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QUASI-ISOMETRY GROUPS

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We extend some notions and properties concerning actions by isometries to the case of quasi-isometries. A generalization of a classical result of Gromov is proved.

1. Introduction

Quasi-isometry is an important equivalence relation for metric spaces. It is used in geometry and group theory to characterize the large-scale behaviour of a metric space or a finitely-generated group ([Gr1], [Gr2]).

As quasi-isometry is the isomorphism in a special category, it seems to be natural to investigate the action of groups on metric spaces by quasi-isometries. But quasi-isometries are defined up to finite distances and thus different definitions of an action by quasi-isometries are possible.

In this paper one of such definitions is proposed. This definition allows us to generalize the notions and the properties of an action by isometries to the quasi-isometry action. In fact the actions by quasi-isometries are similar to some actions by isometries (Theorem 6.1)

In addition, a generalization of a classical fact ([Gr1, P.5]) is proved (Theorem 8.4) The classical theorem deals with a cocompact proper action by isometries, and states the quasi-isometry of the acting group and the space only.

2. Preliminary Definitions

In most cases, we denote by $d_{\mathfrak{X}}(x,y)$ or $d(x,y)$ the distance between points x, y of a metric spase $\mathfrak{X}$.

Two maps $h_i\colon A \to \mathfrak{X}$, $i = 1,2$, where A is a set and $\mathfrak{X}$ a metric space, are equivalent if

$$\sup_{x\in\mathfrak{X}} d_{\mathfrak{X}}(h_1(x), h_2(x)) < \infty.$$

A map $f\colon \mathfrak{X} \to \mathfrak{Y}$ is said to be L-Lipschitz on the δ-scale if

$$d_{\mathfrak{Y}}(f(x_1), f(x_2)) < L d_{\mathfrak{X}}(x_1, x_2) + \delta$$

for all $x_1, x_2 \in \mathfrak{X}$.

A set $N \subseteq \mathfrak{X}$ is called an E-net if for any $x \in \mathfrak{X}$ there exists a point $x' \in N$ such that $d(x, x') < E$.

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The metric spaces as objects with the Lipschitz on a large-scale maps as morphisms form a category. The equivalence agrees with the superposition of the morphisms, so we may consider the classes of equivalent morphisms as morphisms of a new category. The isomorphisms of the latter category are called quasi-isometries. In other words, two spases $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are quasi-isometric, if there exist two L-Lipschitz on δ-scale maps $f_1: \mathfrak{X}_1 \to \mathfrak{X}_2$ and $f_2: \mathfrak{X}_2 \to \mathfrak{X}_1$ (which are called C-inverse (L, δ)-quasi-isometries) satisfying the inequalities

$$d_{\mathfrak{X}_1}(f_2(f_1(x)), x) < C, \qquad d_{\mathfrak{X}_2}(f_2(f_1(x)), x) < C.$$

Two metrics d_1, d_2 on the same set $\mathfrak{X}$ are called id-quasi-isometric, if id: $(\mathfrak{X}, d_1) \to (\mathfrak{X}, d_2)$ and id: $(\mathfrak{X}, d_2) \to (\mathfrak{X}, d_1)$ is a pair of inverse quasi-isometries. (Here $\mathrm{id}(x) \equiv x$).

3. Definition of Action by Quasi-isometries

Definition 3.1. Let G be a group, $\mathfrak{X}$ a metric space. A map $\varphi: G \to \mathfrak{X}^{\mathfrak{X}}$ is called an action by (uniform) quasi-isometries if the following conditions hold:

1) There exist positive constants L, δ such that

$$d(\varphi(g)(x_1), \varphi(g)(x_2)) < Ld(x_1, x_2) + \delta;$$

2) $\sup_{g_1, g_2 \in G, x \in \mathfrak{X}} d(\varphi(g_1)(\varphi(g_2)(x)), \varphi(g_1 g_2)(x)) = C_1 < \infty;$

3) $\sup_{x \in \mathfrak{X}} d(\varphi(1)(x), x) = C_2 < \infty.$

Two actions φ_1, φ_2 are equivalent if

$$\sup_{g \in G, x \in \mathfrak{X}} d(\varphi_1(g)(x), \varphi_2(g)(x)) < \infty \tag{1}$$

Remark. From the definition it follows that $\varphi(G)$ is a set of quasi-isometries. It is obvious that if φ_1 is an action by quasi-isometries and $\varphi_2: G \to \mathfrak{X}^{\mathfrak{X}}$ is a map satisfying the condition (1), then φ_2 is also an action by quasi-isometries.

