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ON USE OF ITERATES OF ENDOMORPHISMS FOR CONSTRUCTING GROUPS WITH SPECIFIC PROPERTIES

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We produce a new approach for constructing groups with unusual properties such for example as to have intermediate growth or to be amenable. It is based on the use of iterations of endomorphism of a group.

The particular case of the construction is when one starts the limit procedure with non-hopfian group. The example of Baumslag-Solitar group is considered carefully.

The goal of this paper is to describe some constructions in group theory, and to give a few interesting examples.

We start with a simple version of the construction.

I. Let G be a group, $K \lhd G$ be a normal subgroup, and $\varphi\colon G \to G$ be an endomorphism such that $\varphi^{-1}(K) \supset K$ and $\varphi^{-1}(K) \neq K$. Denote by φ^n the n-th iteration of φ and let $L_n \lhd G$, $n \geq 1$ be defined as $L_n = \varphi^{-n}(K)$. Then $\{L_n\}$ is an increasing sequence of normal subgroups, and we can define the group $G_{\varphi,K} = G/L$, where $L = \bigcup_{n=1}^{\infty} L_n$. If $K = 1$ we denote this group by G_φ. It seems to us that groups of this form can possess specific growth and other properties.

Let us explain this in more detail. Let G a finitely-generated group with a finite system A of generators. Then the length $\partial^A(g)$ of $g \in G$, the growth function

$$\gamma_G^A(n) = \#\{g \in G : \partial^A(g) \leq n\},$$

and the number $\lambda_G^A = \lim_{n\to\infty} \sqrt[n]{\gamma_G^A(n)}$ are determined. G is said to have *exponential growth* if $\lambda_G^A > 1$ and G has *subexponential growth* if $\lambda_G^A = 1$. If there are constants c and $d > 0$ such that $\gamma_G^A(n) \leq cn^d$, $n = 1, 2, \dots$ then G is said to have *polynomial growth.* If the growth of G is neither exponential nor polynomial, then G is said to have intermediate growth.

Denote $\lambda(G) = \inf_A \lambda_G^A$, where the infimum is taken over all finite generating systems.

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A group is said to have *uniformly exponential growth* if $\lambda(G) > 1$. The question of the existence of groups of intermediate growth was posed by J. Milnor in [16] and positively solved in [5]. Up to now the only known examples of groups of intermediate growth are essentially those costructed in [4],[6] and [7]. The question remains to find new such examples. A more recent question is whether finitely generated groups of exponential but not of uniformly exponential growth exist.

One possible way for the constrution of such groups is the following. Let G be a finitely generated non-hopfian group with a system of generators A, i.e., a group for which there is a surjective homomorphism $\varphi\colon G \to G$ with non-trivial kernel. Let $G_n = G/L_n$, where $L_n = \varphi^{-n}(1)$ is the increasing sequence of normal subgroups and $A^{(n)} = \varphi^{(n)}(A)$ is the system of generators of $G_n \simeq G$, $n = 1, 2, \ldots$.

Define $G_\varphi = G/L$, where $L = \bigcup_1^\infty L_n$ and let $\bar{A}$ be the image of A under the canonical homomorphism $G \to G/L$. Then the pair $(G_\varphi, \bar{A})$ is the limit of the sequence $\{(G_n, A^{(n)})\}$ in the sense that will be made precise below. Because the exponent of growth does not increase when passing to factor group we have a decreasing sequence of numbers $\lambda_G^A \geq \lambda_G^{A^{(1)}} \geq \cdots \geq \lambda_G^{A^{(n)}} \geq \ldots$.

