \approx \Pi\mathbf{A}Q$, $\mathbf{A} \in {}^{st}\mathcal{B}(\mathbf{H})$. Then $Ax \approx \Pi\mathbf{A}Q\Pi\xi$, hence ${}^{\bullet}(Ax) = \mathbf{A}\xi$ □

The following assertion is evident.

**4.14. Proposition.** *Let* $A, B \in {}^{nst}\mathcal{B}(H)$*,* $c, d \in \mathbb{C}$*,* $|c| \ll \infty$*,* $|d| \ll \infty$*. Then* ${}^{\bullet}(cA + dB) = {}^{\circ}c{}^{\bullet}A + {}^{\circ}d{}^{\bullet}B$*.*

But the situation with multiplication is not so good.

**4.15. Proposition.** *Let* $n \in {}^{st}\mathbb{N}$*,* $A_1, \ldots, A_n \in {}^{nst}\mathcal{B}(H)$*. Then* $A_1, \ldots, A_n x \in {}^{nst}H$ *and* ${}^{\bullet}(A_1 \ldots A_n x) = ({}^{\bullet}A_1) \ldots ({}^{\bullet}A_n)({}^{\bullet}x)$*.*

*Proof.* It suffices to consider the case $n = 2$. Let $\xi = {}^{\bullet}x$, then

$$A_1A_2x \approx \Pi{}^{\bullet}A_1Q\Pi{}^{\bullet}A_2Q\Pi\xi = \Pi{}^{\bullet}A_1P{}^{\bullet}A_2P\xi \approx \Pi{}^{\bullet}A_1^{\bullet}A_2\xi.$$

Therefore, $A_1A_2x \in {}^{nst}H$ and ${}^{\bullet}(A_1A_2x) = {}^{\bullet}A_1{}^{\bullet}A_2{}^{\bullet}x$. □

**4.16. Corollary.** *Let* $A, B \in {}^{nst}\mathcal{B}(H)$*, then*

$$AB = 1_H \implies ({}^{\bullet}A)({}^{\bullet}B) = \mathbb{I}_{\mathbf{H}}. \tag{4.14}$$

*Proof.* $AB = \mathbb{I}_H$ implies $\forall x \in {}^{nst}H$ ${}^{\bullet}x = {}^{\bullet}(ABx)$. By 4.15, $\forall x \in {}^{nst}H$ ${}^{\bullet}x = {}^{\bullet}A{}^{\bullet}B{}^{\bullet}x$. But $\{{}^{\bullet}x :\ x \in {}^{nst}H\} = {}^{st}\mathbf{H}$. Hence, by transfer, we get (4.14). □

**4.17. Example.** Let $A$ be the operator of multiplication by a function $a \in \mathbb{C}^T$, that is $\forall x \in H \;\; \forall t \in T \;\; Ax(t) = a(t)x(t)$. Note that $\|A\| = \|a\|_\infty := \max_{t\in T} |a(t)|$. We claim that $A \in {}^{st}\mathcal{B}(H)$ iff the function $a$ is $\|\cdot\|_\infty$-standard, that is

$$A \in {}^{st}\mathcal{B}(H) \Leftrightarrow \|a\|_\infty \ll \infty \;\text{ and }\; (\exists b \in {}^{st}\mathbb{C}^{\mathbb{N}})\; (\forall t \in T)\; (a(t) = b(t)). \tag{4.15}$$

Note that such function $b$ coincides with the standard extension to $\mathbb{N}$ of $({}^\circ[a(t)])_{t\in {}^{st}\mathbb{N}}$. Let $\mathbf{B}$ be the operator of multiplication in $\mathbf{H}$ by $b$. Then ${}^\bullet A = \mathbf{B}$.