Further, instead of $\varphi(g)(x)$ we shall sometimes write $g(x)$. We distinguish $\varphi(g_1 g_2)(x)$ and $\varphi(g_1)(\varphi(g_2)(x))$ writing $g_1 g_2(x)$ in the first case and $g_1 \circ g_2(x)$ in the second one.

Theorem 3.1. *Suppose that G acts on $\mathfrak{X}$ by quasi-isometries, $f_1: \mathfrak{X} \to \mathfrak{Y}$ and $f_2: \mathfrak{Y} \to \mathfrak{X}$ are C'-inverse (L', δ')-quasi-isometries. Then the action of G on $\mathfrak{Y}$ defined by the rule $g(y) := f_1 \circ g \circ f_2(y)$ is an action by quasi-isometries.*

Definition 3.2. The action $f_1 \circ g \circ f_2(y)$ is called induced by the pair of quasi-isometries f_1, f_2.

Proof of Theorem 3.1. Let us check the axioms of Definition 3.1:

1. $d_{\mathfrak{Y}}(f_1 \circ g \circ f_2(y_1), f_1 \circ g \circ f_2(y_2)) < L' d_{\mathfrak{X}}(g \circ f_2(y_1) . g \circ f_2(y_2)) + \delta' < L' L d_{\mathfrak{X}}(f_2(y_1), f_2(y_2)) + L'\delta + \delta' < L' L L' d_{\mathfrak{Y}}(y_1, y_2) + L' L \delta' + L'\delta + \delta$.
2. $d_{\mathfrak{Y}}(f_1 \circ (g_1 g_2) \circ f_2(y), f_1 \circ g_1 \circ f_2 \circ f_1 \circ g_2 \circ f_2(y)) < L' d_{\mathfrak{X}}((g_1 g_2) \circ f_2(y), g_1 \circ f_2 \circ f_1 \circ g_2 \circ f_2(y)) + \delta' \leq L' d_{\mathfrak{X}}(g_1 \circ g_2 \circ f_2(y), g_1 \circ f_2 \circ f_1 \circ g_2 \circ f_2(y)) + L' C_1 + \delta' < L' L d_{\mathfrak{X}}(g_2 \circ f_2(y), f_2 \circ f_1 \circ g_2 \circ f_2(y)) + L' C_1 + \delta' + L'\delta < L' L C' + L' C_1 + \delta' + L'\delta$.
3. $d_{\mathfrak{Y}}(f_1 \circ 1 \circ f_2(y), y) < d_{\mathfrak{Y}}(f_1 \circ 1 \circ f_2, f_1 \circ f_2(y)) + C' < L' d_{\mathfrak{X}}(1 \circ f_2(y), f_2(y)) + C' + \delta' < L' C_2 + C' + \delta'$.

Remark. In the conditions of the previous theorem, if $\tilde{f}_1, \tilde{f}_2$ are equivalent to f_1, f_2 respectively, then the action induced by $\tilde{f}_1, \tilde{f}_2$ is equivalent to the action induced by f_1, f_2.

Definition 3.3. An action $\varphi: G \to \mathfrak{X}^{\mathfrak{X}}$ is conjugated to an action $\psi: G \to \mathfrak{Y}^{\mathfrak{Y}}$ if ψ is equivalent to an action induced by some pair of inverse quasi-isometries $f_1: \mathfrak{X} \to \mathfrak{Y}$, $f_2: \mathfrak{Y} \to \mathfrak{X}$.

4. Orbit-space

Theorem 4.1. *Suppose G acts by quasi-isometries on a space $\mathfrak{X}$. Then the function*

$$d_G(x_1, x_2) := \inf_{g \in G} \sup_{h \in G} d(hg(x_1), h(x_2))$$

is a metric on $\mathfrak{X}$.