Theorem 1. *The relation* $\lambda_{G_\varphi}^{\bar{A}} = \lim_{n\to\infty} \lambda_{G_n}^{A^{(n)}}$ *holds.*

For the proof of this theorem and the following result we will use the topology in the space of groups with a given number of generators first considered in [6]. This topology implies the convergence $\lim_{n\to\infty}(G_n, A^{(n)}) = (G_\varphi, \bar{A})$ in the following sense. Let Γ_n and Γ be, the Cayley graphs of G_n and G_φ with respect to the system of generators $A^{(n)}$ and $\bar{A}$, and let Γ_n^k and Γ^k be balls of radius k around the identity element in the corresponding graphs. Then for any $k \in \mathbb{N}$ there is $N(k) \in \mathbb{N}$ such that the graphs Γ_n^k and Γ^k are isomorphic when $n > N(k)$.

Proof of Theorem 1. Let $\lambda_* = \lim_{n\to\infty} \lambda_{G_n}^{A^{(n)}}$. We are going to show that $\lambda_{G_\varphi}^{\bar{A}} = \lambda_*$. By the Polya-Szego lemma [17] and sub-multiplicativity of the growth function, we have $\lambda_G^A = \lim_{n\to\infty} \sqrt{[n]}\gamma_G^A(n) = \inf_n \sqrt{[n]}\gamma_G^A(n)$. Let $\varepsilon > 0$ and p be such that $\lambda_{G_\varphi}^{\bar{A}} > \sqrt{[p]}\gamma_{G_\varphi(p)}^{\bar{A}} - \varepsilon$.

We can find the number $N(p)$ such that $\Gamma_n^{(p)}$ and $\Gamma^{(p)}$ are isomorphic whenever $n > N(p)$. Because the growth function $\gamma(n)$ counts the number of points in the ball of radius n in the Cayley graph $\lambda_{G_n}^{A^{(n)}} \leq \sqrt{[p]}\gamma_{G_n}^{A^{(n)}}(p) = \sqrt{[p]}\gamma_{G_\varphi}^{\bar{A}}(p)$. if $n > N(p)$, and so $\gamma_{G_\varphi}^{\bar{A}} \geq \lambda_{G_n}^{A^{(n)}} - \varepsilon \geq \lambda_* - \varepsilon$, which implies that $\gamma_{G_\varphi}^{\bar{A}} \geq \lambda_*$.

The opposite inequality is obvious because $(G_\varphi, \overline{A})$ is the image of $(G_n, A^{(n)})$ for any $n \geq 1$. $\square$

Theorem 2. *If* $\gamma_{G_\varphi}^{\bar{A}} = 1$, *then the group* G_φ *has intermediate growth.*

Proof. Suppose the growth of G_φ is polynomial. Then by a theorem of Gromov [9], the group G_φ is virtually nilpotent and is therefore finitely presented. Let $r_1, r_2, \ldots, r_l$ be the set of relators of G_φ with respect to the generating system $\bar{A}$, and let $Q = \mathrm{Max}_{1\leq i\leq l} |r_i|$, where $|r|$ is the length of the word r.

To each relator r_i corresponds a loop p_i with unit 1 as base point and all loops p_i, $1 \leq i \leq l$ lie in $\Gamma^{(Q)}$. As $\Gamma_n^{(Q)}$ coincides with $\Gamma^{(Q)}$ whenever $n > N(Q)$, this implies that the relations $r_1 = 1, r_2 = 1, \ldots, r_l = 1$ hold in the group G_n. Hence G_n is the image of G_φ and therefore G, being isomorphic to G_n, is virtually nilpotent. But a virtually nilpotent group is residually finite and by a theorem of A.I. Malcev [13] G cannot be non-hopfian (we use the fact that G is finitely generated). We have shown that the growth of G_φ is not polynomial, and by the assumption of the theorem, G_φ has intermediate growth. $\square$

Corollary 1. *Suppose that G is a finitely generated non-hopfian group of exponential growth. If $\lambda^{\bar{A}}_{G_\varphi} = 1$, then G is not of uniformly exponential growth.*

Proof. We have $\lambda(G) = \inf_A \lambda^A_G \leq \inf_n \lambda^{A^{(n)}}_G = 1$ as follows from Theorem 1. $\square$

The last statement shows that if one is able to produce an example of a finitely generated non-hopfian group G of exponential growth such that $\lambda_{G_\varphi} = 1$, then this will give simultaneously an example of a group of exponential but not of uniformly exponential growth (the group G itself) as well as an example of a group of intermediate growth (the group G_φ).

