УДК 517.544

THE SYMMETRICAL CARLEMAN PROBLEM FOR THE BAND WITH THE PARALLEL SHIFT ON THE REAL AXIS

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P.V. Kerekesha. *The symmetrical Carleman problem for the band with the parallel shift on the real axis*, Matematychni Studii, **10**(1998) 33–40.

The Carleman problem (PC) for the symmetrical band with the parallel shift on the real axis is considered. Due to received integral representation of analytical function in the band the normal solvability, the existence, the uniqueness and the construction of approximate PC solution with a corresponding error estimation are studied. PC is also studied in the space of distributions. In the particular case the PC solution with application is received.

П.В. Керекеша. *Симметрическая задача Карлемана для полосы с паралельным сдвигом на вещественую ось* // Математичні Студії. – 1998. – Т.10, № 1. – С.33–40.

Рассматривается симметрическая задача Карлемана (ЗК) для полосы с паралельным сдвигом на вещественную ось. Благодаря полученному интегральному представлению аналитической функции в полосе, исследованы вопросы нормальной разрешимости, существования, единственности, асимптотического поведения и построения приближенного решения ЗК с соответствующей оценкой погрешности. Исследована также ЗК в пространстве обобщенных функций. В частном случае получено точное решение ЗК с приложением.

1. Introduction. In the paper [1] the Carleman problem has been considered in the form

$$A(x)\Phi(x+i)+B(x)\Phi(x-i)+C(x)\Phi(x)=G(x),\quad x\in\mathbb{R} \tag{1}$$

and the necessary and sufficient conditions of its normal solvability has been given. In the case when $B(x)\equiv 0$ the Carleman problem (1) was studied in the paper [2].

In the present paper the Carleman problem (1) is considered in the case when

$$A(x)=B(x)=1. \tag{2}$$

The Carleman problem (1)–(2) will be called the symmetrical Carleman problem for band with the parallel shift on the real axis.

2. Definitions and formulation of the symmetrical Carleman problem. Let h be a positive number.

1991 *Mathematics Subject Classification.* 30E25.

Definition 1. By $\{-h, h\}$ we denote the space of the functions $\varphi(t)$ such that

$$\big(\exp(hx) + \exp(-hx)\big)\varphi(x) \in L^2(\mathbb{R}). \tag{3}$$

Definition 2. By $\{\{-h, h\}\}$ denoted the space of functions $\Phi(z)$ analytical in the band $|\operatorname{Im} z| < h$ for which there exists a constant C such that for all $y \in [-h, h]$

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^2 \, dx \le C.$$

Theorem 1. *In order that a function $\varphi(x) \in \{-h, h\}$ it is necessary and sufficient that the Fourier transform $\Phi(x) = (V\varphi)(x) \in \{\{-h, h\}\}$.*

This result follows from the theorem [3, p.173].

Problem. *Find a function $\Phi(x)$ which belongs to the space $\{\{-h, h\}\}$ and satisfies the condition*

$$\Phi(x+i) + \Phi(x-i) + C(x)\Phi(x) = G(x), \quad x \in \mathbb{R}. \tag{4}$$

Here $C(x)$ is the continuous function on the all real axis and $G(x)$ belongs to the space $L^2(\mathbb{R})$.

3. The integral representation of the analytical function in the symmetrical band and its boundary values.

Theorem 2. *Let the function $\Phi(z) \in \{\{-h, h\}\}$. Then*
1) the integral representation

$$\Phi(z) = \frac{1}{4h} \int_{-\infty}^{\infty} \frac{N(t)\, dt}{\cosh \frac{\pi}{2h}(t-z)}, \quad |\operatorname{Im} z| < h \tag{5}$$

is valid;
2) the limit values of the function $\Phi(z)$ exist almost everywhere on the boundary and the following formulae are true

$$\Phi(x \pm ih) = \frac{N(x)}{2} \pm \frac{i}{4h} \int_{-\infty}^{\infty} \frac{N(t)\, dt}{\sinh \frac{\pi}{2h}(t-x)}, \tag{6}$$

where $N(x)$ is an arbitrary function from the space $L^2(\mathbb{R})$.

Proof. As the function $\Phi(x) \in \{\{-h, h\}\}$, under the conditions of Theorem 1, $\varphi \in \{-h, h\}$ and hence the function $\varphi(x)$ satisfies condition (3).

