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APPROXIMATION PROPERTIES OF SOME OPERATORS IN WEIGHTED SPACES OF FUNCTIONS OF TWO VARIABLES

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We consider the operators $L^{\{i\}}_{m,n}$ of the Szasz-Mirakjan type in some weighted spaces of continuous functions of two variables. For these operators we give theorems of degree of approximation, the Voronovkaya type theorem, the Bernstein type inequality and theorem on convergence of partial derivatives of $L^{\{i\}}_{m,n}(f;\cdot,\cdot)$.

These theorems are some analogues of the results given in [3]–[8].

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Рассматриваются операторы $L^{\{i\}}_{m,n}$ типа Саса-Миракьяна в некоторых весовых пространствах непрерывных функций двух переменных. Для этих операторов приводятся теоремы о степени аппроксимации, теорема типа Вороновской, неравенство типа Бернштейна и теорема о сходимости частичных производных функций вида $L^{\{i\}}_{m,n}(f;\cdot,\cdot)$.

Эти теоремы аналогичны некоторым результатам работ [3]–[8].

1. Preliminaries

In the papers [1] and [2] the approximation theorems are given for the Szasz-Mirakjan operators in polynomial and exponential weighted spaces of functions of one variable.

Some approximation theorems for the Szasz-Mirakjan operators in the space of continuous and bounded functions of two variables are given in [9].

In the papers [5–8] we introduced and investigated some operators of the Szasz-Mirakjan type in polynomial and exponential weighted spaces of functions of one variable. Some properties of $L^{\{i\}}_{m,n}$, $i = 1, 2$, were given in [3] and [4].

In the present paper we extend the results given in [5–8] to the operators $L^{\{i\}}_{m,n}$ introduced in the polynomial-exponential weighted space of functions of two variables.

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1.1. Let $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := (0, +\infty)$, $\mathbb{R}_0 := \mathbb{R}_+ \cup \{0\}$ and $\mathbb{R}_0^2 := \mathbb{R}_0 \times \mathbb{R}_0$. Similarly as in [1, 2], for fixed $p \in \mathbb{N}_0$ and $q \in \mathbb{R}_+$, we define the weighted functions on $\mathbb{R}_0$

$$w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1} \text{ if } p \geq 1, \tag{1}$$

$$\nu_q(x) := e^{-qx}. \tag{2}$$

Next, for these fixed p, q, we define the weighted function of two variables

$$w_{p,q}(x, y) := w_p(x)\nu_q(y), \qquad (x, y) \in \mathbb{R}_0^2, \tag{3}$$

and the weighted space $C_{p,q}$ of all real-valued functions f defined on $\mathbb{R}_0^2$ for which $w_{p,q}(\cdot, \cdot) \times$
$\times f(\cdot, \cdot)$ is uniformly continuous and bounded on $\mathbb{R}_0^2$ and the norm is given by

$$||f||_{p,q} \equiv ||f(\cdot, \cdot)||_{p,q} := \sup_{(x,y)\in\mathbb{R}_0^2} w_{p,q}(x, y)|f(x, y)|. \tag{4}$$

For $f \in C_{p,q}$ we define the modulus of continuity

$$\omega(f, C_{p,q}; t, s) := \sup_{0\leq h\leq t,\, 0\leq\delta\leq s} ||\Delta_{h,\delta} f(\cdot, \cdot)||_{p,q}, \qquad t, s \geq 0, \tag{5}$$

where $\Delta_{h,\delta} f(x, y) = f(x+h, y+\delta) - f(x, y)$. Moreover, for a fixed $m \in \mathbb{N}$ we denote by $C^m_{p,q}$ the set of all functions $f \in C_{p,q}$ having partial derivatives of the order $\leq m$ belonging also to $C_{p,q}$.

1.2. In this paper we shall consider the following operators $L^{\{i\}}_{m,n}$, $m, n \in \mathbb{N}$, $1 \leq i \leq 4$, in the space $C_{p,q}$ $(p \in \mathbb{N}_0, q \in \mathbb{R}_+)$:

$$L^{\{1\}}_{m,n}(f; x, y) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{m,j}(x)a_{n,k}(y) f\big(\tfrac{2j}{m}, \tfrac{2k}{n}\big), \tag{6}$$

$$L^{\{2\}}_{m,n}(f; x, y) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{m,j}(x)a_{n,k}(y) \frac{mn}{4} \int_{\frac{2j}{m}}^{\frac{2j+2}{m}} \int_{\frac{2k}{n}}^{\frac{2k+2}{n}} f(t, z)\, dt\, dz, \tag{7}$$

$$\begin{aligned} L^{\{3\}}_{m,n}(f; x, y) := {}& c_m(x)c_n(y)f(0, 0) + c_m(x) \sum_{k=0}^{\infty} b_{n,k}(y) f\big(0, \tfrac{2k+1}{n}\big)+ \\ &+ c_n(y) \sum_{j=0}^{\infty} b_{m,j}(x) f\big(\tfrac{2j+1}{m}, 0\big)+ \\ &+ \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{m,j}(x)b_{n,k}(y) f\big(\tfrac{2j+1}{m}, \tfrac{2k+1}{n}\big), \end{aligned} \tag{8}$$

$$\begin{aligned} L^{\{4\}}_{m,n}(f; x, y) := {}& c_m(x)c_n(y)f(0, 0) + c_m(x) \sum_{k=0}^{\infty} b_{n,k}(y) \frac{n}{2} \int_{\frac{2k+1}{n}}^{\frac{2k+3}{n}} f(0, z)\, dz+ \\ &+ c_n(y) \sum_{j=0}^{\infty} b_{m,j}(x) \frac{m}{2} \int_{\frac{2j+1}{m}}^{\frac{2j+3}{m}} f(t, 0)\, dt+ \\ &+ \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{m,j}(x)b_{n,k}(y) \frac{mn}{4} \int_{\frac{2j+1}{m}}^{\frac{2j+3}{m}} \int_{\frac{2k+1}{n}}^{\frac{2k+3}{n}} f(t, z)\, dt\, dz, \end{aligned} \tag{9}$$

$(x,y)\in\mathbb{R}_0^2$, $f\in C_{p,q}$, where

$$a_{n,k} := \frac{1}{\cosh nx}\frac{(nx)^{2k}}{(2k)!}, \tag{10}$$

$$b_{n,k} := \frac{1}{\sinh nx}\frac{(nx)^{2k+1}}{(2k+1)!}, \quad c_n(x) := \frac{1}{1+\sinh nx}, \tag{11}$$

for $k\in\mathbb{N}_0$, $n\in\mathbb{N}$, $x\in\mathbb{R}_0$ and $\sinh x$, $\cosh x$ are the elementary hyperbolic functions. These operators are analogues of the operators $L_n^{\{i\}}$, $n\in\mathbb{N}$, $1\le i\le 4$, introduced for functions of one variable in the papers [3,5], i.e.

