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FINITE GROUPS IN βG

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We give a characterization of finite groups in the Stone-Čech compactification βG of countable discrete group G and present some applications of this characterization.

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Получена характеризация конечных групп в стоун-чеховской компактификации βG счетной дискретной группы G, а также представлены некоторые приложения этой характеризации.

0. Introduction

Let G be a discrete group and let βG be the Stone-Čech compactification of G. The elements of the space βG will be considered as ultrafilters on G. Identify the group G with the set of all principal ultrafilters on G and denote by G^* the set $\beta G \setminus G$ of all free ultrafilters on G. Given ay subset $A \subset G$, put $\bar{A} = \{p \in \beta G : A \in p\}$. Then the family $\{\bar{A} : A \subset G\}$ forms an open base of the topology on βG. Given any filter φ on G, denote by $\overline{\varphi} = \bigcap\{\bar{A} : A \in \varphi\}$, so $\overline{\varphi}$ is a closed subset in βG and every closed subset in βG can be represented in such form.

The multiplication on G has a natural extension to βG. We define the product pq of ultrafilters p, q by the determination of all subsets $A \subset G$ which are the elements of the ultrafilters pq,

$$A \in pq \Leftrightarrow \{g \in G : g^{-1}A \in q\} \in p.$$

The multiplication on βG is associative and G^* is a closed subsemigroup of βG. Pick $X \in p$ and, for each $x \in X$, take some $Y_x \in q$. Then the subset $\bigcup\{xY_x : x \in X\}$ is an element of the ultrafilter pq. The family of such subsets forms a base of the ultrafilter pq.

The map $p_q\colon \beta G \to \beta G$ defined by $p_q(x) = xq$ is continuous for every $q \in \beta G$. Thus βG is a compact right topological semigroup.

For many years, one of the most difficult open questions was whether or not $\beta\mathbb{Z}$ contains a nontrivial finite group. This question has ben answered by Zelenyuk [1]: the finite groups in $\beta\mathbb{Z}$ are trivial. Indeed, Zelenyuk's method gives a triviality of the finite groups in βG in the case of any countable torsion free group [2, Theorem 8.14]. If the group G contains an element of finite order, we cannot state the trivility of finite groups in βG because $G \subset \beta G$ and G contains nontrivial finite subgroups.

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Moreover, it is easy to show a group G such that G^* has a nontrivial finite group. In this paper we give a characterization of the finite groups in βG for every countable group G and represent some applications of this characterization. As in the seminal Zelenyuk's preprint [1] we widely use the technique of the left topological groups.

1. Left topological and local left topological groups

A *left topological group* is a group provided with a topology in which left translations are continuous. Since the topology of the left topological group is uniquely determined by the filter φ of the neighborhoods at the identity, we denote the left topological group G by the pair (G, φ). A filter φ on a group G is called *left topological* if φ can be taken as the filter of neighborhoods of the identity of a left invariant topology on G. Some characterizations of the left topological filters are obtained in [3]. If φ is a left topological filter on G, then φ is a closed subsemigroup in the Stone-Čech compactification βG of the discrete group G.

Consider two types of the left invariant topologies on the group G defined by idempotents in βG. Let G be a group with the identity e and let p be an idempotent in βG. Then the family of subsets $\varphi_p = \{PY\{e\} : P \in p\}$ is a left topological filter. Denote the left topological group (G, φ_p) by $G(p)$. The topological space $G(p)$ is Hausdorff, but not necessarily regular. Put $S_p = \{x \in \beta G : xp = p\}$ and take a filter ψ_p such that $\bar{\psi}_p = S_p$. Then ψ_p is a left topological filter. Denote (G, ψ_p) by $G[p]$. By [13], the topology of $G[p]$ is Hausdorff and it has a base consisting of clopen (closed and open) subsets, so $G[p]$ is zero-dimensional.

A topological space X with a distinguished element e (the identity) and a partial binary operation (multiplication) is called a *local topological group* if there is a left topological group G such that

(1) e is the identity of G,
(2) X is an open neighborhood of e in G,
(3) the partial multiplication on X is precisely the partial operation induced on X by the multiplication of G.