Further, we enter in consideration the function

$$\nu(x) = \varphi(x)\big(\exp(-hx) + \exp(hx)\big)$$

Obviously, $\nu(x) \in L^2(\mathbb{R})$. Now to the function $\varphi(x)$ the Fourier operator is applicable,

$$(V\varphi)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(t) \exp(itx)\, dt,$$

As a result we obtain

$$\Phi(x) = \frac{1}{4h}\int_{-\infty}^{+\infty}\frac{N(t)\,dt}{\cosh\frac{\pi}{2h}(t-x)}.$$

Here $N(x) = (V\nu)(x)$, $\Phi(x) = (V\varphi)(x)$.
From the formulae

$$\left(V^{-1}\Phi\right)(x - iy) = \varphi(x)\exp(-xy), \quad -h < y < h,$$

$$\varphi(x)\exp(-xy) = \frac{\nu(x)}{2\cosh(xh)}\exp(-xy),$$

$$V\Big(\frac{1}{2\cosh(xh)}\exp(-xy)\Big)(x) = \sqrt{\frac{\pi}{2}}\cdot\frac{1}{2h\cosh\left(\frac{\pi}{2h}\right)(x+iy)}$$

we have

$$\Phi(x+iy) = \frac{1}{4h}\int_{-\infty}^{+\infty}\frac{N(t)\,dt}{\cosh\left(\frac{\pi}{2h}\right)(t-x-iy)},$$

that is, formula (5).

We proceed to prove the second parts of Theorem 2. First, the boundary values $\Phi(x \pm ih)$ are nearly everywhere guaranteed. From this it results that the space $\{\{-h, h\}\}$ is a subspace of the space $H[-b, b]$ [4]. Then

$$\Phi(x \pm ih) = V\{\varphi(x)\exp(\pm x)\}(x) = V\Big[\frac{\nu(x)}{1+\exp(\pm 2hx)}\Big](x) =$$
$$= \frac{N(x)}{2} \pm \frac{i}{4h}\int_{-\infty}^{+\infty}\frac{N(t)\,dt}{\sinh\frac{\pi}{2h}(t-x)}.$$

In establishing the obtained boundary relations, the formula

$$V\Big(\frac{1}{1+\exp(\pm 2hx)}\Big)(x) = \mp\frac{i}{2h}\sqrt{\frac{\pi}{2}}\,\frac{1}{\sinh\frac{\pi}{2}(hx)} + \delta(x)\sqrt{\frac{\pi}{2}}$$

is used ($\delta(x)$ is the Dirac delta-function). Theorem 2 is proved.

4. The existence, uniqueness, asymptotic behavior and normal solvability of the Carleman problem in the space $\{\{-1, 1\}\}$.

Theorem 3. *Let*

$$\gamma = 2^{-1}\max_{x\in R}|C(x)| < 1. \tag{7}$$

Then the Carleman problem (4) *in the space* $\{-1, 1\}$ *has a unique solution.*

Proof. On the basis of results of section 3, the Carleman problem (4) is equivalent to the following integral equation

$$N(x) + \frac{C(x)}{4}\int_{-\infty}^{+\infty}\frac{N(t)\,dt}{\cosh\frac{\pi}{2}(t-x)} = G(x), \quad x \in \mathbb{R}. \tag{8}$$

The operator form of this equation is

$$N(x) = (\mathbf{K}N)(x), \quad x \in \mathbb{R},$$

where

$$(\mathbf{K}N)(x) = -\frac{C(x)}{4} \int_{-\infty}^{+\infty} \frac{N(t)\,dt}{\cosh\frac{\pi}{2}(t-x)} + G(x), \quad x \in \mathbb{R}. \tag{9}$$

Let us now consider two functions $N_{1,2}(x)$ from the space $L^2(\mathbb{R})$. We substitute them in (9) and we produce the valuation of the norm

$$\begin{aligned} \big\|\mathbf{K}(N_2(x) - N_1(x))\big\|_2 \equiv \left(\int_{-\infty}^{+\infty} \left|\frac{C(x)}{4} \int_{-\infty}^{+\infty} \frac{\big(N_2(t) - N_1(t)\big)\,dt}{\cosh\frac{\pi}{2}(t-x)}\right| dx\right)^{1/2} \leq \\ \leq \max_{x\in\mathbb{R}} \frac{|C(x)|}{4} \left(\int_{-\infty}^{+\infty} \left|\big(\nu_2(t) - \nu_1(t)\big)\frac{2}{\cosh x}\right|^2 dx\right)^{1/2} \leq \\ \leq \gamma\|\nu_2(x) - \nu_1(x)\|_2 = \gamma\|N_2(x) - N_1(x)\|_2, \end{aligned} \tag{10}$$

here $(V\nu_{1,2})(x) = N_{1,2}(x)$.