$$L_n^{\{1\}}(f;x) := \sum_{k=0}^{\infty} a_{n,k}(x) f\big(\tfrac{2k}{n}\big), \tag{12}$$

$$L_n^{\{2\}}(f;x) := \sum_{k=0}^{\infty} a_{n,k}(x)\tfrac{n}{2}\int_{\frac{2k}{n}}^{\frac{2k+2}{n}} f(t)\,dt, \tag{13}$$

$$L_n^{\{3\}}(f;x) := c_n(x)f(0) + \sum_{k=0}^{\infty} b_{n,k}(x) f\big(\tfrac{2k+1}{n}\big), \tag{14}$$

$$L_n^{\{4\}}(f;x) := c_n(x)f(0) + \sum_{k=0}^{\infty} b_{n,k}(x)\tfrac{n}{2}\int_{\frac{2k+1}{n}}^{\frac{2k+3}{n}} f(t)\,dt, \tag{15}$$

for $x\in\mathbb{R}_0$ and $n\in\mathbb{N}$ ($L_n^{\{i\}}$, $i=1,2$, were introduced in [3], $L_n^{\{i\}}$, $i=3,4$ were defined in [5]).

In the present paper we shall examine the degree of approximation of functions $f\in C_{p,q}$ by $L_{m,n}^{\{i\}}$, the Voronovskaya type theorem, the Bernstein type inequality and the theorem on convergence of sequences $\big\{\frac{\partial}{\partial x}L_{n,n}^{\{i\}}(f;x,y)\big\}$ and $\big\{\frac{\partial}{\partial y}L_{n,n}^{\{i\}}(f;x,y)\big\}$.

In Section 2 we shall give some auxilary results. The main theorems will be given in Section 3.

Below by $M_k(a,b)$, $k=1,2,\ldots$, we shall denote the suitable positive constants depending only on indicated parameters a,b.

2. Auxiliary results

2.1. First we shall give some results on the operators $L_n^{\{i\}}$ proved in [3–8].

Lemma 1. *For all $x\in\mathbb{R}_0$, $n\in\mathbb{N}$ and $1\le i\le 4$*

$$L_n^{\{i\}}(1,x)=1, \quad |L_n^{\{i\}}(t-x;x)|\le\frac{5}{n}, \quad L_n^{\{i\}}\big((t-x)^2;x\big)\le 11\frac{x+1}{n}. \tag{16}$$

Lemma 2. *For every fixed $x_0\in\mathbb{R}_0$ there exists a positive constant $M_1(x_0)$ such that*

$$L_n^{\{i\}}\big((t-x_0)^4;x_0\big)\le M_1(x_0)\cdot n^{-2}$$

for all $n\in\mathbb{N}$ and $1\le i\le 4$.

Lemma 3. *For every fixed $x_0 \in \mathbb{R}_0$ we have*

$$\lim_{n\to\infty} nL_n^{\{i\}}(t-x_0;x_0) = \begin{cases} 0 & \text{if } i=1,3, \\ 1 & \text{if } i=2, \end{cases}$$

$$\lim_{n\to\infty} nL_n^{\{4\}}(t-x_0;x_0) = \begin{cases} 0 & \text{if } x_0=0, \\ 1 & \text{if } x_0>0, \end{cases}$$

$$\lim_{n\to\infty} nL_n^{\{i\}}\big((t-x_0)^2;x_0\big) = x_0 \qquad \text{for } 1\le i\le 4.$$

Lemma 4. *For every fixed $p \in \mathbb{N}_0$ there exist positive constants $M_k(p)$, $k=2,3$, such that for all $x\in\mathbb{R}_0$, $n\in\mathbb{N}$ and $1\le i\le 4$*

$$w_p(x)L_n^{\{i\}}\Big(\frac{1}{w_p(t)};x\Big) \le M_2(p),$$

$$w_p(x)L_n^{\{i\}}\Big(\frac{(t-x)^2}{w_p(t)};x\Big) \le M_3(p)\frac{x+1}{n}.$$

Lemma 5. *For every fixed $q>0$ and $r>q$ there exist positive constants $M_k(q,r)$, $k=4,5$, and a natural number n_0,*

$$n_0 > q\big(\ln \tfrac{r}{q}\big)^{-1}, \tag{17}$$

such that for all $n>n_0$, $x\in\mathbb{R}_0$ and $1\le i\le 4$

$$v_r(x)L_n^{\{i\}}\Big(\frac{1}{v_q(t)};x\Big) \le M_4(q,r),$$

$$v_r(x)L_n^{\{i\}}\Big(\frac{(t-x)^2}{v_q(t)};x\Big) \le M_5(q,r)\frac{x+1}{n}.$$

Lemma 6. *For every fixed $p\in\mathbb{N}_0$ and $s\in\mathbb{N}$ there exists positive constants $M_6(s)$ and $M_7(p,s)$ such that for all $n\in\mathbb{N}$ and $x\in\mathbb{R}_0$*

$$\Big|\frac{d^s}{dx^s}\frac{1}{1+\sinh nx}\Big| \le M_6(s)\frac{n^s}{1+\sinh nx},$$

$$w_p(x)\sum_{k=0}^{\infty}\Big|\frac{d^s}{dx^s}a_{n,k}(x)\Big|\Big(w_p\Big(\frac{2k}{n}\Big)\Big)^{-1} \le M_7(p,s)n^s,$$

$$w_p(x)\sum_{k=0}^{\infty}\Big|\frac{d^s}{dx^s}b_{n,k}(x)\Big|\Big(w_p\Big(\frac{2k+1}{n}\Big)\Big)^{-1} \le M_7(p,s)n^s.$$

Lemma 7. *For every fixed $s\in\mathbb{N}$ and $r>q>0$ there exist a positive constant $M_8(q,r,s)$ and a natural number n_0 satisfying condition (17) such that for all $x\in\mathbb{R}_0$ and $n>n_0$*

$$v_r(x)\sum_{k=0}^{\infty}\Big|\frac{d^s}{dx^s}a_{n,k}(x)\Big|\Big(v_q\Big(\frac{2k}{n}\Big)\Big)^{-1} \le M_8(q,r,s)n^s,$$

$$v_r(x)\sum_{k=0}^{\infty}\Big|\frac{d^s}{dx^s}b_{n,k}(x)\Big|\Big(v_q\Big(\frac{2k+1}{n}\Big)\Big)^{-1} \le M_8(q,r,s)n^s.$$

2.2. In this part we shall give some basic properties of the operators $L_{m,n}^{\{i\}}$ in the space $C_{p,q}$.

From (6)–(15) it follows that

$$L_{m,n}^{\{i\}}(1;x,y)=1 \quad \text{for} \quad (x,y)\in\mathbb{R}_0^2,\ m,n\in\mathbb{N},\ 1\le i\le 4. \tag{18}$$

Moreover, if $f\in C_{p,q}$ with some $p\in\mathbb{N}_0$, $q\in\mathbb{R}_+$, and $f(x,y)=f_1(x)f_2(y)$ for $(x,y)\in\mathbb{R}_0^2$, then

$$L_{m,n}^{\{i\}}\big(f_1(t)f_2(z);x,y\big)=L_m^{\{i\}}\big(f_1(t);x\big)L_n^{\{i\}}\big(f_2(z);y\big) \tag{19}$$

for all $(x,y)\in\mathbb{R}_0^2$, $m,n\in\mathbb{N}$ and $1\le i\le 4$.