*Proof.* Let $a \in \mathbb{C}^T$, $\|a\|_\infty \ll \infty$, $b \in {}^{st}\mathbb{C}^{\mathbb{N}}$ and $\forall t \in T \;\; a(t) = b(t)$. Then $\forall t \in {}^{st}\mathbb{N}$ $|b(t)| \le \|a\|_\infty$, hence

$$\mathbf{B} \in {}^{st}\mathcal{B}(\mathbf{H}) \;\text{ and }\; \|\mathbf{B}\| = {}^\circ\|A\|. \tag{4.16}$$

Obviously, $A = \Pi\mathbf{B}Q$. Conversely, let $A \in {}^{st}\mathcal{B}(H)$ and $\mathbf{B} := {}^\bullet A$. Denote by $b(\cdot,\cdot)$ the matrix of $\mathbf{B}$ relative to the natural basis of the space $\mathbf{H} = \ell_2(\mathbb{N})$, that is $\mathbf{B}\xi(t) = \sum_{u\in\mathbb{N}} b(t,u)\xi(u)$. Let $\delta_t$ be the Kronecker delta at point $t$. From $A\delta_t = \Pi\mathbf{B}Q\delta_t$ we obtain: $b(t,u) = 0$ for $u \neq t$ and $b(t,t) = a(t)$. The matrix $b(\cdot,\cdot)$ is standard, because so is $B$. Hence if we put $\forall t \in \mathbb{N} \;\; b(t) = b(t,t)$, we obtain $b \in {}^{st}\mathbb{C}^{\mathbb{N}}$. □

*4.18. Remark.* Equality (4.16) is a special case of a more general one:

$$\forall A \in {}^{nst}\mathcal{B}(H) \quad \|{}^\bullet A\| = {}^\circ\|A\|. \tag{4.17}$$

*Proof.* Since $\|\Pi\| = \|Q\| = 1$, $A \approx \Pi {}^\bullet A Q$ implies ${}^\circ\|A\| \le \|{}^\bullet A\|$. For $QA\Pi \approx P{}^\bullet AP$, we have $({}^\bullet A)\xi \approx P{}^\bullet AP\xi \approx QA\Pi\xi$. Therefore, $\|({}^\bullet A)\xi\| \le {}^\circ\|A\|\|\xi\|$ and, by transfer,
$\|{}^\bullet A\| \le {}^\circ\|A\|$. □

As a corollary from 4.17 we have

**4.19. Proposition.** *Let $a \in \mathbb{C}^T$ be the potential of $L$ (see (1.1)). Let conjecture 3.1 be satisfied and let $A$ be the operator of multiplication by $a$ in $H$. Then $A \in {}^{nst}\mathcal{B}(H)$ and ${}^\bullet A = \mathbf{B}$, where $\mathbf{B}$ is the operator of multiplication by $b$:*

$$b = (b(t))_{t\in\mathbb{N}} := \textit{ standard extension of } ({}^\circ[a(t)])_{t\in {}^{st}\mathbb{N}}. \tag{4.18}$$

*Proof.* We have

$$\|a\|_\infty \le \|a\|_1, \tag{4.19}$$

therefore, $\|a\|_\infty \ll \infty$ (see 3.1.1°). For $\forall t \in^{st} \mathbb{N}$ $|b(1)| + \cdots + |b(t)| \le \|a\|_1 + 1$, by transfer we find $\|b\|_1 \ll \infty$. Next, $\forall t \in {}^{st}\mathbb{N}$ $a(t) \approx b(t)$ and $\forall t \in T \;\; t \approx \infty \implies a(t) \approx 0 \approx b(t)$. Let $\mathbf{B}$ be the operator of multiplication in $\mathbf{H}$ by the (standard) function $b$. This $\mathbf{B}$ is standard and $(\forall x \in H)$ $(\|x\| \ll \infty \implies \|Ax - \Pi\mathbf{B}Qx\|^2 = \sum_{t\in T} |a(t) - b(t)|^2|x(t)| \approx 0$. Thus, $\|A - \Pi\mathbf{B}Q\| \approx 0$. □

Form the expression $\mathbf{l}$:

$$\forall \xi \in \mathbf{H} \;\; \forall t \in \mathbb{N} \;\; \mathbf{l}\xi(t) := -\frac{1}{2}[\xi(t-1) + \xi(t+1)] + b(t)\xi(t). \tag{4.20}$$

Denote by $\mathbf{L}$ the operator generated on $\mathbf{H}$ by expression (4.20) and the boundary condition

$$\xi(0) = 0. \tag{4.21}$$

Obviously, $\mathbf{L} \in {}^{st}\mathcal{B}(\mathbf{H})$.