A mapping f of a local left topological group X into a local left topological group Y is called a *homomorphism* if for any $x \in X$ there is a neighborhood U of th identity e such that for all $y \in U$ the products xy, $f(x)f(y)$ are defined and $f(xy) = f(x)f(y)$.

A homomorphism $f\colon X \to Y$ is called a *topological isomorphism* if f is a homeomorphism. A topological isomorphism $f\colon X \to X$ is called an *automorphism*.

An automorphism f of a local left topological group X is called a *homogeneous* automorphism of order m, $m \in \mathbb{N}$, if the f-orbit $O_f(x) = \{f^n(x) : x = 0, 1, \dots\}$ of every element $x \in X$, $x \neq e$ has a cardinality m.

Consider the following example. Let $\mathbb{Z}(m+1) = \{0, 1, \dots, m\}$ be a cyclic group and let $H(m) = \bigoplus_{n=1}^{\infty} \mathbb{Z}(m+1)$ be a direct sum provided with the topology of pointwise convergence. Thus $H(m)$ is a contable zero-dimensional topological group with countable base for its topology. Denote by g the permutation on $\mathbb{Z}(m+1)$ defined by $g(0) = 0$, $g(1) = 2$, $g(2) = 3, \dots, g(m) = 1$. The permutation g acts on the coordinates of the elements of $H(m)$, so g induces a permutation on $H(m)$. We use the same denotation g for the latter permutation. It is evident that g is a homogeneous automorphism of order m.

1.1. Theorem. *Let X be a countable nondiscrete zero-dimensional local left topological group and let f be a homogeneous automorphism of order m. Then there exists a continuous bijection $h\colon X \to H(m)$ such that h is a homomorphism and* *the following diagram is commutative*

$$\begin{array}{ccc} X & \xrightarrow{h} & H(m) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{h} & H(m). \end{array}$$

Moreover, if X has a countable base for its topology, then there exists a homeomorphism h with these properties.

Proof. [4, Theorem 1].

1.2. Corollary. *Let X, Y be countable zero-dimensional local left topological groups with the countable bases for their topologies. Then X, Y are topologically isomorphic.*

2. Characterization of finite groups in βG

2.1. Theorem. *Let G be a countable group and let Y be a finite subgroup in βG. Then $H = \{x \in G : xY = Y\}$ is a finite subgroup in G. If* $\operatorname{card} H = 1$ *then* $\operatorname{card} Y = 1$.

Proof. This is Zelenyuk's Theorem in the form given in [5].

2.2. Lemma. *Let G be a countable group and let Y be a finite subgroup in G. If p is an idempotent in βG and Yp is a subgroup in βG, then $yp = py$ for every $y \in Y$.*

Proof. Put $S_p = \{q \in \beta G : qp = p\}$ and show that $y^{-1}S_p y = S_p$ for every $y \in Y$. Then $(y^{-1}qy)p = y^{-1}q(yp) = y^{-1}q(pyp) = y^{-1}(qp)yp = y^{-1}pyp = y^{-1}(pyp) = y^{-1}yp = p$. Suppose that $y^{-1}py \neq y$ for some $y \in Y$. Consider the left topological group $G[p]$ and define a mapping $f\colon G[p] \to G[p]$ by setting $f(x) = y^{-1}xy$. Since $y^{-1}S_p y = S_p$, for every $U \in \psi_p$ ($\psi_p = S_p$) one can find $V \in \psi_p$ such that $f(V) \subset U$. The element y has a finite order. Therefore, there exists an f-invariant base of the topology of the group $G[p]$ at the identity.

Since $y^{-1}py \neq y$, there exist $P \in p$ and $m \in \mathbb{N}$, $m > 1$ such that $\operatorname{card} O_f(x) = m$ for every $x \in P$. Put $M = \{x \in G : \operatorname{card} O_f(x) = m\}$. The existence of the f-invariant base of topology of group $G[p]$ at the identity e implies that the subset M is open. Put $X = M \cup \{e\}$.