Applying Lemmas 1–7 and (18), (19), we shall prove four lemmas.

Lemma 8. *For every fixed $p\in\mathbb{N}_0$ and $r>q>0$ there exist a positive constant $M_9(p,q,r)$ and a natural number n_0 satisfying (17) such that*

$$\Big\|L_{m,n}^{\{i\}}\Big(\frac{1}{w_{p,q}(t,z)};\cdot,\cdot\Big)\Big\|_{p,r}\le M_9(p,q,r), \tag{20}$$

for all $m\ge 1$, $n>n_0$ and $1\le i\le 4$.

Proof. From (1)–(3), (6)–(15) and (19) we get for all $(x,y)\in\mathbb{R}_0^2$, $m,n\in\mathbb{N}$ and $1\le i\le 4$

$$w_{p,r}(x,y)L_{m,n}^{\{i\}}\Big(\frac{1}{w_{p,q}(t,z)};x,y\Big)=\Big\{w_p(x)L_m^{\{i\}}\Big(\frac{1}{w_p(t)};x\Big)\Big\}\Big\{v_r(x)L_n^{\{i\}}\Big(\frac{1}{v_q(t)};y\Big)\Big\}.$$

Now, using the suitable inequalities given in Lemmas 4 and 5 and by (4), we immediately obtain (20) for $m,n\in\mathbb{N}$ but $n>n_0$ and $1\le i\le 4$. □

Lemma 9. *Let p,q,r,n_0 be fixed positive numbers as in Lemma 8. Then there exists a positive constant $M_{10}(p,q,r)$ such that for every $f\in C_{p,q}$, $1\le i\le 4$ and for all $m\ge 1$, $n>n_0$*

$$\|L_{m,n}^{\{i\}}(f;\cdot,\cdot)\|_{p,r}\le M_{10}(p,q,r)\|f\|_{p,q}. \tag{21}$$

This inequality and (6)–(11) show that $L_{m,n}^{\{i\}}$ with fixed $m\ge 1$, $n>n_0$ and $1\le i\le 4$ is a positive linear operator from the space $C_{p,q}$ into $C_{p,r}$, $p\in\mathbb{N}_0$, $r>q>0$.

Proof. From (6)–(11) and (1)–(4) it follows that

$$\|L_{m,n}^{\{i\}}(f;\cdot,\cdot)\|_{p,r}\le\|f\|_{p,q}\Big\|L_{m,n}^{\{i\}}\Big(\frac{1}{w_{p,q}(t,z)};\cdot,\cdot\Big)\Big\|_{p,r}$$

for any $f\in C_{p,q}$, $m,n\in\mathbb{N}$ and $1\le i\le 4$. Now using Lemma 8, we obtain (21) for $m\ge 1$, $n>n_0$ and $1\le i\le 4$. □

Lemma 10. *Suppose that $(x_0,y_0)\in\mathbb{R}_0^2$ is a fixed point and let φ be a given function from the space $C_{p,q}$ ($p\in\mathbb{N}_0$, $q\in\mathbb{R}_+$) such that $\varphi(x_0,y_0)=0$. Then*

$$\lim_{n\to\infty} L_{n,n}^{\{i\}}\big(\varphi(t,z);x_0,y_0\big)=0, \quad 1\le i\le 4.$$

Proof. Let $i=1$. Choose $\varepsilon>0$, $r>q$ and $n_0\in\mathbb{N}$ satisfying (17). By the assumtions on φ there exist two positive constants $\delta=\delta(\varepsilon)$ and M_{11} such that

$$w_{p,q}(t,z)|\varphi(t,z)|\le M_{11} \quad \text{for all } \ (t,z)\in\mathbb{R}_0^2, \tag{23}$$

$$w_{p,q}(t,z)|\varphi(t,z)|\le \frac{\varepsilon}{4M_9^*} \quad \text{for } \ |t-x_0|<\delta,\ |z-y_0|<\delta, \tag{24}$$

where $M_9^*\equiv M_9(p,q,r)$ is a fixed positive constant given by Lemma 8. Moreover, by (6) and (10), we can write for every $n\in\mathbb{N}$

$$w_{p,r}(x_0,y_0)\big|L_{n,n}^{\{1\}}\big(\varphi(t,z);x_0,y_0\big)\big|\le w_{p,r}(x_0,y_0)\Bigg\{\sum_{|\frac{2j}{n}-x_0|<\delta}\ \sum_{|\frac{2k}{n}-y_0|<\delta}+$$
$$+\sum_{|\frac{2j}{n}-x_0|<\delta}\ \sum_{|\frac{2k}{n}-y_0|\ge\delta}+\sum_{|\frac{2j}{n}-x_0|\ge\delta}\ \sum_{|\frac{2k}{n}-y_0|<\delta}+$$
$$+\sum_{|\frac{2j}{n}-x_0|\ge\delta}\ \sum_{|\frac{2k}{n}-y_0|\ge\delta}\Bigg\}a_{n,j}(x_0)a_{n,k}(y_0)\Big|\varphi\Big(\frac{2j}{n},\frac{2k}{n}\Big)\Big|:=S_1+S_2+S_3+S_4. \tag{25}$$

Applying (24) and (20), we have

$$S_1\le\frac{\varepsilon}{4M_9^*}w_{p,r}(x_0,y_0)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{n,j}(x_0)a_{n,k}(y_0)\Big(w_{p,q}\Big(\frac{2j}{n},\frac{2k}{n}\Big)\Big)^{-1}\le$$
$$\le\frac{\varepsilon}{4M_9^*}\Big\|L_{n,n}^{\{1\}}\Big(\frac{1}{w_{p,q}(t,z)};\cdot,\cdot\Big)\Big\|_{p,r}\le\frac{\varepsilon}{4} \quad \text{for } n>n_0.$$