With this perspective in mind, we propose to consider known examples of finitely generated non-hopfian groups and to investigate the limit groups G_φ.

Example 1. Let us consider the first such example namely the Baumslag-Solitar non-hopfian group $G = \langle b, t | t^{-1} b^2 t = b^3 \rangle$ with surjective but not injective homomorphism $\varphi\colon G \to G$, $\varphi\colon b \to b^2, t \to t$. But before we are going to prove the following statement that will be used below.

Proposition 1. *Let G be a torsion-free non-hopfian group and let $\varphi\colon G \to G$ be a surjective but not injective homomorphism. Then the group G_φ is torsion-free.*

Proof. Suppose $g \in G_\varphi$, $g \neq 1$ and $g^p = 1$ for some $p > 1$. Then there is a number q such that $\varphi^q(g^p) \overset{G}{=} 1$. But $\varphi^q(g^p) = [\varphi^q(g)]^p \overset{G}{=} 1$, and because G is torsion-free, it follows that $\varphi^q(g) \overset{G}{=} 1$ which implies that $g \overset{G_\varphi}{=} 1$. $\square$

To formulate the next statement we denote by $Q_{2,3}$ the subgroup of additive group of rationals Q consisting of elements of the form $\{\frac{m}{2^p 3^q} : m, p, q \in \mathbb{Z}\}$. Let $\psi\colon Q_{2,3} \to Q_{2,3}$ be the automorphism $x \to \frac{3}{2}x$.

Theorem 3. *The isomorphism $\Gamma_\varphi \simeq Q_{2,3} \rtimes_\psi \mathbb{Z}$ holds.*

Remark. This group is known. It is not finitely presented. See [3], where the cohomological properties of this group are studied.

Proof. Let $\{\bar{b}, \bar{t}\}$ be the system of generators of G_φ which is the image of the system $\{b, t\}$ under the canonical homomorphism $G \to G/L$, where, as before, $L = \bigcup_n \varphi^{-n}(1)$. Denote by $B = \langle \bar{b} \rangle^{(\#)}_{G_\varphi}$ the normal closure of the element $\bar{b}$ in G_φ.

Lemma 1. *The group B is commutative.*

Proof. It is enough to show that the elements $b_k = (\bar{t})^{-k} \bar{b} (\bar{t})^k, k \in \mathbb{Z}$ commute with each other. Take the word $w_{k,l} = [b_k, b_l] = t^{-k} b^{-1} t^{k-l} b^{-1} t^{l-k} b t^{k-l} b t^l$ and let us show that for some $p \in N$, $\varphi^p(w_{k,l}) \overset{G}{=} 1$. This will mean that $[\bar{b}_k, \bar{b}_l] \overset{G_\varphi}{=} 1$. Suppose $k - l \leq 0$ and put $p = l - k$. We have ($\sim$ means conjugation) $\varphi^p(w_{k,l}) = t^{-k} b^{-2^p} t^{k-l} b^{-2^p} t^{l-k} b^{2^p} t^{k-l} b^{2^p} t^l \sim b^{-2^p} t^{-p} b^{-2^p} t^p b^{2^p} t^{-p} b^{2^p} t^p \sim b^{-2^p} b^{-3^p} b^{2^p} b^{3^p} = 1$ The case when $k - l > 0$ can be reduced to the previous one because $[b_l, b_k] = [b_k, b_l]^{-1}$. $\square$

Lemma 2. *The group $Q_{2,3}$ has a presentation*

$$Q_{2,3} = \langle a_n | n \in \mathbb{Z},\ \textit{all generators commute}\ , 2a_n = 3a_{n-1}, n \in \mathbb{Z} \rangle. \tag{1}$$

Proof. Let Q' be the group with presentation (1). We are going to prove that Q' is a group of rank 1. First we mention that Q' is torsion-free. Indeed, below it will be shown that Q' is the union of an increasing sequence of cyclic groups. Therefore, either Q' is torsion or torsion-free. Suppose Q' is a torsion group and $ma_i \overset{Q'}{=} 0$, for some $m \in \mathbb{Z}$, $m \neq 0$, and some i, $-\infty < i < \infty$. Without loss of generality, we may assume $i = 0$ and so $ma_0 \overset{Q'}{=} 0$.