**4.20. Proposition.** *The operator $L$ (with integrable potential $a$; see 3.1) is near-standard (that is $L \in {}^{nst}\mathcal{B}(H)$) and ${}^\bullet L = \mathbf{L}$.*

*Proof.* Let $\mathbf{L}_0$ be the operator $\mathbf{L}$ corresponding to $b \equiv 0$. Evidently, $L_0 = \Pi \mathbf{L}_0 Q$, where $L_0$ is the operator $L$ relative to $a \equiv 0$. Therefore, $L_0$ is standard and its standardized image is ${}^\bullet L_0 = \mathbf{L}_0$. Since (in notation of 4.19) $L = L_0 + A$, $\mathbf{L} = \mathbf{L}_0 + \mathbf{B}$, our assertion follows from 4.14. □

**5. Condition at infinity**. Recall that we suppose that $\forall t \in \mathbb{Z} \setminus T \quad a(t) = 0$. Denote by $e(\cdot, \varrho)$ the solution of equation $\ell x = \lambda x$, $\lambda = -\frac{1}{2}(\varrho + \varrho^{-1})$ determined by

$$\forall t \geq m \quad e(t, \varrho) = \varrho^{-t}. \tag{5.1}$$

It is easy to verify that condition (5.1) is equivalent to the following:

$$e(m+1, \varrho) = \varrho^{-m-1}, \quad e(m, \varrho) = \varrho^{-m}. \tag{5.2}$$

*5.1. Remark.* The function $e(\cdot, \varrho)$ is the (unique) solution of the integral equation

$$e(t, \varrho) = \varrho^{-t} - \frac{2}{\varrho - \varrho^{-1}} \sum_{u=t+1}^{m} a(u) e(u, \varrho) Z_0(t-u, \varrho), \ t \in \mathbb{Z}, \tag{5.3}$$

where $Z_0(t, \varrho) = \varrho^t - \varrho^{-t}$.

*Proof.* In virtue of $\forall t > m \ a(t) = 0$, the function $e(\cdot, \varrho)$, which is defined by (5.3), satisfies (5.1). It is easy to check that it satisfies the equation $\ell e = \lambda e$ too. □

Denote by $\alpha(t)$ the remainder

$$\alpha(t) := \sum_{u=t+1}^{m} |a(u)|. \tag{5.4}$$

Whenever $|\varrho - \varrho^{-1}| \gg 0$, there exists $t_\varrho \in {}^{st}\mathbb{N}$ such that

$$\forall t \geq t_\varrho \quad 4\alpha(t) < |\varrho - \varrho^{-1}|. \tag{5.5}$$

This follows by permanence from conjecture 3.1. Put

$$W := \{\varrho \in \mathbb{C} : \ |\varrho - \varrho^{-1}| \gg 0 \ \text{ and } \ |\varrho| \geq 1\}. \tag{5.6}$$

**5.2. Proposition.** *Let $\varrho \in W$, then $\forall t \geq t_\varrho$*

$$|e(t, \varrho)| \leq \frac{|\varrho - \varrho^{-1}|}{|\varrho - \varrho^{-1}| - 4\alpha(t)} |\varrho|^{-t} \tag{5.7}$$

*and*

$$|e(t, \varrho) - \varrho^{-t}| \leq \frac{4\alpha(t)}{|\varrho - \varrho^{-1}| - 4\alpha(t)} |\varrho|^{-t}. \tag{5.8}$$

*Proof.* Suppose that there exists a real $C_t > 0$ such that $\forall u > t \ |e(t, \varrho) < C_t |\varrho|^{-u}$. Then (5.3) implies that $|e(t, \varrho)| \leq |\varrho|^{-t} + (4C_t)/(|\varrho - \varrho^{-1}|)\alpha(t)|\varrho|^{-t}$ (for $\forall u > t$ $|Z_0(t-u, \varrho)| \leq 2|\varrho|^{u-t}$). We require that $C_t \leq 1 + (4C_t)/(|\varrho - \varrho^{-1}|)\alpha(t)$. If (5.5) is fulfilled, the last inequality is satisfied. This proves (5.7). From (5.3) and (5.7) we obtain (5.8). □