Denote by ψ the filter on G with the base $\{X \cap U : U \in \psi_p\}$. Then (G, ψ) is a left topological group and X is an open neighborhood of the identity. If $U \in \psi_p$ is clopen in $G[p]$ then $U \cap X$ is clopen in (G, ψ), so X is a zero-dimensional local left topological group. It is clear that the restriction of f to X is a homogeneous automorphism of order m.

Consider the map $h\colon X \to H(m)$ given by Theorem 1.1. Let βX and $\beta(H(m))$ be the Stone-Čech compactifications of the discrete spaces X and $H(m)$. Denote by $h^\beta\colon \beta X \to \beta(H(m))$ and $g^\beta\colon \beta(H(m)) \to \beta(H(m))$ the Stone-Čech extensions of the map $h\colon X \to H(m)$ and the permutation $g\colon H(m) \to H(m)$. By the commutativity of the diagram in Theorem 1.1, one has

$$h^\beta(p) = h^\beta(y^{-1}pyp) = h^\beta(y^{-1}py) = h^\beta(p) = g^\beta(h^\beta(p))h^\beta(p).$$

Given any $z \in H(m)$, $z \neq 0$, let $\alpha(z) = \min\{n \in \mathbb{N} : z(n) \neq 0\}$. For $i \in \{1, \dots, m\}$ put $H_i = \{z \in G \setminus \{0\} : z(\alpha(z)) = i\}$. Then $G \setminus \{0\} = H_1 \cup \dots \cup H_m$, so there exists $j \in \{1, \dots, m\}$ such that $H_j \in h^\beta(p)$. Since $h^\beta(p)$ is an idempotent, $H_{g(j)} \in g^\beta(h^\beta(p))h^\beta(p)$. But $H_j \cap H_{g(j)} = \varnothing$, so $g^\beta(h^\beta(p))h^\beta(p) \neq h^\beta(p)$. This contradiction shows that $y^{-1}py = p$ for every $y \in Y$.

Now we are going to prove the main result of this paper.

2.3. Theorem. *Let G be a countable group and F a finite subgroup with the identity p in βG. Then there exists a finite subgroup Y in G such that $F = Yp$ and $yp = py$ for each $y \in Y$.*

Proof. It suffices to find, for every $q \in F$, an element $y \in G$ such that $q = yp$ and $yp = py$.

Let Q be a subgroup of F generated by q, $Z = \{z \in G : zQ = Q\}$. Take any $z_1, z_2 \in Z$. Since $Zp \subset Q$ and p is the identity of Q, we have $(z_1p)(z_2p) = z_1(pz_2p) = z_1z_2p$. It implies that Zp is a subsemigroup of Q. Since Q is finite, Zp is a supgroup. By Lemma 2.2, $zp = pz$ for every $z \in Z$.

Take any $z \in Z$ and show that $zq = qz$. Indeed, $zq = z(pq) = (zp)q$. By the commutativity of Q, $(zp)q = q(zp)$, so $zq = qzp = qpz = qz$.

Fix $A \in q$ such that $zx = xz$ for every $x \in A$ and every $z \in Z$. Let S be a subgroup of G generated by the set $Z \cup A$. Then Z is an invariant subgroup of S and $Q \subset \beta S$. Making the factorization S/Z and using Theorem 2.1, we get $Zp = Q$.

3. Examples

In an early draft of the survey [5] Hindman posed the following problem: "We do not know whether there can be a countable noncommutative group G which has a nontrivial finite subgroup, but G^* does not." The following examples give an affirmative answer to this question.