Observing that $1\le\big(\frac{2k}{n}-y_0\big)^2\delta^{-2}$ if $\big|\frac{2k}{n}-y_0\big|\ge\delta$, we get by (23), (3) and (19)

$$S_2\le M_{11}w_{p,r}(x_0,y_0)\sum_{|\frac{2j}{n}-x_0|<\delta}\ \sum_{|\frac{2k}{n}-y_0|\ge\delta}a_{n,j}(x_0)a_{n,k}(y_0)\Big(w_{p,q}\Big(\frac{2j}{n},\frac{2k}{n}\Big)\Big)^{-1}\le$$
$$\le M_{11}\delta^{-2}\Big\{w_p(x_0)\sum_{j=0}^{\infty}a_{n,j}(x_0)\Big(w_p\Big(\frac{2j}{n}\Big)\Big)^{-1}\Big\}\times$$
$$\times\Big\{v_r(y_0)\sum_{|\frac{2k}{n}-y_0|\ge\delta}a_{n,k}(y_0)\Big(\frac{2k}{n}-y_0\Big)^2\Big(v_q\Big(\frac{2k}{n}\Big)\Big)^{-1}\Big\}\le$$
$$\le M_{11}\delta^{-2}\Big\{w_p(x_0)L_n^{\{1\}}\Big(\frac{1}{w_p(t)};x_0\Big)\Big\}\Big\{v_r(y_0)L_n^{\{1\}}\Big(\frac{(t-y_0)^2}{v_q(z)};y_0\Big)\Big\}$$

for $n\in\mathbb{N}$, which by Lemma 4 and Lemma 5 yields

$$S_2\le M_{11}M_2(p)M_5(q,r)\frac{y_0+1}{n\delta^2}\equiv M_{12}(p,q,r)\frac{y_0+1}{n\delta^2}$$

for all $n > n_0$. Analogously, by Lemma 4 and Lemma 5 we obtain for $n > n_0$

$$S_3 \le M_{13}(p,q,r)\frac{x_0+1}{n\delta^2},$$
$$S_4 \le M_{14}(p,q,r)\frac{(x_0+1)(y_0+1)}{n^2\delta^2}.$$

But for fixed $(x_0, y_0) \in \mathbb{R}_0^2$ and for fixed positive numbers ε, δ, $M_k(p,q,r)$, $12 \le k \le 14$, there exist natural numbers $n_i > n_0$, $i = 1,2,3$, such that

$$M_{12}(p,q,r)\frac{y_0+1}{n\delta^2} < \frac{\varepsilon}{4} \qquad \text{for } n > n_1,$$
$$M_{13}(p,q,r)\frac{x_0+1}{n\delta^2} < \frac{\varepsilon}{4} \qquad \text{for } n > n_2,$$
$$M_{14}(p,q,r)\frac{(x_0+1)(y_0+1)}{n^2\delta^2} < \frac{\varepsilon}{4} \qquad \text{for } n > n_3.$$

Hence, denoting by $n_4 = \max\{n_1, n_2, n_3\}$, we get

$$S_k < \frac{\varepsilon}{4} \quad \text{for all } n > n_4 \text{ and } 1 \le k \le 4$$

which by (25) implies

$$w_{p,r}(x_0,y_0)\big|L_{n,n}^{\{1\}}\big(\varphi(t,z);x_0,y_0\big)\big| < \varepsilon \qquad \text{for all } n > n_4.$$

This fact and (3) prove that (22) is satisfied for $i = 1$. The proof of (22) for $i = 2,3,4$ is similar. □

3. Main theorems

3.1. First we shall prove two theorems on the degree of approximation of functions $f \in C_{p,q}$ by the operators $L_{m,n}^{\{i\}}$.

Theorem 1. *Suppose that $g \in C_{p,q}^1$ with some $p \in \mathbb{N}_0$, $q > 0$. Then for every fixed $r > q$ there exist a positive constant $M_{15}(p,q,r)$ and a natural number n_0 satisfying the condition (17) such that for all $(x,y) \in \mathbb{R}_0^2$, $m \ge 1$, $n > n_0$ and $1 \le i \le 4$*

$$w_{p,r}(x,y)|L_{m,n}^{\{i\}}(g;x,y) - g(x,y)| \le$$
$$\le M_{15}(p,q,r)\left\{\left\|\frac{\partial g}{\partial x}\right\|_{p,q}\sqrt{\frac{x+1}{m}} + \left\|\frac{\partial g}{\partial y}\right\|_{p,q}\sqrt{\frac{y+1}{n}}\right\}. \quad (26)$$

Proof. We shall prove (26) only for $i = 4$ because the proof of (26) for $i = 1,2,3$ is analogous.

Let $i = 4$ and let (x,y) be a fixed point in $\mathbb{R}_0^2$. For $g \in C_{p,q}^1$ we have

$$g(t,z) - g(x,y) = \int_x^t \frac{\partial g}{\partial u}(u,z)\,du + \int_y^z \frac{\partial g}{\partial v}(x,v)\,dv, \qquad (t,z) \in \mathbb{R}_0^2.$$

From this and by (18) we get for $m,n\in\mathbb{N}$

$$L^{\{4\}}_{m,n}\big(g(t,z);x,y\big)-g(x,y)= \\ =L^{\{4\}}_{m,n}\Big(\int_x^t \frac{\partial g}{\partial u}(u,z)\,du;x,y\Big)+L^{\{4\}}_{m,n}\Big(\int_y^z \frac{\partial g}{\partial v}(v,z)\,dv;x,y\Big). \quad (27)$$

Moreover, by (1)–(4), we have

$$\Big|\int_x^t \frac{\partial g}{\partial u}(u,z)\,du\Big|\le\Big\|\frac{\partial g}{\partial x}\Big\|_{p,q}\Big|\int_x^t\frac{du}{w_{p,q}(u,z)}\Big|\le\Big\|\frac{\partial g}{\partial x}\Big\|_{p,q}\Big(\frac{1}{w_{p,q}(t,z)}+\frac{1}{w_{p,q}(x,z)}\Big)|t-x|$$

and analogously

$$\Big|\int_y^z \frac{\partial g}{\partial v}(x,v)\,dv\Big|\le\Big\|\frac{\partial g}{\partial y}\Big\|_{p,q}\Big(\frac{1}{w_{p,q}(x,z)}+\frac{1}{w_{p,q}(x,y)}\Big)|z-y|.$$

Using the above inequalities and by (3), (19) and (16), we get for $m,n\in\mathbb{N}$

$$w_{p,r}(x,y)\,\Big|L^{\{4\}}_{m,n}\Big(\int_x^t\frac{\partial g}{\partial u}(u,z)\,du;x,y\Big)\Big|\le \\ \le w_{p,r}(x,y)L^{\{4\}}_{m,n}\Big(\Big|\int_x^t\frac{\partial g}{\partial u}(u,z)\,du\Big|;x,y\Big)\le \\ \le\Big\|\frac{\partial g}{\partial x}\Big\|_{p,q}w_{p,r}(x,y)\Big\{L^{\{4\}}_{m,n}\Big(\frac{|t-x|}{w_{p,q}(t,z)};x,y\Big)+L^{\{4\}}_{m,n}\Big(\frac{|t-x|}{w_{p,q}(x,z)};x,y\Big)\Big\}\le \\ \le\Big\|\frac{\partial g}{\partial x}\Big\|_{p,q}v_r(y)L^{\{4\}}_{n}\Big(\frac{1}{v_q(z)};y\Big)\Big\{w_p(x)L^{\{4\}}_{m}\Big(\frac{|t-x|}{w_p(t)};x\Big)+L^{\{4\}}_{m}\big(|t-x|;x\big)\Big\}$$

and analogously

$$w_{p,r}(x,y)\,\Big|L^{\{4\}}_{m,n}\Big(\int_y^z\frac{\partial g}{\partial v}(x,v)\,dv;x,y\Big)\Big|\le \\ \le\Big\|\frac{\partial g}{\partial y}\Big\|_{p,q}\Big\{v_r(y)L^{\{4\}}_{n}\Big(\frac{|z-y|}{v_q(z)};y\Big)+L^{\{4\}}_{n}\big(|z-y|;y\big)\Big\}.$$