**5.3. Proposition.** *Let $\varrho \in W$, then $e(\cdot, \varrho) \in {}^{nst}H$ iff $|\varrho| \gg 1$.*

*Proof.* Let $|\varrho| \gg 1$ and $\varrho \in W$. From (5.7) we see that there exists some $C \ll \infty$ such that $|e(t, \varrho)| \leq C|\varrho|^{-t}$. The equation $\ell e = \lambda e$ defines $e(t, \varrho)$ for $t < t_\varrho$ whenever $e(t, \varrho)$ is known for $t \geq t_\varrho$. Since $t_\varrho$ is standard, in order to find $e(t, \varrho)$, $t < t_\varrho$, one needs a finite quantity of steps only. As $\forall t > t_\varrho \quad |e(t, \varrho)| \ll \infty$, we have

$$\forall t \in T \quad |e(t, \varrho)| \ll \infty. \tag{5.9}$$

Therefore, in virtue of (5.7) $\|e(\cdot, \varrho)\| \ll \infty$. Since $(\forall t \in T)$ $(t \approx \infty \implies \sum_{u>t} |e(u, \varrho)|^2 \approx \infty)$, in view of 4.4.2 $e(\cdot, \varrho) \in {}^{nst}H$. On the other hand, by (5.8),

$$(\forall t \in T) \ \ (t \approx \infty \implies |e(t, \varrho)| \geq \frac{1}{2}|\varrho|^{-t}).$$

This shows that for $|\varrho| \approx 1$ there exists a $t \approx \infty$ such that $|e(t, \varrho)| \gg \infty$, hence $e(\cdot, \varrho) \notin {}^{nst}H$. □

Let $\varrho \in W$ and $|\varrho| \gg 1$. Let us describe the shadow of $e(\cdot, \varrho)$. It is known (see [15, th.2]) that the equation $\ell\varepsilon = \lambda\varepsilon$, $\lambda = -\frac{1}{2}(\varrho + \varrho^{-1})$ ($\ell$ is the same as in (4.20); it corresponds to $b = {}^\bullet a$) has a solution $\varepsilon(\cdot, \varrho)$ such that $\forall \delta > 0$ uniformly in $\{\varrho \in \mathbb{C} : |\varrho^2 - 1| > \delta, \ |\varrho| \geq 1\}$

$$\varepsilon(t, \varrho) = \varrho^{-t}[1 + o(1)], \ \ t \to \infty. \tag{5.10}$$

This $\varepsilon(\cdot, \varrho)$ coincides with the (unique) solution of the "integral" equation

$$\varepsilon(t, \varrho) = \varrho^{-t} - \frac{2}{\varrho - \varrho^{-1}} \sum_{u=t+1}^{\infty} b(u)\varepsilon(u, \varrho)Z_0(t - u, \varrho), \ \ t \in \mathbb{N}. \tag{5.11}$$

**5.4. Proposition.** *Let $|\varrho| \gg 1$, then ${}^\bullet e(\cdot, \varrho) = \varepsilon(\cdot, {}^\circ\varrho)$.*

*Proof.* In view of Definitions 4.1 and 4.3 ${}^\bullet e(\cdot, \varrho)$ is equal to the standard extension to $\mathbb{N}$ of $({}^\circ[e(t, \varrho)])_{t \in {}^{st}\mathbb{N}}$. We see from (3.7) and (5.7) that $\forall u > t \ \ |e(u, \varrho)||Z_0(t - u, \varrho)||\varrho|^t \ll \infty$. Therefore, we have in (5.3) $\forall t \in {}^{st}\mathbb{N} \ {}^\circ[e(t, \varrho)] \approx \varrho^{-t} - 2/(\varrho - \varrho^{-t}) \sum_{u=t+1}^{m} b(u)e(u, \varrho)Z_0(t - u, \varrho)$ (we recall that $\forall t \in T \ b(t) \approx a(t)$). Since $\|b\| \ll \infty$, we get $\sum_{u=t+1}^{m} b(u)e(u, \varrho)Z_0(t - u, \varrho) \approx 0$. Then $\forall t \in {}^{st}\mathbb{N} \ {}^\circ[e(t, \varrho)] = \varepsilon(t, \varrho)$ and by transfer ${}^\bullet e(\cdot, \varrho) = \varepsilon(\cdot, {}^\circ\varrho)$. □

