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ON AUTOMORPHISMS PERMUTING GENERATORS IN GROUPS OF RANK TWO

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Let F be a free group on two generators, and let σ be the automorphism of F permuting the generators. Let N be a σ-invariant, normal subgroup of F, and let $\bar{\sigma}$ be the automorphism of F/N induced by σ. Let G be a group of rank two, and let α be its automorphism of order two. A question we consider is: when is α induced by σ? We give some examples of groups in which every automorphism of order two is induced by σ and examples of groups with an automorphism of order two which is not induced by σ. We also give an example of a group which has two non-conjugated automorphisms, both induced by σ.

Automorphisms of finite order in groups were considered by many authors: B. Neumann in [5] considered automorphisms of order two and three and their fixed points, G. Higman [3] considered automorphisms of prime order p. E. Płonka in [6] found all fixed points of an automorphism interchanging generators in free nilpotent groups of class less or equal 3, O. Macedońska and D. Solitar in [4] found all fixed points of automorhism interchanging generators in a free metabelian two-generator group. We concentrate here on automorphisms of order two permuting generators in two-generator groups. Let F be a free group of rank two, generated by x and y, and let σ be the automorphism of F permuting x and y. Let N be a normal, σ-invariant subgroup of F. There exists an automorphism $\bar{\sigma}$ of F/N induced by σ. Then the group F/N has a presentation $\langle x, y \mid N\rangle$, which will be called a symmetric presentation [4]. So we have a definition:

Definition 1 [4]. A presentation: $\langle x, y \mid R_i\rangle$, is called symmetric if the mapping: $x \leftrightarrow y$ defines an automorphism.

Definition 2. We say that an automorphism of order two α in a two generator group G is induced by σ if there exists an element u in G such that $G = \mathrm{gp}(u, u^\alpha)$.

In such a case α acts in G as the automorphism permuting generators u and u^α. Let φ denote an epimorphism $F \to G$:

$$\varphi\ :\ \begin{cases} x \to u, \\ y \to u^\alpha, \end{cases}$$

and let $N = \ker\varphi$, then $\langle x, y \mid N\rangle$ is a symmetric presentation of G.

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Lemma 1. *Let G be a two generator group with an automorphism α of order two. If α is induced by σ then every conjugate to α is induced by σ.*

Proof. Let $\beta = \gamma^{-1}\alpha\gamma$. We have to show that there exists b such that $G = \mathrm{gp}(b, b^\beta)$. By assumption there exists a such that $G = \mathrm{gp}(a, a^\alpha)$. Let $b = a^\gamma$, then $b^\beta = a^{\gamma\beta} = a^{\gamma\beta\gamma^{-1}\gamma} = a^{\alpha\gamma}$, so the automorphism γ acts as follows:

$$\gamma : \begin{cases} a \to b \\ a^\alpha \to b^\beta, \end{cases}$$

which means that the pair b, b^β generates G. □

We show that the converse statement is not true, that is two automorphisms induced by σ are not necessarily conjugated.

Example 1. There exist two non-conjugate automorphisms in A_4 both induced by σ.

Proof. Existence of such automorphisms is connected with two different symmetric presentations of A_4:

$$\langle x, y \mid x^3 = y^3 = (xy)^2 = (yx)^2 = 1 \rangle, \tag{1}$$

$$\langle x, y \mid x^3 = y^3 = (xy)^3 = (yx)^3 = 1, x^2y = y^2x \rangle. \tag{2}$$

The first presentation can be obtained by the mapping:

$$x \to (1,2,3), \quad y \to (1,2,4),$$

and the second by the mapping:

$$x \to (1,2,3), \quad y \to (1,3,4).$$

It can be checked, using Coxeter-Moser algorithm [1], that these two presentations really define the group A_4.

Automorphisms corresponding to these presentations are induced by σ and are not conjugate because their subgroups of fixed points are not isomorphic, which can be checked by computation. □

It follows from the proof of Lemma 1 that if α is induced by σ, then there exists a such that $G = \langle a, a^\sigma \rangle$ and if $\beta = \gamma^{-1}\alpha\gamma$, then for $b = a^\gamma$ we have $G = \langle b, b^\beta \rangle$. The pair $\{a, a^\alpha\}$ satisfies a relation $w(x, y) = 1$ if and only if so does the pair $\{b, b^\beta\}$.