In the chain of inequalities (10) we take in account the Parseval equality. Further, according to (10), (7) and the contraction mapping principle [5, p.72], the proof of Theorem 3 follows.

Theorem 4. *Let there*

$$1 + C(\pm\infty)/2\cosh x \neq 0, \quad x \in \mathbb{R}. \tag{11}$$

Then equation (8) is normally solvable and its index is equal to zero.

The proof is based on the results of the paper [6].

From Theorem 4 it follows

Theorem 5. *Let $C(x)$ satisfy condition (11). Then the Carleman problem (4) is normally solvable and its index is equal to zero.*

The validity of Theorem 5 follows from the equivalence of the Carleman problem (4) and the integral equation (8).

Theorem 6. *Let $C(x)$ be continuous on the real axis and $G(x)$ satisfy the conditions*

$$G(x)\exp(\alpha x) \in L^\infty(\mathbb{R}), \tag{12}$$

$$\lim_{x\to\infty}\big(G(x)\exp(\alpha x)\big) = B, \quad \alpha \in [0, \tfrac{\pi}{2}). \tag{13}$$

In order that a unique fixed solution of equation (8) satisfying the requirement

$$\lim_{x\to\infty}\big(N(x)\exp(\alpha x)\big) = A \tag{14}$$

exist, it is necessary that

$$A = \frac{B}{1 + (2\cos\alpha)^{-1}C(\infty)}. \tag{15}$$

Proof. Suppose that a unique resolution of equation (8) satisfying to condition (14) exists. We enter in consideration the function

$$\psi(x) = N(x)\exp(\alpha x), \quad g(x) = G(x)\exp(\alpha x), \quad x \in \mathbb{R}. \tag{16}$$

Then with reference to (16), equation (8) accepts the form

$$\psi(x) + \frac{C(x)}{4}\int_{-\infty}^{\infty} \frac{\psi(t)\exp\alpha(x-t)\,dt}{\cosh\frac{\pi}{2}(x-t)} = g(x), \quad x \in \mathbb{R}. \tag{17}$$

Taking in account the Tauberian theorem and conditions (13), (14), we receive

$$A = B - \frac{C(\infty)}{2\cos\alpha}A. \tag{18}$$

Relation (15) follows from (18). Establishing of formula (18), we take in consideration that

$$\lim_{x\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\psi(t)\exp\alpha(x-t)\,dt}{\cosh\frac{\pi}{2}(x-t)} = \psi(\infty)K(0), \tag{19}$$

where

$$K(x) = \sqrt{\frac{2}{\pi}}\frac{1}{\cosh(x+i\alpha)}.$$

Theorem 6 is proved.

Theorem 7. *Suppose that the function $C(x)$ is continuous on the real axis and the inequality*

$$\sup_{x\in\mathbb{R}}\frac{|C(x)|}{2\cos\alpha} < 1, \quad \alpha \in [0, \tfrac{\pi}{2}) \tag{20}$$

holds, and the function $G(x)$ satisfies conditions (12), (13) of Theorem 6. Then a unique resolution of equation (8) with the given asymptotic behavior

$$N(x) = O(A\exp(-\alpha x)), \quad x\to\infty, \quad A = \frac{B}{1 + C(\infty)(2\cos\alpha)^{-1}}, \quad \alpha \in [0, \tfrac{\pi}{2}) \tag{21}$$

exists.