Proofs of (5.7) and (5.8) are not valid, whenever $|\varrho| < 1$, because in this case $\forall u > t \ |Z_0(t - u, \varrho)| \leq 2|\varrho|^{t-u}$ (but not $\leq 2|\varrho|^{u-t}$). But an analogous reasoning is applicable for $|\varrho|^m \approx 1$. Put

$$W_m := \{\varrho \in \mathbb{C} : \ |\varrho^2 - 1| \gg 0, \ |\varrho|^m \approx 1\}. \tag{5.12}$$

**5.5. Proposition.** *Assume that (5.5) is satisfied. Then evaluations (5.7) and (5.8) are valid also for $\varrho \in W_m$.*

*Proof.* Put $C_t := (|\varrho - \varrho^{-1}|)/(|\varrho - \varrho^{-1}| - 4\alpha(t))$. As $C_t \geq 1$, we find $\forall u \geq m$ $|e(u, \varrho)| \leq C_t|\varrho|^{-u}$. Suppose that for some $t \in T$ $\forall u > t$ $|e(u, \varrho)| \leq C_t|\varrho|^{-u}$. Then (5.3) implies that $|e(t, \varrho)| \leq C_t|\varrho|^{-t}$. As before, from (5.3) and (5.7) we obtain (5.8). □

For $\varrho \neq 0$ put

$$\forall t \in \mathbb{Z} \quad \tilde{e}(t, \varrho) := e(t, \tfrac{1}{\varrho}). \tag{5.13}$$

*5.6. Remark.* $\tilde e(\cdot,\varrho)$ is a solution of the equation $\ell\tilde e=\lambda\tilde e$, $\lambda=-\frac12(\varrho+\varrho^{-1})$, with the condition at infinity

$$\forall t\ge m\quad \tilde e(t,\varrho)=\varrho^t.$$

Suppose that (5.5) is satisfied. Put

$$W_+:=\{\varrho\in\mathbb{C}:\ |\varrho|\le 1,\ |\varrho^2-1|\gg 0\}. \tag{5.15}$$

Then (5.7) and (5.8) remain to be true for $\varrho\in W_+\cap W_m$, if we change $e(\cdot,\varrho)$ with $\tilde e(\cdot,\varrho)$, and $\varrho^{-t}$ with $\varrho^t$.

*5.7. Remark.* Let $f_1,f_2\in\mathbb{C}^{\mathbb{Z}}$ be two solutions of the equation $\ell f=\lambda f$. The Wronskian

$$w(f_1,f_2,\lambda):=f_1(t)f_2(t+1)-f_1(t=1)f_2(t) \tag{5.16}$$

is independent of $t$. In view of (5.1) and (5.14) we have

$$w(e,\tilde e,\lambda)=\varrho-\varrho^{-1},\ \ \lambda=-\frac12(\varrho+\varrho^{-1}), \tag{5.17}$$

where $e:=e(\cdot,\varrho)$, $\tilde e:=e(\cdot,\frac1\varrho)$, $\varrho\ne 0,\pm1$. Therefore, if $\varrho\ne 0,1,-1$, the general solution of the equation $\ell x=\lambda x$ is the following

$$x(t)=Ce(t,\varrho)+\tilde C\tilde e(t,\varrho), \tag{5.18}$$

where $C,\tilde C$ are arbitrary complex constants. For instance, the solution $Z(\cdot,\varrho)$ defined by condition (3.3) can be represented as

$$Z(t,\varrho)=e(\varrho)\tilde e(t,\varrho)-\tilde e(\varrho)e(t,\varrho), \tag{5.19}$$

where

$$e(\varrho):=e(0,\varrho),\quad \tilde e(\varrho)=e(\tfrac1\varrho). \tag{5.20}$$

**5.8. Corollary.** *For $\varrho\in W_m$ (see (5.12)) none solution $x$ of the equation $\ell x=\lambda x$ such that $\|x\|\gg 0$ is $\|\cdot\|$-nearstandard.*