3.1. Example. Let G be the semidirect product of $\mathbb{Z}$ and the group of order 2. We take the elements of G to be the pairs (z, i), $z \in \mathbb{Z}$, $i = 0, 1$, with the multiplication $(z_1, i_1)(z_2, i_1) = (z_1 + (-1)^{i_1}, z_2, i_1 + i_2)$. Each element of the form $(z, 1)$ has order 2. Suppose that F is a nontrivial finite subgroup in G^*. By Theorem 2.3, $F = Yp$, where Y is a nontrivial subgroup in G, p is an idempotent in G^* and $yp = py$ for every $y \in Y$. It is easy to see that $\{(z, 0) : z \in \mathbb{Z}\} \in p$. Take $y \in Y$, $y \neq (0, 0)$. Since $y = (z, 1)$ for some $z \in \mathbb{Z}$, we have $y^{-1}py = -p$, a contradiction.

This example also shows that, given any idempotent $p \in \mathbb{Z}^*$, the system of equations

$$-x + x = p, \quad x + p = x, \quad -x + p = -x$$

is unsolvable in $\mathbb{Z}^*$.

3.2. Example. We show that there exists a countable torsion group G with only trivial centre such that every proper subgroup in G is finite. Such a group has been constructed by Ol'shanskii in [6, §28]. Suppose that G^* has a nontrivial finite subgroup F. By Theorem 2.3, there exists a finite subgroup $Y \subset G$ and an idempotent $p \in G^*$ such that $F = Yp$ and $yp = py$ for every $y \in Y$. Pick $P \in p$ with $xy = yx$ for each $x \in P$. Since P generates G, y is a central element in G, a contradiction.

4. Maximal topologies on groups

A topological space X is called *maximal* if it has no isolated points and any strictly stronger topology on X has isolated points. The obstacles to progress in understanding the nature of maximality were discovered at the down of their investigation about fifty years ago. For example, it was not clear until the 80s whether there exists a ZFC example of a maximal Tychonov space. The breakthrough is due to van Douwen [7], who established that there is a countable regular maximal space in ZFC. In this connection the following question arose. Does there exist a maximal regular homogeneous space in ZFC? Such example was given in [8] (see also [2, Theorem 10.9]). We just show the way to this example. Let G be an infinite group and p an idempotent in G^*. Then the left topological group $G(p)$ is maximal. The group $G(p)$ is regular if and only if the equation $xp = p$ has only trivial solution $x = p$ in G^*. In this case p is called a regular idempotent (in [2] a regular idempotent is renamed a strongly right maximal idempotent). Using Corollary 1.2, it is proved in [8] that for every infinite group G there exists a regular idempotent $p \in G^*$. Consequently, every infinite group admits a maximal regular left invariant topology. Since a left topological group is homogeneous, we have an example of regular maximal homogeneous space. But it is still unknown whether there exists a regular maximal homogeneous space of uncountable dispersion character of a topological space is the minimal cardinality of its nonempty open subsets.

What about the maximal topological groups? Under Martin's Axiom an example of such a group has been constructed by Malykhin [9]. But as distinct from the left topological case, there exist models of ZFC without maximal topological groups [3]. In what follows we apply Theorem 2.3 to show that every maximal topological group is complete.

An ultrafilter q on a topological group (G, τ) is called *left (right) fundamental* if for every neighborhood of the identity $U \in \tau$ there exists $Q \in q$ with $Q^{-1}Q \subset U$ ($QQ^{-1} \subset U$). If q is left fundamental then q^{-1} is right fundamental.

A topological group (G, τ) is called *complete in the left (right) uniformly* if every left (right) fundamental ultrafilter on (G, τ) convergens. A topological group (G, τ) is called complete if every left and right fundamental ultrafilter on (G, τ) converges. If G is complete in the left (right) uniformity then G is complete. It should be noted that the left and right uniformities on a maximal topological group need to be equal [8]. We omit the proof of the following two simple Lemmas.

4.1. Lemma. *For every ultrafilter q on a topological group (G, τ) the following conditions are equivalent*

(1) q is left fundamental,

(2) for every $U \in \tau$ there is an element $a \in G$ such that $U \in aq$,

(3) there is an ultrafilter r on G such that $rq \in \bar{\tau}$.