Proof. $d_G(x, x) = 0$ — it is obvious.

$\inf_{g\in G} \sup_{h\in G} d(hg(x_1), h(x_2)) = \inf_{g_1, g_2 \in G} \sup_{h\in G} d(hg_1(x_1), hg_2(x_2))$, hence

$$d_G(x_1, x_2) = d_G(x_2, x_1),$$

$d_G(x, z) \le \inf_{g_1 \in G} \sup_{h\in G}[d(hg_1(x), hg_2(y)) + d(hg_2(y), h(z))]$ for any $g_2 \in G, y \in \mathfrak{X}$, hence

$$\begin{aligned} d_G(x, z) \le \inf_{g_1, g_2 \in G} \sup_{h\in G} [d(hg_1(x), hg_2(y)) + d(hg_2(y), h(z))] \le \\ \inf_{g_1, g_2 \in G} [\sup_{h\in G} d(hg_1(x), hg_2(y)) + \sup_{h \in G} d(hg_2(y), h(z))] = \\ \inf_{g_1, g_2 \in G} [\sup_{h\in G} d(hg_2^{-1} g_1(x), h(y)) + \sup_{h \in G} d(hg_2(y), h(z))] = \\ \inf_{g_3, g_2 \in G} [\sup_{h\in G} d(hg_3(x), h(y)) + \sup_{h \in G} d(hg_2(y), h(z))] = \\ \inf_{g_3 \in G} \sup_{g \in G} d(hg_3(x), h(y)) + \inf_{g_2 \in G} \sup_{h \in G} d(hg_2(y), h(z)) = d_G(x, y) + d_G(y, z). \end{aligned}$$

Thus $d_G(x, z) \le d_G(x, y) + d_G(y, z)$.

Proposition 4.2. *The map* $\mathrm{id}: (\mathfrak{X}, d) \to (\mathfrak{X}, d_G)$ *is Lipschitz on a large scale.*

Proof. $\inf_{g\in G} \sup_{h \in G} d(hg(x), h(y)) \le \inf_{g \in G}(Ld(g(x), y) + \delta + C) \le Ld(1(x), y) + \delta + 2C$.

Definition 4.1. The space $(\mathfrak{X}, d_G)$ is called an orbit-space and is denoted by $\mathfrak{X}/G$.

Theorem 4.3. *The orbit-space of an action is quasi-isometric to the orbit-space of a conjugated action.*

Proof. If we change the action to an equivalent one, the metric d_G changes to a metric $\tilde{d_G}$ such that $|d_G - \tilde{d_G}| < D$ for some constant D.

Let $f_1: \mathfrak{Y} \to \mathfrak{X}, f_2: \mathfrak{X} \to \mathfrak{Y}$ be a pair of C'-inverse (L', δ')-quasi-isometries. Suppose G acts on $\mathfrak{Y}$ by the induced action. Let us prove that f_1, f_2 is also a pair of inverse quasi-isometries for the spaces $\mathfrak{X}/G$ and $\mathfrak{Y}/G$.

$$d_G(f_1(y_1), f_1(y_2)) = \inf_{g\in G} \sup_{h\in G} d(hg(f_1(y_1)), h(f_1(y_2))),$$

$d_{\mathfrak{X}}(hg \circ f_1(y_1), h \circ f_1(y_2)) < d_{\mathfrak{X}}(f_1 \circ f_2 \circ hg \circ f_1(y_1), f_1 \circ f_2 \circ h \circ f_1(y_2)) + 2C' < L' d_{\mathfrak{Y}}(f_2 \circ hg \circ f_1(y_1), f_2 \circ h \circ f_1(y_2)) + \delta' + 2C' = L' d_G(y_1, y_2) + \delta' + 2C'$. Thus $d_G(f_1(y_1), f_1(y_2)) \leq L' d_G(y_1, y_2) + \delta' + 2C'$.