Consider the free Z-module M with the basis $\{a_n\}_{-\infty}^{\infty}$. For some $N \in \mathbb{N}$ and $k_n \in \mathbb{Z}$, $-N \leq n \leq N$ we have the relation

$$ma_0 \overset{M}{=} \sum_{-N}^{N} k_n(2a_n - 3a_n). \tag{2}$$

Equation (2) is equivalent to the system of relations

$$\begin{cases} 2k_N = 0, \\ 2k_i - 3k_{i+1} = 0, \quad i = \pm 1, \dots, \pm N, \\ 2k_0 - 3k_1 = m, \\ -3k_{-N} = 0 \end{cases}$$

which has a solution if and only if $m = 0$. Therefore, Q' is torsion-free.

Now let us consider the sequence of subgroups $H_n < Q'$ defined by $H_n = \langle a_{-n}, \dots a_n | a_i + a_j = a_j + a_i, 2a_i = 3a_{i-1}, -n+1 \leq i \leq n \rangle$.

Using induction on $n = 0, 1, 2 \dots$ we are going to prove that H_n is a cyclic group. Let $A_n = \langle \xi_n \rangle$ be the cyclic group generated by ξ_n where $\xi_n = \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} a_{-n+r}$. ξ_n is a linear combination of generators of H_n and hence $A_n \leq H_n$. On the other hand, the relations in H_n imply that

$$3^{j+n-r} a_{-n+r} = 2^{j+n-r} a_j, \ 0 \leq r \leq 2n, \ -n \leq j \leq n \tag{3}$$

Therefore, for $j = \pm 0, \pm 1, \dots, \pm n$ we have

$$\begin{aligned} 3^{n+j} 2^{n-j} \xi_n &= \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} 3^{n+j} 2^{n-j} a_{-n+r} = \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} 3^{n+j-r} 2^{n-j} 3^r a_{-n+r} \\ &= \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} 3^r 2^{n-j} 2^{n+j-r} a_j = \Big\{ \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} 3^r 2^{2n-r} \Big\} a_j = a_j. \end{aligned}$$

Hence $H_n \leq A_n$, and the group $Q' = \cup_1^{\infty} H_n$ is the union of cyclic subgroups.

The mapping $\alpha \colon Q' \to Q_{2,3}$, $a_n \to (\frac{3}{2})^n$ determines a homomorphism, because relations in (1) obviously hold in $Q_{2,3}$. This shows that Q' is torsion-free and H_n, $n \geq 1$ are infinite cyclic group. Hence Q' is the group of rank 1. But every non-zero subgroup of a torsion-free group of rank 1 has rank 1 and the corresponding factor group has rank 0 (i.e. is a torsion group) [12]. Therefore, α is an isomorphism.

Because N is commutative and is generated by elements $b_n, n \in \mathbb{Z}$ satisfying the relations $b_n^2 = b_{n-1}^3$, which are equivalent to the relations in (1) (in additive notation), we get that the mapping $\beta \colon a_n \to b_n, n \in \mathbb{Z}$ determines the homomorphism $Q_{2,3} \to B$. And again, because $Q_{2,3}$ is of rank 1, every factor group of $Q_{2,3}$ has torsion, but B is torsion-free, as follows from Proposition 1. Therefore, β is an isomorphism.