Using the Hölder inequality, Lemma 1 and Lemma 4, we get

$$L^{\{4\}}_{m}\big(|t-x|;x\big)\le 2\big\{L^{\{4\}}_{m}\big((t-x)^2;x\big)\big\}^{1/2}\le 8\Big(\frac{x+1}{m}\Big)^{1/2},$$

$$w_p(x)L^{\{4\}}_{m}\Big(\frac{|t-x|}{w_p(t)};x\Big)\le\Big\{w_p(x)L^{\{4\}}_{m}\Big(\frac{(t-x)^2}{w_p(t)};x\Big)\Big\}^{1/2}\Big\{w_p(x)L^{\{4\}}_{m}\Big(\frac{1}{w_p(t)};x\Big)\Big\}^{1/2}\le \\ \le M_{16}(p)\Big(\frac{x+1}{m}\Big)^{1/2},$$

for all $m\in\mathbb{N}$, and analogously for all $n>n_0$

$$L^{\{4\}}_{n}\big(|z-y|;y\big)\le 8\Big(\frac{y+1}{m}\Big)^{1/2},$$

$$v_r(y)L^{\{4\}}_{n}\Big(\frac{|z-y|}{v_q(z)};y\Big)\le M_{17}(q,r)\Big(\frac{y+1}{n}\Big)^{1/2}.$$

Combining these, we derive from (27)

$$w_{p,r}(x,y)|L^{\{4\}}_{m,n}\big((g(t,z);x,y\big)-g(x,y)| \leq$$
$$\leq M_{18}(p,q,r)\Big\{\Big\|\frac{\partial g}{\partial x}\Big\|_{p,q}\sqrt{\frac{x+1}{m}}+\Big\|\frac{\partial g}{\partial y}\Big\|_{p,q}\sqrt{\frac{x+1}{n}}\Big\}.$$

for $m \geq 1$ and $n > n_0$. Thus the proof of (26) for $i = 4$ is complete. □

Applying Theorem 1 and the above lemmas, we shall prove the main approximation theorem.

Theorem 2. *Let $p \in \mathbb{N}_0$, $r > q > 0$ and let $n_0 \in \mathbb{N}$ be a fixed number satisfying the condition (17). Then there exists a positive constant $M_{19}(p,q,r)$ such that for any $f \in C_{p,q}$, $(x,y) \in \mathbb{R}^2_0$, $m \geq 1$, $n > n_0$ and $1 \leq i \leq 4$*

$$w_{p,r}(x,y)|L^{\{i\}}_{m,n}(f;x,y)-f(x,y)| \leq M_{19}(p,q,r)\,\omega\Big(f,C_{p,q};\sqrt{\frac{x+1}{m}},\sqrt{\frac{y+1}{n}}\Big), \tag{28}$$

where ω is the modulus of continuity of f defined by (5).

Proof. We use the Steklov mean $f_{h,\delta}$ of function $f \in C_{p,q}$ defined by the formula

$$f_{h,\delta} := \frac{1}{h\delta}\int_0^h\int_0^\delta f(x+u,y+v)\,du\,dv, \quad (x,y) \in \mathbb{R}^2_0,\ h,\delta > 0.$$

From this we get for $(x,y) \in \mathbb{R}^2_0$ and $h,\delta > 0$

$$f_{h,\delta}(x,y)-f(x,y) = \frac{1}{h\delta}\int_0^h\int_0^\delta \Delta_{u,v}f(x,y)\,du\,dv,$$
$$\frac{\partial}{\partial x}f_{h,\delta}(x,y) = \frac{1}{h\delta}\int_0^\delta \Delta_{h,0}f(x,y+v)\,dv,$$
$$\frac{\partial}{\partial y}f_{h,\delta}(x,y) = \frac{1}{h\delta}\int_0^h \Delta_{0,\delta}f(x+u,y)\,du,$$

which show that if $f \in C_{p,q}$ ($p \in \mathbb{N}_0$, $q \in \mathbb{R}_+$), then $f_{h,\delta} \in C^1_{p,q}$ for any fixed $h,\delta > 0$. Moreover, for every $r > q$ we have

$$\|f_{h,\delta}-f\|_{p.r} \leq \|f_{h,\delta}-f\|_{p,q} \leq \omega(f,C_{p,q};h,\delta), \tag{29}$$
$$\Big\|\frac{\partial f_{h,\delta}}{\partial x}\Big\|_{p,q} \leq 2h^{-1}\omega(f,C_{p,q};h,\delta), \tag{30}$$
$$\Big\|\frac{\partial f_{h,\delta}}{\partial y}\Big\|_{p,q} \leq 2\delta^{-1}\omega(f,C_{p,q};h,\delta). \tag{31}$$

Next, by (1)–(11), we can write

$$w_{p,r}(x,y)\big|L^{\{i\}}_{m,n}(f;x,y)-f(x,y)\big| \leq w_{p,r}(x,y)\Big\{\big|L^{\{i\}}_{m,n}\big(f(t,z)-f_{h,\delta}(t,z);x,y\big)\big|+$$
$$+\big|L^{\{i\}}_{m,n}\big(f_{h,\delta}(t,z);x,y\big)-f_{h,\delta}(x,y)\big|+\big|f_{h,\delta}(x,y)-f(x,y)\big|\Big\} := A_1+A_2+A_3, \tag{32}$$

for $(x,y)\in\mathbb{R}_0^2$, $m,n\in\mathbb{N}$, $h,\delta\in\mathbb{R}_+$ and $1\le i\le 4$.