$$d_G(f_2(x_1), f_2(x_2)) = \inf_{g\in G} \sup_{h\in G} d(hg(f_2(x_1)), h(f_2(x_2))) =$$
$$\inf_{g\in G} \sup_{h\in G} d(f_2 \circ hg \circ f_1 \circ f_2 \circ \ (x_1), f_2 \circ h \circ f_1 \circ f_2(x_2)) \leq$$
$$\inf_{g\in G} \sup_{h\in G} L' d_{\mathfrak{X}}(hg \circ f_1 \circ f_2(x_1), h \circ f_1 \circ f_2(x_2)) + \delta' \leq$$
$$\inf_{g\in G} \sup_{h\in G} L' d_{\mathfrak{X}}(hg(x_1), h(x_2)) + 2LC' + 2\delta + \delta' = L' d_G(x_1, x_2) + 2LC' + 2\delta + \delta'.$$

Thus f_2 is also Lipschitz on a large scale.

Definition 4.2. An action is trivial if $\sup\{d(g(x), x) : g \in G, x \in \mathfrak{X}\} < \infty$. An action is quasi-transitive (or cobounded) if $\mathfrak{X}/G$ is bounded.

Corollary 4.4. *Any action conjugated to a quasi-transitive action is quasi-transitive.*

Proposition 4.5. *G acts on $\mathfrak{X}/G$ trivially. (The action of G on $\mathfrak{X}/G$ pointwisely coincides with the original action of G on $\mathfrak{X}$).*

Proposition 4.6. *If G acts trivially on $\mathfrak{X}$, then d_G is id-quasi-isometrical to d.*

Proposition 4.7. *Let $\tilde{d}$ be a metric on $\mathfrak{X}$ and suppose* $\mathrm{id}\colon (\mathfrak{X}, d) \to (\mathfrak{X}, \tilde{d})$ *is Lipschitz on a large scale. If G acts trivialy on $(\mathfrak{X}, \tilde{d})$, then the map* $\mathrm{id}\colon (\mathfrak{X}, d_G) \to (\mathfrak{X}, \tilde{d})$ *is Lipschitz on a large-scale.*

Proposition 4.8. *G acts quasi-transitively iff the set $\{g(x) : g \in G\}$ is a net for some (and, consequently, for all) $x \in \mathfrak{X}$.*

5. The Translation Group

Suppose G acts by quasi-isometries on $\mathfrak{X}$. The maps $g\colon \mathfrak{X} \to \mathfrak{X}$ which are equivalent to $\mathrm{id}\colon \mathfrak{X} \to \mathfrak{X}$ are called translations.

It is clear that
$$T(G) = \{g \in G : \sup_{x\in\mathfrak{X}} d(g(x), x) < \infty\}$$
is a subgroup of G. This subgroup is called *the translation group*.

Proposition 5.1. *$T(G)$ doesn't depend on the choise of a conjugated action.*

Proof. It is obvious that $T(G)$ doesn't depend on the choise of an equivalent action. Thus it is sufficient to prove that if $T'(G)$ is the translation group of an action induced by a pair $f_1\colon \mathfrak{X} \to \mathfrak{Y}$, $f_2\colon \mathfrak{Y} \to \mathfrak{X}$ of inverse quasi-isometries, then $T'(G) \supseteq T(G)$.

Let $s \in T(G)$, then $d(f_1 \circ s \circ f_2(y), y) \leq d(f_1 \circ s \circ f_2(y), f_1 \circ f_2(y)) + d(f_1 \circ f_2(y), y) < L' d(s \circ f_2(y), f_2(y)) + \delta' + C' \leq L' C_s + \delta' + C'$, where $C_s = \sup_{x\in\mathfrak{X}} d(s(x), x)$.

Proposition 5.2. *$T(G) \trianglelefteq G$.*

Proof. The proposition can be proved by direct calculation, but it also follows from the fact that $T(G)$ is the kernel of the natural homomorphism $\pi\colon G \to \mathrm{Aut}(\mathfrak{X})$, where $\mathrm{Aut}(\mathfrak{X})$ is the automorfism group of $\mathfrak{X}$ as an object of the category of Lipschitz on a large scale maps defined in the first section, and $\pi(g)$ is the class of quasi-isometries equivalent to g.