Theorem 1. *Let α be induced by σ and let β be an arbitrary automorphism of order two in a group G. Then β and α are conjugated if and only if following two statements hold:*
1. β is induced by σ,
2. there exist a and b such that $G = \langle a, a^\alpha \rangle = \langle b, b^\beta \rangle$, and the mapping:

$$\gamma : \begin{cases} a \to b \\ a^\alpha \to b^\beta \end{cases},$$

is an automorphism of G.

Proof. If β and α are conjugated then by Lemma 1 β is induced by σ. Conversely, let γ be the automorphism defined above and let β be induced by σ. We have to show that β and α are conjugated. We show that $\beta = \gamma^{-1}\alpha\gamma$. Indeed, $b^{\gamma^{-1}\alpha\gamma} = a^{\alpha\gamma} = (a^\alpha)^\gamma = b^\beta$ and $(b^\beta)^{\gamma^{-1}\alpha\gamma} = (a^\alpha)^{\alpha\gamma} = (a^{\alpha^2})^\gamma = a^\gamma = b$. □

We consider the following problem:

Problem 1. *Describe all groups of rank two in which every automorphism of order two is induced by* σ.

We first give an example of a group with automorphism of order two which is not induced by σ.

Example 2. Let G be the quaternion group Q_8. Then G has the symmetric presentation:

$$\langle x, y \mid x^2 = y^2, x^4 = y^4 = 1, x^{-1}yx = y^{-1}, y^{-1}xy = x^{-1}\rangle.$$

Let δ be the automorphism which maps:

$$\delta : \begin{cases} x \to x^{-1} \\ y \to y^{-1} \end{cases}$$

then δ is not induced by σ.

Proof. It is easy to check that for every element $a \in Q_8$, a^δ is equal either to a or to a^{-1}. So $\langle a, a^\delta\rangle = \langle a\rangle \neq Q_8$. □

Now we give one example of a group with every automorphism induced by σ.

Example 3. Every automorphism of order two of the group A_4 is induced by σ.

Proof. We can show more: for every automorphism α, not necessarily of order two, in A_4, there exists s such that

$$A_4 = \langle s, s^\alpha\rangle. \tag{3}$$

Let $a = (1\ 2\ 3)$ and $b = (1\ 2\ 4)$. The automorphism α is given by a mapping:

$$\alpha : \begin{cases} a \to c, \\ b \to d. \end{cases}$$

Elements c and d must have orders equal three and must satisfy the same relations as a and b. It is easy to check that if $c \neq a^{\pm 1}$ then $A_4 = \langle a, c\rangle$ and then for $s = a$ the property 3 is fulfilled. It is enough to find the element s for the automorphism given by mapping:

$$\delta : \begin{cases} a \to a^{-1}, \\ b \to b^{-1}. \end{cases}$$

Let $s = a^2b = (1\ 3\ 2)(1\ 2\ 4) = (2\ 4\ 3)$, then $s^\alpha = a^{-2}b^{-1} = (1\ 2\ 3)(1\ 4\ 2) = (1\ 4\ 3)$, and hence the elements s, s^α generate A_4. □

We give now an example of relatively free group in which every automorphism of order two is induced by σ.

Example 4. Every automorphism of order two of the Klein 4-group V_4 is induced by σ and V_4 has only one symmetric presentation: $V_4 = \langle x, y \mid x^2 = y^2 = (xy)^2 = 1\rangle$.

The Klein 4-group is a relatively free group of exponent two. Now we show that in every relatively free group of the exponent not equal two there exists an automorphism of order two that is not induced by σ:

Theorem 2. *Let G be a two-generator, relatively free group. Then either G has an automorphism of order two that is not induced by σ or G has the exponent two.*