Applying Lemma 9 and (29), we get

$$A_1\le \big\|L_{m,n}^{\{i\}}(f-f_{h,\delta};\cdot,\cdot)\big\|_{p,r}\le M_{10}(p,q,r)\|f-f_{h,\delta}\|_{p,q}\le M_{10}(p,q,r)\,\omega(f,C_{p,q,r};h,\delta),$$

for all $m\ge 1$, $n>n_0$, $1\le i\le 4$ and $h,\delta\in\mathbb{R}_+$. By Theorem 1 and (29)–(31) we have

$$\begin{aligned}A_2&\le M_{15}(p,q,r)\Big\{\Big\|\frac{\partial f_{h,\delta}}{\partial x}\Big\|_{p,q}\sqrt{\frac{x+1}{m}}+\Big\|\frac{\partial f_{h,\delta}}{\partial y}\Big\|_{p,q}\sqrt{\frac{y+1}{n}}\Big\}\le\\&\le 2M_{15}(p,q,r)\,\omega(f,C_{p,q};h,\delta)\Big\{h^{-1}\sqrt{\frac{x+1}{m}}+\delta^{-1}\sqrt{\frac{y+1}{n}}\Big\},\\A_3&\le\omega(f,C_{p,q};h,\delta),\end{aligned}$$

for all $(x,y)\in\mathbb{R}_0^2$, $m\ge 1$, $n>n_0$, $1\le i\le 4$ and $h,\delta\in\mathbb{R}_+$. Hence we get from (32)

$$\begin{aligned}&w_{p,r}(x,y)|L_{m,n}^{\{i\}}(f;x,y)-f(x,y)|\le\\&\qquad\le M_{20}(p,q,r)\,\omega(f,C_{p,q};h,\delta)\Big\{1+h^{-1}\sqrt{\frac{x+1}{m}}+\delta^{-1}\sqrt{\frac{y+1}{n}}\Big\}\end{aligned}\tag{33}$$

for all $(x,y)\in\mathbb{R}_0^2$, $m\ge 1$, $n>n_0$, $1\le i\le 4$ and $h,\delta>0$.

Now, for every fixed (x,y) and m,n, setting $h=\sqrt{\frac{x+1}{m}}$ and $\delta=\sqrt{\frac{y+1}{n}}$ to (33), we obtain the desired estimation (27). □

Theorem 2 implies the following

Corollary 1. *Let $f\in C_{p,q}$ with some $p\in\mathbb{N}_0$ and $q\in\mathbb{R}_+$. Then for every $(x,y)\in\mathbb{R}_0^2$ and $1\le i\le 4$*

$$\lim_{m,n\to\infty}L_{m,n}^{\{i\}}(f,x,y)=f(x,y).\tag{34}$$

Moreover, (34) holds uniformly on every rectangle $0\le x\le a$, $0\le y\le b$.

3.2. Now we shall prove the Voronovskaya type theorem

Theorem 3. *Suppose that $f\in C_{p,q}^2$ with some $p\in\mathbb{N}_0$ and $q\in\mathbb{R}_+$. Then for every $(x,y)\in\mathbb{R}_0^2$*

$$\begin{aligned}&\lim_{n\to\infty}n\big\{L_{n,n}^{\{i\}}(f;x,y)-f(x,y)\big\}=\\&\qquad=\begin{cases}\frac{x}{2}\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{y}{2}\frac{\partial^2 f}{\partial y^2}(x,y) & \text{if }\ i=1,3,\\ \frac{\partial f}{\partial x}(x,y)+\frac{\partial f}{\partial y}(x,y)+\frac{x}{2}\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{y}{2}\frac{\partial^2 f}{\partial y^2}(x,y) & \text{if }\ i=2.\end{cases}\end{aligned}\tag{35}$$

Moreover,

$$\begin{aligned}&\lim_{n\to\infty}n\big\{L_{n,n}^{\{4\}}(f;x,y)-f(x,y)\big\}=\\&\qquad=\begin{cases}0 & \text{if }\ x=y=0,\\ \frac{\partial f}{\partial x}(x,y)+\frac{\partial f}{\partial y}(x,y)+\frac{x}{2}\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{y}{2}\frac{\partial^2 f}{\partial y^2}(x,y) & \text{if }\ x^2+y^2>0.\end{cases}\end{aligned}\tag{36}$$

Proof. Fix $(x_0, y_0) \in \mathbb{R}_0^2$. Then, by the Taylor formula we can write for $(t, z) \in \mathbb{R}_0^2$

$$\begin{aligned} f(t,z) - f(x_0,y_0) &= \frac{\partial f}{\partial x}(x_0,y_0)(t-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(z-y_0) + \\ &+ \frac{1}{2}\left\{ \frac{\partial^2 f}{\partial x^2}(x_0,y_0)(t-x_0)^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}(x_0,y_0)(t-x_0)(z-y_0) + \right. \\ &+ \left. \frac{\partial^2 f}{\partial y^2}(x_0,y_0)(z-y_0)^2 \right\} + \psi(t,z;x_0,y_0)\sqrt{(t-x_0)^4 + (z-y_0)^4}, \end{aligned} \tag{37}$$

where $\psi(\cdot,\cdot) \equiv \psi(\cdot,\cdot;x_0,y_0) \in C_{p,q}$ and $\lim\limits_{(t,z)-(x_0,y_0)} \psi(t,z) = 0$. Applying (16), (18) and (19) we derive from (37)

$$\begin{aligned} L_{n,n}^{\{i\}}\big(f(t,z);&x_0,y_0\big) - f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0) L_n^{\{i\}}(t-x_0;x_0) + \\ &+ \frac{\partial f}{\partial y}(x_0,y_0) L_n^{\{i\}}(z-y_0;y_0) + \frac{1}{2}\left\{ \frac{\partial^2 f}{\partial x^2}(x_0,y_0) L_n^{\{i\}}\big((t-x_0)^2;x_0\big) + \right. \\ &+ 2\frac{\partial^2 f}{\partial x\,\partial y}(x_0,y_0) L_n^{\{i\}}(t-x_0;x_0) L_n^{\{i\}}(z-y_0;y_0) + \\ &+ \left. \frac{\partial^2 f}{\partial y^2}(x_0,y_0) L_n^{\{i\}}\big((z-y_0)^2;y_0\big) \right\} + \\ &+ L_{n,n}^{\{i\}}\big(\psi(t,z)\sqrt{(t-x_0)^4 + (z-y_0)^4};x_0,y_0\big) \end{aligned} \tag{38}$$

for $n \in \mathbb{N}$ and $1 \le i \le 4$.