*Proof.* By 5.5 and (5.8), for $\varrho\in W_m$ solution (5.8) can be represented as $x(t)=x_0(t)+\alpha(t)x_1(t)$, where $x_0(t):=C\varrho^{-t}+\tilde C\varrho^t$ and $|x_1(t)|\ll\infty$. As we have observed in example (4.7), if $|C|+|\tilde C|\gg 0$, then there is a set $S\subset\mathbb{N}$ such that card $S\approx\infty$ and $\forall t\in S\ |x_0(t)|\gg 0$. Since $\forall t\approx\infty\ \ \alpha(t)\approx 0$, there exists a $t\approx\infty$ such that $x(t)\not\approx 0$. □

We need a solution of the equation $\ell x=\lambda x$, for which an evaluation such as (5.8) is true not only for $\varrho\in W_m$ and $|\varrho|<1$. If $a\equiv 0$, this solution is $\tilde e_0(t,\varrho):=\varrho^t$. In the general case we define $\hat e(t,\varrho)\equiv\hat e(t,t_0,\varrho)$ by the "integral" equation

$$\hat e(t,\varrho)=\varrho^t-\frac{2}{\varrho-\varrho^{-1}}\left[\sum_{u=t_0}^{t-1}a(u)\hat e(u,\varrho)\varrho^{u-t}+\sum_{u=t}^{m}a(u)\hat e(u,\varrho)\varrho^{t-u}\right],\ \ t>t_0, \tag{5.21}$$

with a sufficiently large $t_0\ll\infty$. One can obtain this equation from the formula $x(t)=C_-\varrho^{-t}+C_+\varrho^{+t}-2/(\varrho-\varrho^{-1})\sum_{u=1}^{t-1}f(u)Z_0(t-u,\varrho)$ (for the general solution of the equation $(\ell_0-\lambda)u=f$) by setting $x(t)=\hat e(t,\varrho)$, $f(u)=-a(u)\hat e(u,\varrho)$ and by the suitable choice of $C_\pm$. Hence the solution of (5.21) (its existence will be proved below) satisfies also $\ell\hat e=\lambda\hat e$.

**5.9. Proposition.** *Let $|\varrho| \gg 1$ and $t_\varrho \in T$ be such that (compare with (5.5))*

$$t_\varrho \ll \infty \quad and \quad \alpha(t_\varrho) \ll \frac{1}{2}|\varrho - \varrho^{-1}|, \tag{5.22}$$

*where $\alpha(t)$ is remainder (5.4). Then equation (5.2) has a unique solution $\hat{e}(\cdot, \varrho) = \hat{e}(\cdot, t_0, \varrho)$ (which satisfies also the equation $\ell\hat{e} = \lambda\hat{e}$). That equation has a property*

$$(\forall t \approx \infty) \ \ (\varrho^{-t}\hat{e}(t, \varrho) \approx 1). \tag{5.23}$$

*Proof.* For simplicity we shall write $\hat{e}(t)$ in place of $\hat{e}(t, t_0, \varrho)$. After the substitution $\hat{e}(t) = \varrho^t \breve{e}(t)$, $t_0 = t_\varrho$, equation (5.21) takes the form:

$$\breve{e}(t) = 1 - \frac{2}{\varrho - \varrho^{-1}} \left[ \sum_{u=t_\varrho}^{t-1} a(u)\varrho^{2(u-t)}\breve{e}(u) + \sum_{u=t}^{m} a(u)\breve{(}u) \right], \ t > t_\varrho. \tag{5.21$'$}$$