Let us show that $B \cap \langle \bar{t} \rangle = \{1\}$ and that $\langle \bar{t} \rangle$ is infinite. If we suppose that $t^k \overset{G_\varphi}{=} w$, $k \neq 0$, where w is a word of the form $w = \prod_{i=1}^{l} u_i^{-1} b^{\pm 1} u_i$, then for some large power t^N of t $t^{-N}(t^{-k}w)t^N = t^{-N}t^{-k}wt^N$ and $t^{-N}wt^N \overset{G}{=} b^s$, for some s (because G is an HNN-extension). But $\varphi^n(t^{-k}b^s) = t^{-k}b^{2^n s} \overset{G}{\neq} 1$ for $n = 0, 1, \dots,$ so $B \cap \langle \bar{t} \rangle = \{1\}$.

For any n and $k \neq 0$, $\varphi^n(t^k) = t^k \overset{G}{\neq} 1$, therefore $\langle \bar{t} \rangle \simeq \mathbb{Z}$.
Conjugating by t in B gives an action equivalent to that of ψ on $Q_{2,3}$, thus $G_\varphi \simeq B \rtimes \langle \bar{t} \rangle \simeq Q_{2,3} \rtimes_\psi \mathbb{Z}$ and the proof is now complete. □

Remark 2. In [1] the criterion of hopficity for a group which is an HNN-extension with a free abelian base group of finite rank is given. It would be interesting to apply the construction I to non-hopfian groups which are HNN-extensions of this form and to analyse limit groups on growth and other properties. One can find a number of interesting examples of non-hopfian groups in [15] and [19].
We are going to discuss the word problem in the group $G_{\varphi,K}$ obtained by construction I.

Suppose that the group G/K has solvable word problem. Then the following procedure can be viewed as a generalized algorithm for solving the word problem in G.

To check if $w \overset{G_{\varphi,K}}{=} 1$, do the following

(a) Test if $w \in K$, that is, if $w \overset{G/K}{=} 1$. If "yes", then $w \overset{G_{\varphi,K}}{=} 1$, if "no" go to (b)
(b) Compute $w' = \varphi(w)$ and go to (a), replacing w by w'.

If there is $N = N(w)$ such that $\varphi^N(w) \in K$ then $w \overset{G_{\varphi,K}}{=} 1$. Otherwise, $w \overset{G_{\varphi,K}}{\neq} 1$.

The following diagram is a geometric expression of this procedure. $w \overset{\varphi}{\to} w_1 \overset{\varphi}{\to} w_2 \to \dots \overset{\varphi}{\to} w_n \in K$?

Question. *When there is an effective estimate on $N(w)$? In other words, when is this procedure a real algorithm?*

Question. *Suppose a free group F_m of rank $m \geq 2$, a normal subgroup $K \lhd F_m$ with solvable membership problem, and an endomorphism $\varphi\colon F_m \to F_m$ are given. Is there an algorithm which checks for every $w \in F_m$ if there is $N = N(w)$ such that $\varphi^N(w) \in H$?*

The construction I is also interesting for the theory of amenable groups. The notion of amenable group was introduced by von Neumann [18]. A group G is called amenable if there is a finitely additive left invariant measure μ on the σ-algebra of all subsets of G such that $\mu(G) = 1$.

Finite and commutative groups are amenable and the class AG of amenable groups is closed with respect to the operations of taking subgroup, taking factor-group, group-extesion, and direct limit. Let EG be the least class of groups containing finite and commutative groups and closed with respect to the above operations. For a long time it was open problem of M. Day whether the relation $AG = EG$ hold. In [8] this question was solved in negative, because every group of intermediate growth belongs to the difference $AG \setminus EG$.

Still yet any new example of a group in $AG \setminus EG$ is of interest. One possible way for constructing such examples is based on the construction I.