Using the Hölder inequality, we get for $n \in \mathbb{N}$ and $1 \le i \le 4$

$$\begin{aligned} &\left| L_{n,n}^{\{i\}}\big(\psi(t,z)\sqrt{(t-x_0)^4 + (z-y_0)^4};x_0,y_0\big) \right| \le \\ &\quad \le M^* \big\{ L_{n,n}^{\{i\}}\big(\psi^2(t,z);x_0,y_0\big) \big\}^{1/2} \big\{ L_{n,n}^{\{i\}}\big((t-x_0)^4 + (z-y_0)^4;x_0,y_0\big) \big\}^{1/2}, \end{aligned} \tag{39}$$

where $M^* = 1$ if $i = 1,2$ and $M^* = 4$ if $i = 3,4$. Next, by the linearity of $L_{n,n}^{\{i\}}$ and by (16), (18), (19) and Lemma 2, we have for $n \in \mathbb{N}$ and $1 \le i \le 4$

$$\begin{aligned} &L_{n,n}^{\{i\}}\big((t-x_0)^4 + (z-y_0)^4;x_0,y_0\big) = \\ &\quad = L_n^{\{i\}}\big((t-x_0)^4;x_0\big) + L_n^{\{i\}}\big((z-y_0)^4;y_0\big) \le M_{21}(x_0,y_0)\cdot n^{-2}. \end{aligned} \tag{40}$$

The properties of $\psi(\cdot,\cdot)$ imply that we can apply Lemma 10 for $\varphi(t,z) = \psi^2(t,z)$. Hence

$$\lim_{n\to\infty} L_{n,n}^{\{i\}}\big(\psi^2(t,z);x_0,y_0\big) = 0 \quad \text{for} \quad 1 \le i \le 4,$$

and from (39)–(41) we get for $1 \le i \le 4$

$$\lim_{n\to\infty} n L_{n,n}^{\{i\}}\big(\psi(t,z)\sqrt{(t-x_0)^4 + (z-y_0)^4};x_0,y_0\big) = 0 \quad .$$

Now, using (42) and Lemma 3 to (38), we obtain (35) and (36) for given $(x_0, y_0) \in \mathbb{R}_0^2$. Thus the proof is completed. □

3.3. Applying Lemma 6 and Lemma 7 we shall prove the Bernstein type inequality for the operators $L_{m,n}^{\{i\}}$.

Theorem 4. *Let p, q, r and n_0 be fixed numbers satisfying the conditions of Theorem 2 and $s_1, s_2 \in \mathbb{N}_0$. Then there exists a positive constant $M^*_{22} \equiv M_{22}(p,q,r,s_1,s_2)$ such that for any $f \in C_{p,q}$, $1 \le i \le 4$, $m \ge 1$ and $n > n_0$*

$$\Big\|\frac{\partial^{s_1+s_2}}{\partial x^{s_1}\partial y^{s_2}} L^{\{i\}}_{m,n}(f;x,y)\Big\|_{p,r} \le M^*_{22} m^{s_1} n^{s_2} \|f\|_{p,q}. \tag{43}$$

Proof. Inequality (43) for $s_1 = s_2 = 0$ and $1 \le i \le 4$ is given in Lemma 9. Let $i = 1$ and $s_1^2 + s_2^2 \ge 1$. By (1)–(4) we get from (6)

$$\begin{aligned} w_{p,r}(x,y)\Big|\frac{\partial^{s_1+s_2}}{\partial x^{s_1}\partial y^{s_2}} L^{\{1\}}_{m,n}(f;x,y)\Big| &\le \\ &\le w_{p,r}(x,y)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Big|\frac{d^{s_1}}{dx^{s_1}}a_{m,j}(x)\Big|\Big|\frac{d^{s_2}}{dx^{s_2}}b_{n,k}(x)\Big|\Big|f\Big(\frac{2j}{m},\frac{2k}{n}\Big)\Big| \le \\ &\le \|f\|_{p,q}\Big(w_p(x)\sum_{j=0}^{\infty}\big|a^{(s_1)}_{m,j}(x)\big|\frac{1}{w_p\big(\frac{2j}{m}\big)}\Big)\Big(v_r(x)\sum_{k=0}^{\infty}\big|b^{(s_2)}_{n,k}(y)\big|\frac{1}{v_q\big(\frac{2k}{n}\big)}\Big) \end{aligned}$$

for $(x,y) \in \mathbb{R}^2_0$ and $m,n \in \mathbb{N}$. Next applying Lemma 6 and Lemma 7, we immediately obtain (43) for $m \ge 1$, $n \ge n_0 + 1$ and $i = 1$.

The proof of (43) for $i = 2, 3, 4$ is analogous. □

3.4. In final part we shall give some analogue of (34) for partial derivatives of $L^{\{i\}}_{n,n}(f;\cdot,\cdot)$.

Theorem 5. *Let $f \in C^1_{p,q}$ with some $p \in \mathbb{N}_0$ and $q \in \mathbb{R}_+$. Then for every $(x,y) \in \mathbb{R}^2_+$ and $1 \le i \le 4$*

$$\lim_{n\to\infty}\frac{\partial}{\partial x}L^{\{i\}}_{n,n}(f;x,y) = \frac{\partial f}{\partial x}(x,y), \tag{44}$$

$$\lim_{n\to\infty}\frac{\partial}{\partial y}L^{\{i\}}_{n,n}(f;x,y) = \frac{\partial f}{\partial y}(x,y). \tag{45}$$

Proof. We shall prove only (44) for $i = 1$ and $i = 3$, because the proof of (44) for $i = 2, 4$ and (45) for $1 \le i \le 4$ is analogous.

Let $(x,y) \in \mathbb{R}^2_+$ be a fixed point. Denoting by

$$T(nx) := \frac{\cosh nx}{1 + \sinh nx} \quad \text{for} \quad x \in \mathbb{R}_0,\ n \in \mathbb{N}, \tag{46}$$

we get from (6) and (8) by (10)–(19)

$$\frac{\partial}{\partial x}L^{\{1\}}_{n,n}\big(f(t,z);x,y\big) = -n\tanh nx L^{\{1\}}_{n,n}\big(f(t,z);x,y\big) + \frac{n}{x}L^{\{1\}}_{n,n}\big(tf(t,z);x,y\big),\ n\in\mathbb{N}, \tag{47}$$

$$\frac{\partial}{\partial y}L^{\{3\}}_{n,n}\big(f(t,z);x,y\big) = -nT(nx)L^{\{3\}}_{n,n}\big(f(t,z);x,y\big) + \frac{n}{x}L^{\{3\}}_{n,n}\big(tf(t,z);x,y\big),\ n\in\mathbb{N}. \tag{48}$$

By the Taylor formula for $f \in C^1_{p,q}$ we have

$$f(t,z) = f(x,y)+\frac{\partial f}{\partial x}(x,y)(t-x)+\frac{\partial f}{\partial y}(x,y)(z-y)+\varphi(t,z;x,y)\sqrt{(t-x)^2+(z-y)^2} \tag{49}$$

for $(t,z) \in \mathbb{R}^2_0$, where $\varphi(t,z) \equiv \varphi(t,z;x,y)$ is a function from the space $C_{p,q}$ and $\lim\limits_{(t,z)\to(x,y)} \varphi(t,z) = 0$. From (47) and (49) and by (6)–(19) we get

$$\begin{aligned}\frac{\partial}{\partial x}L_n^{\{1\}}\big(f(t,z);x,y\big) &= -n\tanh nx\Big\{f(x,y)+\frac{\partial f}{\partial x}(x,y)L_n^{\{1\}}(t-x;x)+\\ &+\frac{\partial f}{\partial y}(x,y)L_n^{\{1\}}(z-y;y)+L_{n,n}^{\{1\}}\big(\varphi(t,z)\sqrt{(t-x)^2+(z-y)^2};x,y\big)\Big\}+\\ &+\frac{n}{x}\Big\{f(x,y)L_n^{\{1\}}(t;x)+\frac{\partial f}{\partial x}(x,y)L_n^{\{1\}}\big(t(t-x);x\big)+\frac{\partial f}{\partial y}(x,y)L_n^{\{1\}}(z-y;y)+\\ &+L_{n,n}^{\{1\}}\big(t\varphi(t,z)\sqrt{(t-x)^2+(z-y)^2};x,y\big)\Big\},\quad n\in\mathbb{N}.\end{aligned}$$