6. The Isometrisation Theorem

Let $\mathfrak{X}$ be a metric space. Every action by isometries on $\mathfrak{X}$ (in an ordinary sence) is an action by quasi-isometries.

Theorem 6.1. *Every action by quasi-isometries of a group G on a space $\mathfrak{X}$ is conjugated to some action by isometries on a quasi-isometric to $\mathfrak{X}$ space $\mathfrak{Y}$.*

If G acts quasi-transitively, then there exists a left-invariant metric on G such that the action of G on itself by the left multiplication is conjugate to the action of G on $\mathfrak{X}$.

Proof. Let $\mathfrak{Y}$ be the space $\mathfrak{X} \times G$ with the metric

$$d_{\mathfrak{Y}}((x_1, g_1), (x_2, g_2)) = \sup_{h \in G} d_{\mathfrak{X}}(hg_1(x_1), hg_2(x_2)).$$

Let G be acting on $\mathfrak{Y}$ by the rule $h((x, g)) = (x, hg)$. It is easy to see that this action is an action by isometries. Let us prove that the maps $m: (x, g) \mapsto g(x)$, $p: x \mapsto (x, 1)$ are inverse quasi-isometries.

- $d_{\mathfrak{X}}(m(x_1, g_1), m(x_2, g_2)) = d_{\mathfrak{X}}(g_1(x_1), g_2(x_2)) \leq \sup_{h \in G} d_{\mathfrak{X}}(hg_1(x_1), hg_2(x_2)) = d_{\mathfrak{Y}}((x_1, g_1), (x_2, g_2))$,
- $d_{\mathfrak{Y}}(p(x_1), p(x_2)) = d_{\mathfrak{Y}}((x_1, 1), (x_2, 1)) = \sup_{h \in G} d_{\mathfrak{X}}(h(x_1), h(x_2)) \leq L d_{\mathfrak{X}}(x_1, x_2) + \delta$,
- $d_{\mathfrak{X}}(m(p(x)), x) = d_{\mathfrak{X}}(1(x), x) < \sup_{x \in \mathfrak{X}} d_{\mathfrak{X}}(1(x), x) < \infty$,
- $d_{\mathfrak{Y}}(p(m((x, g))), (x, g)) = d_{\mathfrak{Y}}((g(x), 1), (x, g)) = \sup_{h \in G} d_{\mathfrak{X}}(h \circ g(x), hg(x)) < \sup_{h, g \in G, x \in \mathfrak{X}} d_{\mathfrak{Y}}(h \circ g(x), hg(x)) < \infty$.

Let us prove that the actions of G on $\mathfrak{X}$ and $\mathfrak{Y}$ are conjugated by m and p.

$$d_{\mathfrak{Y}}(g((x_0, g_0)), p \circ g \circ m((x_0, g_0))) = d_{\mathfrak{Y}}((x_0, gg_0), (g \circ g_0(x_0), 1)) =$$
$$\sup_{h \in G} d_{\mathfrak{X}}(hgg_0(x_0), h \circ g \circ g_0(x_0)) < \sup_{h, g, g_0 \in G, x_0 \in \mathfrak{X}} d_{\mathfrak{X}}(hgg_0(x_0), h \circ g \circ g_0(x_0)) < \infty.$$

The first part of the theorem is proved.

If G acts quasi-transitively, then the set $\{(x_0, g) : g \in G\}$ is a G-invariant net of $\mathfrak{Y}$ on which G acts by isometries, and thus the metric $\rho(g_1, g_2) = d_{\mathfrak{Y}}((x_0, g_1), (x_0, g_2))$ is the required left-invariant metric.

7. Discrete Actions

Definition 7.1. An action by quasi-isometries of a group G on a space $\mathfrak{X}$ is discrete, if for all $x \in \mathfrak{X}, R > 0$ the set $\{g \in G : d(g(x), x) < R\}$ is finite.

Proposition 7.1. *An action conjugated to a discrete action is discrete.*

Proof. If φ_1 is a discrete action and φ_2 is D-equivalent to φ_1, then $d(\varphi_2(g)(x), x) < R$ implies $d(\varphi_1(g)(x), x) < d(\varphi_2(g)(x), x) + D$. Thus,

$$\{g \in G : d(\varphi_2(g)(x), x) < R\} \subseteq \{g \in G : d(\varphi_1(g)(x), x) < R + D\},$$

and therefore φ_2 is discrete.

