Математичні Студії. Т.8, №2 Matematychni Studii. V.8, No.2

УДК 519.4

# ON IDENTITIES DEFINING $t$-GROUPS

Olga Macedońska

O. Macedońska. *On identities defining t-groups*, Matematychni Studii, **8**(1997) 189–191.

We discuss the situation leading to the conjecture that a variety of groups consists of $t$-groups if and only if it is pseudoabelian.

A variety of groups is the class of groups satisfying a given set of identities. For example the variety of all abelian groups satisfies the identity $[x, y] = 1$, where $[x, y] = x^{-1}y^{-1}xy$. The variety of all metabelian groups is defined by the identity $[[x, y], [z, t]] = 1$. The variety of 2-nilpotent groups satisfies the identity $[[x, y], z] = 1$. By Birkhoff, a variety of groups is also the class of groups closed with respect to taking subgroups, quotient groups and cartesian products.

A group $G$ is called a *$t$-group* (or is said to have a $t$-property) if every subnormal subgroup in $G$ is normal. By another words, $G$ is a $t$-group if for any two subgroups $A$ and $B$ the fact that $A$ is normal in $B$, and $B$ is normal in $G$ implies that $A$ is normal in $G$. So the "$t$" stands for transitivity of normality. The $t$-groups with additional conditions were studied in [1], [8]. As examples of $t$-groups we have abelian groups, direct products of simple groups, a quaternion group. A nilpotent group $G$ is a $t$-group if and only if every subgroup is normal, i.e. either $G$ is abelian or it is a Dedekind group (a direct product of a quaternion group, an elementary abelian 2-group, and an abelian group with all elements of odd order) [8].

The smallest group, which is not a $t$-group is the dihedral group $D_4$ of order eight (the symmetry group of a square). It is known ([6], p.166) that the quaternion group $Q_8$ and the dihedral group $D_4$ generate the same variety which is defined by the identities $x^4 = 1$, $[x, y]^2 = 1$, $[[x, y], z] = 1$. Since $D_4$ is not a $t$-group, these identities do not imply the $t$-property, while the identity $[x, y] = 1$ does imply the $t$-property.

A natural question arises:

**Question 1.** *What identities imply the $t$-property?*

For conveniens we introduce

1991 *Mathematics Subject Classification.* 20E10, 20E15.

**Definition 1.** A variety is called a t-variety if it is not abelian and every group in the variety is a t-group.

The question is answered in [4]. We formulate the result below, where $\langle x\rangle$ denotes an infinite cyclic group generated by $x$.

**Theorem 1.** *An identity implies the t-property if and only if it is of the form*

$$[x, y] = u(x, y), \tag{1}$$

*where* $u(x, y)$ *belongs to* $\big[[\langle x\rangle, \langle y\rangle], \langle y\rangle\big]$.

It is interesting to notice that simple identities of the form (1), as for example $[x, y] = [[x, y], y]$, imply the abelian identity [2], and hence do not define $t$-varieties. A natural question arises:

**Question 2.** *Does there exist a t-variety?*

This question has a positive answer. To discuss it we need to define a special type of varieties.

**Definition 2.** A variety is called pseudoabelian if it is not abelian, but all its finite groups are abelian.

The pseudoabelian varieties can be defined similarly by means of metabelian groups instead of finite groups [6].

**Theorem 2.** *A variety is pseudoabelian if and only if it is not abelian, but all its metabelian groups are abelian.*

*Proof.* Let $\mathcal{M}$ be a pseudoabelian variety and let $G$ be a metabelian group in it. If $G$ is not abelian, then we can find a two-generator non-abelian subgroup $H$ in $G$. By Lemma 7.2 of B. Neumann [5], $H$ has a non-abelian finite factor group, which gives a contradiction.

Conversely, let $\mathcal{M}$ be a non-abelian variety such that all its metabelian groups are abelian. If $G$ is a finite non-abelian group of minimal order in $\mathcal{M}$, then all subgroups in $G$ are abelian, and by a theorem of O. Schmidt, $G$ is a metabelian (non-abelian) group in $\mathcal{M}$, which gives a contradiction.

The question of existence of pseudoabelian varieties was formulated in 1967 [6] and is known as “The Fifth Problem of Hanna Neumann” The Problem was solved positively in 1985 by A.Yu. Ol’shanskii [7] who gave an example of a pseudoabelian variety. This example gives a positive answer for the Question 2 because of the following theorem [4].

**Theorem 3.** *The Ol’shanskii pseudoabelian variety is a t-variety.*

The proof is based on checking that the Ol’shanskii variety satisfies an identity of the type (1).

The following natural question arises:

**Question 3.** *Is every t-variety pseudoabelian?*

We can show that the answer is positive.

**Theorem 4.** *Every variety of $t$-groups is pseudoabelian.*

*Proof.* In view of Theorems 1, 2, we have to show that if $G$ is metabelian and satisfies identity (1), then $G$ is abelian.

By substituting $[y,z]$ for $y$ in the equation $[x,y] = u(x,y)$, we obtain that for $x,y,z \in G$: $[x,[y,z]] \in G'' = 1$ and hence $G$ is nilpotent of class at most 2. Now $[x,y] = u(x,y) \in [[G,G],G] = 1$ and hence $G$ is abelian as required.

We consider now converse to the above statement.

**Question 4.** *Is every pseudoabelian variety a $t$-variety?*

This Question is open. To give a positive answer it is enough to show that if an element $g \in G$ fails to normalize a subnormal subgroup of $G$, then $G$ has a metabelian factor and so cannot belong to any pseudoabelian variety.
There exists an unpublished partial result obtained independently by L. Kovács and Peter M. Neumann [3]: If an element of *squarefree order* fails to normalize a subnormal subgroup of a group $G$, then $G$ has a metabelian factor. It is shown in [4] that the condition of squarefree order can be weakened to only odd squares.

So, we can only conjecture that *a variety consists of $t$-groups if and only if it is pseudoabelian.*

## REFERENCES


[1] W. Gaschütz, *Gruppen in denen das Normalteilersein transitiv ist*, J. Reine Angew. Math. **198** (1957), 87–92.

[2] N.D. Gupta, *Some group-laws equivalent to the commutative law*, Arch. Math. **17** (1966), 97–102.

[3] L.G. Kovács and M.F. Newman, *Hanna Neumann's problems on varieties of groups*, Proc. Second Internat. Conf. Theory of Groups, Lecture Notes in Math., vol. 372, Springer-Verlag, Berlin, Heidelberg, New York, 1974, pp. 417–431.

[4] O. Macedońska, A. Storozhev, *Varieties of $t$-groups*, Communications in Algebra (to appear).

[5] B.H. Neumann, *Ascending derived series*, Compositio Math. **13** (1956), 47–64.

[6] H. Neumann, *Varieties of groups*, Springer-Verlag, Berlin, Heidelberg, New York, 1967.

[7] A.Yu. Ol'shanskii, *Varieties in which all finite groups are abelian*, Mat. Sbornik **126** (1985), 59–82.

[8] D.J.S. Robinson, *Groups in which normality is a transitive relation*, Proc. Camb. Phil. Soc. **60** (1964), 21–38.


Institute of Mathematics, Silesian Technical University,
ul. Kaszubska 23, 44-100 Gliwice, Poland
e-mail: olga@zeus.polsl.gliwice.pl

*Received 1.02.1997*