But for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$

$$L_n^{\{1\}}(t;x) = x\tanh nx,\quad L_n^{\{1\}}\big(t(t-x);x\big) = L_n^{\{1\}}\big((t-x)^2;x\big)+xL_n^{\{1\}}(t-x;x).$$

Consequently,

$$\begin{aligned}&\frac{\partial}{\partial x}L_n^{\{1\}}\big(f(t,z);x,y\big) =\\ &\quad=\frac{\partial f}{\partial x}(x,y)\Big\{n(1-\tanh nx)L_n^{\{1\}}(t-x;x)+\frac{n}{x}L_n^{\{1\}}\big((t-x)^2;x\big)\Big\}+\\ &\quad+n(1-\tanh nx)L_{n,n}^{\{1\}}\big(\varphi(t,z)\sqrt{(t-x)^2+(z-y)^2};x,y\big)+\\ &\quad+\frac{n}{x}L_{n,n}^{\{1\}}\big(\varphi(t,z)(t-x)\sqrt{(t-x)^2+(z-y)^2};x,y\big),\quad n\in\mathbb{N}.\end{aligned} \tag{50}$$

Analogously, from (48) and (49) and by (6)–(19), we get

$$\begin{aligned}&\frac{\partial}{\partial x}L_n^{\{3\}}\big(f(t,z);x,y\big) =\\ &\quad=\frac{\partial f}{\partial x}(x,y)\Big\{n\big(1-T(nx)\big)L_n^{\{3\}}(t-x;x)+\frac{n}{x}L_n^{\{3\}}\big((t-x)^2;x\big)\Big\}+\\ &\quad+n\big(1-T(nx)\big)L_{n,n}^{\{3\}}\big(\varphi(t,z)\sqrt{(t-x)^2+(z-y)^2};x,y\big)+\\ &\quad+\frac{n}{x}L_{n,n}^{\{3\}}\big(f(t,z)(t-x)\sqrt{(t-x)^2+(z-y)^2};x,y\big),\quad n\in\mathbb{N}.\end{aligned} \tag{51}$$

We observe that for every fixed $x > 0$ and $\alpha \geq 0$

$$\lim_{n\to\infty} n^\alpha(1-\tanh nx) = 0,\quad \lim_{n\to\infty} n^\alpha\big(1-T(nx)\big) = 0. \tag{52}$$

Moreover, by the Hölder inequality and by (19), (16) and (18), we have for $n \in \mathbb{N}$

$$\begin{aligned}&\big|L_{n,n}^{\{1\}}\big(\varphi(t,z)\sqrt{(t-x)^2+(z-y)^2};x,y\big)\big| \leq\\ &\quad\leq\Big\{L_{n,n}^{\{1\}}\big(\varphi^2(t,z);x,y\big)\Big\}^{1/2}\Big\{L_n^{\{1\}}\big((t-x)^2;x\big)+L_n^{\{1\}}\big((z-y)^2;y\big)\Big\}^{1/2},\\ &\big|L_{n,n}^{\{1\}}\big(\varphi(t,z)(t-x)\sqrt{(t-x)^2+(z-y)^2};x,y\big)\big| \leq\\ &\quad\leq\Big\{L_{n,n}^{\{1\}}\big(\varphi^2(t,z);x,y\big)\Big\}^{1/2}\Big\{L_n^{\{1\}}\big((t-x)^4;x\big)+L_n^{\{1\}}\big((t-x)^2;x\big)L_n^{\{1\}}\big((z-y)^2;y\big)\Big\}^{1/2}\end{aligned}$$

and a similar inequality holds for $i = 3$. From the above, applying Lemmas 1–3 and Lemma 10, we deduce that

$$\lim_{n\to\infty} n^{1/2} L_{n,n}^{\{i\}}\big(\varphi(t,z)\sqrt{(t-x)^2+(z-y)^2}; x, y\big) = 0, \qquad i = 1, 3, \tag{53}$$

$$\lim_{n\to\infty} n L_{n,n}^{\{i\}}\big(\varphi(t,z)(t-x)\sqrt{(t-x)^2+(z-y)^2}; x, y\big) = 0, \qquad i = 1, 3. \tag{54}$$

Now, using Lemma 3 and (52)–(54) to (50) and (51), we obtain

$$\lim_{n\to\infty} \frac{\partial}{\partial x} L_{n,n}^{\{i\}}\big(f(t,z); x, y\big) = \frac{\partial f}{\partial x}(x, y) \qquad \text{for} \quad i = 1, 3.$$

Thus the proof is completed. □

REFERENCES

[1] M. Becker, *Global approximation theorems for Szasz-Mirakjan and Baskakov operators in polynomial weight spaces*, Indian Univ. Math. J. **27** (1978), no. 1, 127–142.

[2] M. Becker, D. Kucharski, R.J. Nessel, *Global approximation theorems for Szasz-Mirakjan operators in exponential weight spaces*, Linear Spaces and Approximation (Proc. Conf. Oberwolfach, 1977), vol. 40, Birkhäuser Verlag, Basel, ISNM, 1978, pp. 319–333.

[3] B. Firlej, L. Rempulska, *Approximation of functions by some operators of the Szasz-Mirakjan type*, Fasc. Math. **27** (1997), 17–27.

[4] M. Leśniewicz, L. Rempulska, *Approximation by some operators of the Szasz-Mirakjan type in exponential weight spaces*, Glasnik Matematicki **32** (1997), no. 52, 57–69.

[5] L. Rempulska, M. Skorupka, *Approximation of functions by some operators of the Szasz-Mirakjan type*, Fasc. Math. **26** (1996), 125–137.

[6] L. Rempulska, M. Skorupka, *Approximation theorems for some operators of the Szasz-Mirakjan type in exponential weight spaces*, Publicacije Elektroteh. Fak. (Beograd), Ser. Mat. **7** (1996), 9–18.

[7] L. Rempulska, M. Skorupka, *The Voronovskaya theorem for some operators of the Szasz-Mirakjan type*, Le Matematiche **50** (1995), no. 2, 251–261.

[8] L. Rempulska, M. Skorupka, *The Bershtein inequality for some operators of the Szasz-Mirakjan type*, Note di Matematica **14** (1994), no. 2, 277-290.

[9] V. Totik, *Uniform approximation by Szasz-Mirakjan type operators*, Acta Math. Hung. **41** (1983), no. 3–4, 291–307.

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