Proof. We consider equation (17) equivalent to equation (8). Consider the operator $\mathbf{K}$:

$$(\mathbf{K}\psi)(x) = g(x) - \frac{C(x)}{4}\int_{-\infty}^{+\infty} \frac{\psi(t)\exp\alpha(x-t)\,dt}{\cosh\frac{\pi}{2}(t-x)} \tag{22}$$

and we find out first that it acts from the space $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$

$$\|(\mathbf{K}\psi)(x)\|_\infty \le \|g\|_\infty + \operatorname{vrai}\sup_{x\in\mathbb{R}}\left|\frac{C(x)}{4}\int_{-\infty}^{+\infty}\frac{\exp\alpha(x-t)}{\cosh\frac{\pi}{2}(t-x)}\psi(t)\,dt\right| \le$$
$$\le \|g\|_\infty + \sup_{x\in\mathbb{R}}\frac{|C(x)|}{2\cos\alpha}\|\psi\|_\infty < \|g\|_\infty + l|\psi\|_\infty, \quad \psi(x) \in L^\infty(\mathbb{R}). \tag{23}$$

Secondly, $\mathbf{K}$ is a contractive operator:

$$\|\mathbf{K}(\varphi-\psi)\|_\infty = \operatorname*{vrai\,sup}_{x\in\mathbb{R}}\left|\frac{C(x)}{4}\int_{-\infty}^{+\infty}\frac{\exp\alpha(x-t)}{\cosh\frac{\pi}{2}(t-x)}(\varphi(t)-\psi(t))\,dt\right| \leq$$

$$\leq \sup_{x\in\mathbb{R}}\frac{|C(x)|}{2\cos\alpha}\|\varphi-\psi\|_\infty \quad \varphi,\psi\in L^\infty(R). \quad (24)$$

In the chain of inequalities (23), (24) we account for condition (20) and the equality

$$\int_{-\infty}^{+\infty}\frac{\exp\alpha(x-t)}{\cosh\frac{\pi}{2}(t-x)}\,dt = \frac{2}{\cos\alpha}, \quad \alpha\in[0,\tfrac{\pi}{2}).$$

Since $L^\infty(R)$ is a complete space, according to the Banach theorem on stationary point, a unique resolution of equation (17), which can be constructed by the method of consecutive approximations, exists.

We will show now, that for the resolution $\psi(x)$ the limit at infinity exists.

Since the function $\psi(x)$ is the resolution of equation (17), the identity

$$\psi(x) = (\mathbf{K}\psi)(x) \quad (25)$$

remains true.

According to the Tauberian theorem [7, p.241] and equality (13), the limit in the right-hand part exists

$$\lim_{x\to\infty}(\mathbf{K}\psi)(x) = B - a\frac{C(\infty)}{\cos\alpha}, \quad a\in\mathbb{R}.$$

Then the limit in left-hand part also exists and the equality

$$\lim_{x\to\infty}\psi(x) = A = B - a\frac{C(\infty)}{\cos\alpha}, \quad a\in\mathbb{R} \quad (27)$$

is true.

So, equation (17) has a unique resolution, for which the limit (27) exists. Then according to Theorem 6 the constant A is associated with constant B by relation (15). Substituting it in equality (27), we find out that $a = A = \psi(\infty)$.

At last, according to equalities (15) and (16), a unique resolution of the initial equation (8) with the given asymptotic behavior (21) exists. Thus the resolution structure of equation (8) has the form

$$N(x) = \psi(x)\exp(-\alpha x), \quad \psi(x)\in L^\infty(\mathbb{R}).$$

Remark. The asymptotic behavior of solution of Carleman problem (4) is associated to that of equation (8) by the relation

$$\Phi(x) = O\Big(\frac{G(x)-N(x)}{C(x)}\Big), \quad (x\to\infty)$$

whenever the conditions of Theorem 7 are satisfied.

5. The construction of the approximate solution of the Carleman problem (4) and the error estimation in the space $L^2(R)$.

First, let us consider the construction of the approximate solution of equation (8). For this purpose we write down equation (8) in the operator form

$$(\mathbf{K}N)(x) = G(x), \quad (\mathbf{K}N)(x) \equiv N(x) - \frac{C(x)}{4}\int_{-\infty}^{\infty} \frac{N(t)\,dt}{\cosh\frac{\pi}{2}(x-t)}. \tag{28}$$

Along with equation (28) let us consider the approximate equation corresponded to the approximate Carleman problem (4).

$$\widetilde{\mathbf{K}}\widetilde{N} = \widetilde{G}, \quad (\widetilde{\mathbf{K}}\widetilde{N})(x) \equiv \widetilde{N}(x) - \frac{\widetilde{C}(x)}{4}\int_{-\infty}^{\infty} \frac{\widetilde{N}(t)\,dt}{\cosh\frac{\pi}{2}(x-t)}. \tag{29}$$