4.2. Lemma. *Let S be a semigroup, $s, p \in S$, $s^2 \neq s$, $p^2 = p$. Then $sp = p$ or $\{p, sp\}$ is a subgroup of S.*

4.3. Theorem. *Every maximal topological group (G, τ) is complete in the left uniformity.*

Proof. Fix any idempotent $p \in \bar{\tau} \cap G^*$. Since (G, τ) is maximal, we have $(G, \tau) = G(p)$. By [9], $G(p)$ contains a countable open subgroup B of period 2.

Let q be a left fundamental ultrafilter on $G(p)$. Since $B \in \tau$, by Lemma 4.1, there is $a \in G$ such that $B \in aq$. Put $s = aq$ and consider s as an ultrafilter on B. Since B is an abelian group, s is a fundamental ultrafilter on B. Suppose that s does not converge in B and denote by $\hat{B}$ the completion of the topological abelian group B. Then s converges in $\hat{B}$ to some element $c \in \hat{b} \setminus B$. Since the period of B is equal to 2, we haven $c^2 = e$, where e is the identity of B. It is easy to see that $s^2 = p$, $sp \neq p$. By Lemma 4.2, $\{p, sp\}$ is a subgroup of B^*. By Theorem 2.3, $sp = bp$ for some element $b \in B$. Since B is a topological group, s converges to b in B, a contradiction.

4.4. Corollary. *Let G be a countable group and p an idempotent in G^* such that $G(p)$ is a topological group. Let $r, q \in \beta G$ and $rq = p$. Then there exists $a \in G$ such that $q = ap$.*

Proof. By Lemma 4.1, q is a left fundamental ultrafilter. By Theorem 4.3, q converges in $G(p)$ to some element a, so $q = ap$.

5. Finite regular subsemigroups in G^*

Let G be a group with identity e and K a subsemigroup in G^*. Call K a regular subsemigroup provided that $xK \cap K \neq \varnothing$, $x \in \beta G$, $x \neq e$ implies $x \in K$. If (G, τ) is a regular left topological group then $\bar{\tau} \cap G^*$ is a regular subsemigroup in G^* [3]. We complete the paper with the following result proved by Zelenyuk.

5.1. Theorem. *Let G be a countable group and K a finite regular subsemigroup in G^*. Then K is a semigroup of idempotents.*

Proof. By Theorem 2.3, every finite subgroup in K is trivial. Suppose that K is not a semigroup of idempotents. Then there exist $p, q \in K$ such that $p \neq q$ and $qq = pp = p = pq = qp$. Since $\bar{Q}q \cap \bar{P}p \neq \varnothing$ for every $Q \in q$, $P \in p$, by the Frolik Lemma [10], we have two cases.

Case 1. $Qp \cap \bar{P}p \neq \varnothing$ for every $Q \in q$, $P \in p$. Fix some $Q \in q$, $P \in p$ and take $x \in Q$, $s \in \bar{P}$ with $xq = sp$. Put $r = x^{-1}s$. Then $q = rp = (rp)p = qp = p$, a contradiction.

Case 2. $\bar{Q}q \cap Pp \neq \varnothing$ for every $Q \in q$, $P \in p$. Fix $Q \in q$, $P \in p$ and take $x(Q,P) \in P$, $s(Q,P) \in \bar{Q}$ such that $s(Q,P)q = x(Q,P)p$. Put $r(Q,P) = (x(Q,P))^{-1}s(Q,P)$. Since $r(Q,P)q = p$, by the regularity of K, $r(Q,P) \in K$. Since K is finite, we may assume that there exists $r \in G$ such that $r(Q,P) = r$ for all $Q \in q$, $P \in p$. Then $x(Q,P)r = s(Q,P)$. Passing to the limit, we have $pr = q$. Since $ppr = pr$ and $pq = p$, we have $p = q$, a contradiction.

5.2. Corollary. *Let (G, τ) be a countable Hausdorff topological group. If the semigroup $\bar{\tau}$ is finite then $\bar{\tau}$ is a semigroup of idempotents.*

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