To explain this, let us remind that if the set of generators $A = \{a_1, \dots, a_m, a_1^{-1}, \dots, a_m^{-1}\}$ is given as well as a symmetric probability distribution $p(a_i^\varepsilon) \geq 0$, $\varepsilon = \pm 1$; $i = 1, 2, \dots, m$ $p(a_i) = p(a_i^{-1})$; $\sum_{i=1,\varepsilon=\pm 1}^{m} p(a_i^\varepsilon) = 1$ then one can consider the random walk on G in which the probability of the transition $g \to g a_i^\varepsilon$ is equal to $p(a_i^\varepsilon)$, $\varepsilon = \pm 1$, $1 \leq i \leq m$.

By definition, the spectral radius r of the random walk is the number $r = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{p_{1,1}^{(n)}}$, where $p_{1,1}^{(n)}$ is the probablity of returning to 1 after n steps. It is easy to see that either $p_{1,1}^{(n)} > 0$ for $n = 1, 2, \dots$, or $p_{1,1}^{(2n)} > 0$ and $p_{1,1}^{(2n-1)} = 0$, $n = 1, 2, \dots$. In both cases $p_{1,1}^{2(n+m)} \geq p_{1,1}^{2n} p_{1,1}^{2m}$, therefore $r = \lim_{n\to\infty} \sqrt[2n]{p_{1,1}^{(2n)}} = \sup_n \sqrt[2n]{p_{1,1}^{(2n)}}$.

The Kesten criterion [11] states that G is amenable if and only if $r = 1$. It is easier to apply this theorem in the case when $p(a_i^\varepsilon) = \frac{1}{2m}$. The random walk in this case is called simple, and probabilities $p_{1,1}^{(n)}$ are equal to $\frac{q_n}{(2m)^n}$, where q_n is the number of closed loops of length n with base point at the unit element in the Cayley graph Γ of the group G with respect to the generating system A.

It is clear that if r is the spectral radius of random walk on G and $\overline{r}$ is the spectral radius of random walk on a factor group G/H determined by disribution which arise after factorization, then $\overline{r} \geq r$. Moreover, if H is a non-amenable group then the strict inequality $\overline{r} > r$ holds [11].

Now if G is a non-hopfian group, $\varphi\colon G \to G$ is a non-injective endomorphism, $G_n = G/L_n$, $L_n = \varphi^{-n}(1)$ and r_n is the spectral radius of a random walk on G_n, then we get the increasing sequence of numbers: $r_1 \leq r_2 \leq \dots \leq r_n \leq \dots$, and therefore the limit $r_* = \lim_{n\to\infty} r_n$ exits.

Let r_∞ be the spectral radius of random walk on the group G_φ which is induced by the random walk on G over factorization.

Proposition 2. *The relation $r_\infty = r_*$ holds.*

Proof. Since for any $n = 1, 2, \dots$ the group G_φ is the canonical image of the group G_n, the inequality $r_\infty \geq r_n$ holds and hence $r_\infty \geq r_*$.

We denote by ${}_k p_{1,1}^{(n)}$ the probability $p_{1,1}^{(n)}$ for the group G_k, and by ${}_\infty p_{1,1}^{(n)}$ the probability $p_{1,1}^{(n)}$ for the group G_φ. Let $\varepsilon > 0$ be any positive number and N be such that $r_\infty < \sqrt[N]{{}_\infty p_{1,1}^{(N)}} + \varepsilon$. There is a number $M = M(N)$ such that when $n \geq M$ the Cayley graphs Γ_n of the groups G_n are isomorphic to the Cayley graph Γ_∞ of the group G_φ in the neighbourhood of the unit element of radius N. But for all of these groups the probabilities $p_{1,1}^{(N)}$ coincide: ${}_\infty p_{1,1}^{(N)} = {}_M\, p_{1,1}^{(N)} = {}_{(M+1)}\, p_{1,1}^{(N)} = \dots$, therefore if $n \geq M$ $r_n \geq \sqrt[N]{{}_n p_{1,1}^{(N)}} = \sqrt[N]{{}_\infty p_{1,1}^{(N)}} > r_\infty - \varepsilon$, which implies the inequality $r_* > r_\infty$. $\square$

Corollary 2. *If* $\lim_{n\to\infty} r_n = 1$, *then the group G_φ is amenable.*

Let us go back to the Baumslag-Solitar group G. Being an HNN-extension in which both amalgamated subgroups are proper subgroups of the base group, this group contains a free subgroup with two generators which is non-amenable and hence G is non-amenable as well. The limit group G_φ is isomorphic to a semidirect product $Q_{2,3} \rtimes \mathbb{Z}$, is solvable and therefore is amenable.