Let G be acting on $\mathfrak{X}$ and $f_1: \mathfrak{X} \to \mathfrak{Y}, f_2: \mathfrak{Y} \to \mathfrak{X}$ are C'-inverse (L', δ')-quasi-isometries. If $d_{\mathfrak{Y}}(f_1 \circ g \circ f_2(y), y) < R$ then $d_{\mathfrak{X}}(f_2 \circ f_1 \circ g \circ f_2 \circ (y), f_2(y)) < L'R + \delta'$, hence $d_{\mathfrak{X}}(g \circ f_2(y), f_2(y)) < L'R + \delta' + C'$, therefore

$$\{g \in G : d_{\mathfrak{Y}}(f_1 \circ g \circ f_2(y), y) < R\} \subseteq \{g \in G : d_{\mathfrak{X}}(g \circ f_2(y). f_2(y)) < L'R + \delta' + C'\}$$

and the induced action is discrete.

8. Quasi-geodesic Spaces

Definition 8.1. A space $\mathfrak{X}$ is quasi-geodesic if there exist positive constants C, L, δ such that for any two points $x_1, x_2 \in \mathfrak{X}$ there exists a sequence of points $x_1 = p_0, p_1 \cdots p_n = x_2 \in \mathfrak{X}$, where $n < Ld(x_1, x_2) + \delta$ and $d(p_i, p_{i+1}) < C$. The number C is called the quasi-geodesity radius, and the sequence is called the quasi-geodesic.

Proposition 8.1. *A space quasi-isometric to a quasi-geodesic space is quasi-geodesic.*

Let A be a subset of a group G. By A^n we shall denote the set

$$A^0 = \{1\}, \quad A^n = A \cdot A \ldots A = \{g_1 \cdots g_n : g_i \in A\}.$$

Theorem 8.2. *If a left-invariant metric d on a group G is quasi-geodesic, then there exists a set of generators $T \subseteq G$ such that the metric $w_T(g_1, g_2) = min\{n \geq 0 : g_1^{-1} g_2 \in A^n\}$ is id-quasi-isometric to d.*

Proof. We may put $T = \{g \in G : d(g, 1) < C\}$, where C is the quasi-geodesity radius.

Proposition 8.3. *If T_1 and T_2 are finite sets of generators, then w_{T_1} and w_{T_2} are id-quasi-isometric.*

Definition 8.2. If G is a finitely generated group, then the metric w_T, where T is a finite generating set of G, is called a word metric.

Theorem 8.4. *There exists a quasi-transitive discrete action by quasi-isometries of a group G on a quasi-geodesic space $\mathfrak{X}$ iff the group G is finitely generated and is quasi-isometric as a word-metric space to $\mathfrak{X}$.*

Proof. Any finitely generated group G admits an isometric action on itself as on the word-metric space by the left multiplication. This action is obviously quasi-transitive. The set $\{g \in G : w(gh, h) < R\}$ is finite for any $R > 0$ and $h \in G$. Thus, this action is discrete. So if G is quasi-isometrical to $\mathfrak{X}$, then the induced action of G on $\mathfrak{X}$, accordingly to Corollary 4.4 and Proposition 7.1, is discrete and quasi-transitive.

On the other side, if there exists a discrete quasi-transitive action of G on a quasi-geodesic space $\mathfrak{X}$, then from Theorem 6.1 it follows that there exists a left-invariant metric d on G which is quasi-isometric to $\mathfrak{X}$. From Propositions 8.1 and 8.2 it follows that d is id-quasi-isometric to some w_T, where $T = \{g \in G : d(g, 1) < C\}$ for some $C > 0$. But the action on (G, d) by the left multiplication is conjugated to the action on $\mathfrak{X}$, so from Proposition 7.1 it follows that T is finite, and taking into account Proposition 8.3 we get that d is id-quasi-isometric to the word-metric, so $\mathfrak{X}$ is quasi-isometric to G as the word-metric space.

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