Denote by $\breve{e}_0(\cdot)$ the solution of the homogenous equation corresponding to (5.21$'$). Let $\breve{e}_0 = \max_{t_\varrho \le t \le m} |\breve{e}_0(t)|$. Then that homogenous equation implies $\breve{e}_0 \le 2|\varrho - \varrho^{-1}|^{-1}\alpha(t)\breve{e}_0$. In virtue of (5.22) $\breve{e}_0 = 0$. Therefore, equation (5.21') has a unique solution $\breve{e}(\cdot) = \breve{e}(\cdot, t_\varrho, \varrho)$. Now put $\breve{e} = \max_{t_\varrho \le t \le m} |\breve{e}(t)|$. From (5.21') we obtain $|\breve{e}(t) - 1| \le 2|\varrho - \varrho^{-1}|\breve{e}\alpha(t_\varrho)$. Hence $|\breve{e} - 1| \le 2|\varrho - \varrho^{-1}|^{-1}\alpha(t_\varrho)\breve{e}$ and $\breve{e} \le [1 - 2|\varrho - \varrho^{-1}|^{-1}\alpha(t_\varrho)]^{-1}$. We conclude from (5.22) that $\breve{e} \ll \infty$. Now (5.21') implies $|\breve{e}(t) - 1| \le 2\breve{e}|\varrho - \varrho^{-1}|^{-1}[\alpha(t) + \beta(t)]$, where $\alpha(t)$ is remainder (5.4) and $\beta(t) := \sum_{u=t_\varrho}^{t-1} |a(u)||\varrho|^{2(u-t)}$. Since $\forall t \approx \infty$ $\alpha(t) \approx 0$, we have to prove that $\forall t \approx \infty$ $\beta(t) \approx 0$. To this end let $s$ be the entire party of $\frac{1}{2}t$. We have

$$\begin{aligned}\beta(t) &\le |\varrho|^{-2t} \left[ \Big( \sum_{u=t_\varrho}^{s-1} |a(u)|^2 \Big)^{1/2} \Big( \sum_{u=t_\varrho}^{s-1} |\varrho|^{4u} \Big)^{1/2} + \Big( \sum_{u=1}^{m} |a(u)|^2 \Big)^{1/2} \Big( \sum_{u=s}^{m} |\varrho|^{4u} \Big)^{1/2} \right] \\ &\le |\varrho|^{-2t} \left[ \|a\|_2 \Big( |\varrho|^{4t_\varrho} \frac{|\varrho|^{4(s-t_\varrho)} - 1}{|\varrho|^4 - 1} \Big)^{1/2} + \|a\|_\infty \alpha(s)^{1/2} \Big( |\varrho|^{4s} \frac{|\varrho|^{4(t-s)} - 1}{|\varrho|^4 - 1} \Big)^{1/2} \right].\end{aligned}$$

The last expression is $\approx 0$ whenever $t \approx \infty$, because $\|a\|_1 \ll \infty$ implies $\|a\|_\infty \ll \infty$ and $\|a\|_2 \ll \infty$, and $s \approx \infty$ implies $\alpha(s) \approx 0$. □

*5.10. Remark.* With the aid of the equation $\ell\hat{e} = \lambda\hat{e}$ we extend the solution $\hat{e}(\cdot, t_\varrho, \varrho)$ to the values $t \le t_\varrho$. So this solution is defined for all $t \in T$. We suppose that (5.22) and therefore (5.23) are satisfied. From (5.1) it follows that $e(\cdot, \varrho)$ and $\hat{e}(\cdot, \varrho)$ form some fundamental system of solutions of the equation $\ell x = \lambda x$. Indeed, since $w(e, \hat{e}, \lambda)$ does not depend on $t$, we have (see (5.16))

$$w(e, \hat{e}, \lambda) = \varrho - \varrho^{-1}, \quad \lambda = -\frac{1}{2}(\varrho + \varrho^{-1}). \tag{5.24}$$

**5.11. Corollary.** *For $|\varrho| \gg 1$ the general solution of the equation $\ell x = \lambda x$ can be represented as follows*

$$x(t) = Ce(t, \varrho) + \hat{C}\hat{e}(t, \varrho); \tag{5.25}$$

*here $C, \hat{C} \in \mathbb{C}$ are arbitrary constants. In patricular, we have (see (3.3))*

$$Z(t, \lambda) = e(\varrho)\hat{e}(t\varrho) - \hat{e}(\varrho)e(t, \varrho), \tag{5.26}$$

*where (compare with (5.20))*

$$e(\varrho) := e(0, \varrho), \quad \hat{e}(\varrho) := \hat{e}(0, \varrho). \tag{5.27}$$

*( To be continued.)*

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Department of Mechanics and Mathematics, Lviv State University,
Universytetska str, 1, 290602, Lviv, Ukraine

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