We stress that in this example we started with a non-amenable group and as a result of construction obtained an amenable group. Unfortunately, the group G_φ belongs to the class EG and in this case the construction does not produce a new example of amenable not elementary amenable group. Nevertheless, we hope that taking suitable $G \notin AG$ and $\varphi: G \to G$, it will be possible to get $G_\varphi \in AG \setminus EG$.

II. Now we are going to describe the general costruction. Let G be a group, $K, H \lhd G$ be normal subgroups with $F = G/H$ being finite.

Suppose that for every $l \in F$ a surjective homomorphism $\varphi_l: H \to G$ with the property $\varphi_l^{-1}(K) \supset K$ is given and that the set $\{\varphi_l : l \in F\}$ satisfies the following condition: for any $l \in F$ and any $t \in T$ (T is a system of representatives of right cosets Hl), there is $p = p(l,t) \in F$ such that

$$\varphi_l \circ \mu_t = \varphi_{p(l,t)}, \tag{4}$$

where $\mu_t: H \to H$ is the automorphism $h \to t^{-1}ht$.

Having this data we define the group $G_{H,K,\{\varphi_l\}} = G/L$, where $L = \bigcup_{n=0}^{\infty} L_n$, $L_0 = K$ and if L_n is already defined then $L_{n+1} = \{w \in G : \varphi_l(w) \in L_n, l \in F\}$. The correctness of this construction follows from the following result which is obvious

Lemma 3. *For each $n \geq 0$, L_n is a normal subgroup of G.*

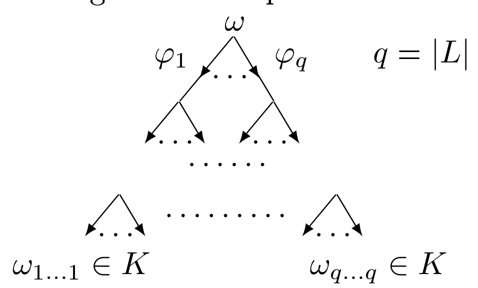
Again, if the word problem in G/H is solvable, then the following procedure can be useful for solving the word problem in the group $G_{H,K,\{\varphi_l\}}$, (which for the moment we shall denote by G_∞).

To check if $w \overset{G_\infty}{=} 1$, do the following

(i) Test if $w \in K$, if "yes" then $w \overset{G_\infty}{=} 1$, if "no", go to (ii).

(ii) Compute $w_l = \varphi_l(w)$, $l \in F$ and go to (i), repeating with each w_l, $l \in F$ the same procedure that we started to do with w. If there is $N = N(w)$ such that for all N-tuples $(l_1, l_2 \dots, l_N)$, $w_{l_1 \dots l_N} \in H$, where $w_{l_1 \dots l_N} = \varphi_{l_N} \varphi_{l_{N-1}} \dots \varphi_{l_1}(w)$, then $w \overset{G_\infty}{=} 1$. Otherwise, $w \overset{G_\infty}{\neq} 1$.

The following picture is the geometric expression of this branched procedure.



Again, if there is an effective bound on $N(w)$, then the procedure becomes the algorithm which solves the word problem in G_∞.

The particular case of the construction II is when instead of (4) we have the relation

$$\varphi_l \circ \mu_g = \varphi_{\hat{g}(l)}, \tag{5}$$

where $\hat{g} \in \operatorname{Aut} L$, and $\wedge: G \to \operatorname{Aut} L$ is the homomorphism induced by the action of G on itself by inner automorphisms.