Suppose that the inverse operator $\widetilde{K}^{-1}$ exists. Then in the presence of the condition

$$\varepsilon = \|\widetilde{\mathbf{K}}^{-1}\| \cdot \|\mathbf{K} - \widetilde{\mathbf{K}}\| < 1.$$

using the theorem of the paper [2, p.152] we obtain the estimation for a mean square error

$$\|N - \widetilde{N}\|_{L^2} \le \frac{\|\widetilde{\mathbf{K}}^{-1}\| \cdot \|\mathbf{K}\widetilde{N} - G\|_{L^2}}{1-\varepsilon}. \tag{30}$$

Then according to (5), the error estimation for an approximate solution of the Carleman problem (4) can be given in the following form

$$\|\Phi - \widetilde{\Phi}\|_{L^2} \le \frac{1}{2}\|N - \widetilde{N}\|_{L^2}. \tag{31}$$

Inequalities (30)–(31) give us the required error estimation.

6. Generalizations. Let us consider the generalizations that consist in the following. It is required to find the solution of the functional equation (4) when the function $C(x)$ belongs to the space of distributions. We consider the case when $G(x)$ belongs to the space $L_2[m, 0]$ [2, p.270].

The solution of equation (4) should be found in the form

$$\Phi(x) = \Phi_0(x) + F(x), \tag{32}$$

where the function must be the solution of the following equation

$$\Phi_0(x+i) + \Phi_0(x-i) = G(x). \tag{33}$$

Applying the inverse Fourier transformation to equality (33) we obtain the following relations

$$\varphi_0(x) = \frac{g(x)}{2\cosh x}, \quad g(x) = \left(V^{-1}G\right)(x), \tag{34}$$

$$\Phi_0(x) = (V\varphi_0)(x). \tag{35}$$

Using the properties of the function $G(x)$ [2, p.270] and relations (34), (35) it is easy to show that $\Phi_0 \in L^2(\mathbb{R})$. Now let us substitute (32) into (4). Then equation (4) becomes

$$F(x+i) + F(x-i) + C(x)F(x) = -C(x)\Phi_0(x), \quad x \in \mathbb{R}. \tag{36}$$

As it is easily seen, the function $C(x)\Phi_0(x) \in L^2(\mathbb{R})$ ($C(x)$ is bounded, $\Phi_0(x) \in L^2(\mathbb{R})$).

Thus, equation (36) can be investigated as the Carleman problem in the usual estimation. It so doing, the solution of the input equation (4) can be constructed by the formula (32).

The considered case has the concrete application [8].

7. The case of the exact solution of the Carleman problem (4) with application.

Let $C(x) = \frac{\lambda}{x}\tanh\frac{\pi x}{2}$, $\lambda > 0$, $G(x) \in L^2(\mathbb{R})$. Then the Carleman problem (4) becomes

$$\Phi(x+i) + \Phi(x-i) + \lambda\frac{\tanh\frac{\pi}{2}x}{x}\Phi(x) = G(x), \quad x \in \mathbb{R}. \tag{37}$$

According to section 2 the solution of the Carleman problem (37) is seeked in the space $\{\{-1,1\}\}$. Taking into account the structure of the function $C(x)$ and the equalities $\cosh\frac{\pi}{2}(x \pm i) = \pm i\sinh\frac{\pi}{2}x$ it can be showed that the solution of the Carleman problem is constructed by the formulae

$$\psi(x) = \frac{\int_0^x \left(\tanh\frac{t}{2}\right)^{\lambda/2} g_1(t)\,dt}{2\left(\tanh\frac{x}{2}\right)^{\lambda/2}\sinh x}, \quad g_1(x) = (V^{-1}G_1)(x), \quad G_1(x) = \frac{xG(x)}{\sinh\frac{\pi}{2}x},$$

$$\psi(x) = (V\psi)(x), \quad \Phi(x) = \psi(x)\cosh\frac{\pi}{2}x.$$

The exact solution of the Carleman problem makes possible to give the exact solution of the torsion problem of the partially supported cylinder with half-circle section [8].

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Received 15.06.1997
Revised 16.02.1998