Proof. Let $\exp(G) \neq 2$. Let a and b be free generators of G. Then the automorphism δ given by mappings:

$$a \to a^{-1}, \quad b \to b^{-1},$$

is not induced by σ. Indeed, suppose $g = g(a,b)$ is an element such that $G = \langle g, g^{\delta} \rangle$. Then because a, $b \in G$, we have the equalities:

$$a = g^{k_1}(g^{\delta})^{l_1} \dots g^{k_s}(g^{\delta})^{l_s}, \tag{4}$$

$$b = g^{m_1}(g^{\delta})^{n_1} \dots g^{m_s}(g^{\delta})^{n_s}, \tag{5}$$

Equalities (4) and (5) are identities in G. Let $d_a = d_a(g)$ and $d_b d_b(g)$ denote the exponent sum of a and b in the word g, then $d_a(g^{\delta}) = -d_a(g)$ and $d_b(g^{\delta}) = -d_b(g)$. Since a and b are free generators in G, we map in (4) $a \to a$ and $b \leftarrow 1$ and we get:

$$a = a^{d_a \sum (k_i - l_i)} \tag{6}$$

If we map in (4) $a \to 1$, $b \to a$ we get

$$1 = a^{d_b \sum (k_i - l_i)} \tag{7}$$

From (6) and (7) we obtain $a^{d_b} = a^{d_b d_a \sum (k_i - l_i)} = 1$. Hence,

$$a^{d_b} = 1 \tag{8}$$

Now we map in (5) $a \to 1$, $b \to a$ and we get $a = a^{d_b \sum (m_i - n_i)}$ and by (8) $a = 1$, so we get a contradiction. □

We show some more examples that illustrate Problem 1.

Example 5. Every automorphism of order two of the symmetric group S_3 is induced by σ. The only one symmetric presentation of S_3 is as follows:

$$\langle x, y \mid x^2 = y^2 = (xy)^3 = (yx)^3 = 1 \rangle.$$

Example 6. Let G be a nonabelian group of order pq, where p and q are odd primes, and $p < q$. Each automorphism of order two is induced by σ and G has the symmetric presentation:

$$\langle x, y \mid x^p = y^p = 1, (x^{-1}y)^k x = x(y^{-1}x)^{kn}, (y^{-1}x)^k y = y(x^{-1}y)^{kn} \rangle,$$

where $p = 2k + 1$, $n^p \equiv 1 \bmod q$. Any two automorphisms of order two in this group are conjugate.

Proof. We shall describe all automorphisms of order two in the group G. The Sylow theorem implies that in G there exists the only one cyclic, normal subgroup of order q. Let a be an arbitrary element of order p and b an arbitrary element of order q. Then a and b generate G and satisfy the presentation:

$$\langle x, y \mid x^p = y^q = 1, yx = xy^n \rangle, \tag{9}$$

where $n^p \equiv 1 \ (\mathrm{mod} q)$.

Let α be an automorphism of order two of G. Since $\langle b \rangle$ is a unique subgroup of order q, so it must be σ-invariant, and since α has order 2, b may have only two values b or b^{-1}. By the Neumann theorem [5] the subgroup of fixed points of α is non-trivial, so it has p or q elements. We show that the first case is impossible. In this case $\langle b \rangle$ is the subgroup of fixed points and by Theorem 1.6.2 (p.17) from [2] in $G/\langle b \rangle$ the automorphism induced by α has no non-trivial fixed point, and it is the inverse automorphism. So, the automorphism α in G is given by the mapping:

$$a \to a^{-1}b^k, \quad b \to b.$$

We show that for $k \neq 1$ the pair $a^{-1}b^k$, b does not satisfy presentation (9). This pair does not fulfill the relation $x^{-1}yx = y^n$ (but the pair a, b does). Indeed, if $b^{-k}aba^{-1}b^k = b^n$ then $aba^{-1} = b^n$ and then $b = (a^{-1}ba)^n = b^{n^2}$ and hence $q|(n^2-1)$, which is impossible. Let now $b^\alpha = b^{-1}$, then α is given by mapping:

$$a \to ab^k, \quad b \to b^{-1},$$

and for every k this mapping defines an automorphism of G. If $k \neq 0$ then the pair a, ab^k generates G, i.e. $G = \langle a, a^\alpha \rangle$. For $k = 0$ we have the automorphism:

$$a \to a, \quad b \to b^{-1},$$

Let $s = ab$, then $s^\alpha = ab^{-1}$ and $G = \langle s, s^\alpha \rangle$. □

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