Example 2. Let $G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1 \rangle \simeq Z_2 * (Z_2 \times Z_2)$, where $Z_2 = \{1, a\}$, $Z_2 \times Z_2 = \{1, b, c, d\}$.

Let $H \lhd G$ be a subgroup of index 2 consisting of elements represented by words with even number of occurrences of the symbol a, $H = \langle b, c, d, aba, aca, ada \rangle$, $L = G/H \simeq \{0, 1\} = Z/2Z$ and let the homomorphisms $\varphi_i: H \to G$, $i = 0, 1$ be defined as

$$\varphi_0 : \begin{cases} b \to a, & aba \to c, \\ c \to a, & aca \to d, \\ d \to 1, & ada \to b \end{cases} \qquad \varphi_1 : \begin{cases} b \to c, & aba \to a, \\ c \to d, & aca \to a, \\ d \to b, & ada \to 1 \end{cases}$$

Then condition (5) is satisfied and we can apply the construction II when $K = 1$.

Theorem 4. *The group $G_\infty = G_{H,K,\{\varphi_i\}}$ arising from this example is isomorphic to the 2-group of intermediate growth constructed in [4].*

Proof. This follows from the algorithm described in [6] for solving the word problem in the mentioned group. The point is that this algorithm is identical to the procedure (i), (ii) above. (In [6] it is proved that at each of its steps the lengths of the words decrease). □

It would be nice to find other examples of such sort (may be with noncommutative L) that produce groups with such properties as to be torsion, of intermediate growth, and may be to have some other interesting properties.

We are going to finish the paper by describing a construction that is in some sense dual to construction II. Let us start from the simple version.

III. Let $\varphi\colon F_m \to F_m$ be an endomorphism of a free group with system of generators $a_1, a_2, \ldots, a_m$. Consider the group of the form

$$G = \langle a_1, a_2, \ldots, a_m | r_1 = r_2 = \cdots = r_k = 1, \varphi^k(r_{k+1}) = \ldots \varphi^k(r_l) = 1, k = 0, 1, \ldots \rangle \tag{6}$$

where $r_1, r_2, \ldots r_k, r_{k+1}, \ldots, r_l \in F_m$ are some elements.

The groups of the form (6) also can have interesting properties as the example below shows.

It may happen that φ induces an endomorphism $G \to G$ for which $\varphi(r_i) \overset{G}{=} 1, 1 \le i \le k$. If φ is injective but not surjective then the group G is non co-hopfian. If φ is surjective but not injective, then G is non hopfian.

Example 3. Let $G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1,$ $\sigma^k((cad)^4) = \sigma^k((adacac)^4) = 1,\ k = 0, 1, \ldots \rangle$, where $\sigma\colon a \to aca,\ b \to d,\ c \to b,\ d \to c$.

Theorem 5. *The group G is isomorphic to the group from the example 2.*

Proof. This was established in [14]. □

It follows that, G is a 2-group of intermediate growth and has many other interesting properties. In this case σ induces an injective endomorphism of G, therefore G is not co-hopfian. (This was used in [8] for constructing finitely presented amenable but not elementary amenable group).

It is not known if there are non-hopfian groups of intermediate growth.

IV. Let $\varphi_i, 1 \le i \le n$ be a system of endomorphisms $F_m \to F_m$. Consider a group given by the presentation.

$$\begin{aligned} G = \langle a_1, \ldots, a_m \mid r_1 = \cdots = r_k = 1, \varphi_{i_1} \ldots \varphi_{i_s}(r_t) = 1, \\ s = 0, 1, \cdots ; k+1 \le t \le l; 1 \le i_1 \le \cdots \le i_s \le k \rangle. \end{aligned} \tag{7}$$

Problem. *To produce interesting examples of groups with a presentation of the form (7).*

Question. *Can Gupta-Sidki groups from [10] be given by presentations of the form* (7) *or even* (6)*?*

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