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ISOMORPHIC EMBEDDINGS OF FINITE METRIC SPACES INTO HAMMING SPACES

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The category of finite metric spaces, morphisms of which are the maps preserving unequalities of distances between pairs of points, is considered. It is proved that every finite metric space admits an isomorphic embedding (in sense of this category) into some Hamming space. The obvious construction of such an embedding is indicated.

1. In this paper we only deal with finite metric spaces. A map φ of a space (X, ϱ) to a space (Y, σ) is called monotone if for all $x, y, u, v \in X$ the inequality $\varrho(x, y) \leq \varrho(u, v)$ implies the inequality $\sigma(\varphi(x), \varphi(y)) \leq \sigma(\varphi(u), \varphi(v))$. All finite metric spaces form a category, in which morfisms are monotone maps. A one-to-one correspondence τ between spaces X and Y will be an isomorphism of this category if it preserves both equalities and strict inequalities of distances between points. An embedding f of a space (X, ϱ) into a space (Y, σ) is isomorphic if spaces (X, ϱ) and $(f(X), \sigma)$ are isomorphic.

At the conference "Algebra and combinatorics. Dresden- Koeningstein, 1994" Professor B. Ganter formulated the following question:

Does every finite metric space admit an isomorphic embeddind for some $m \in \mathbb{N}$ into the Hamming space H_m, the space of all boolean vectors of length m endowed with the Hamming metric (the distance between two points equals to the quantity of their different coordinates).

The main result of this paper is the affirmative answer to this question.

Theorem. *For any n-point metric space X there exists an isomorphic embedding of X into the Hamming space H_m, where*

$$m = \frac{1}{2}\left(\binom{n}{2}^2 - \binom{n}{2} - 2\right)(n^2 - 2n + 7). \tag{1}$$

For a set Ω, $|\Omega| = m$, let 2^Ω denote the set of all subsets of Ω. We define the *halved cube* HQ_m as the graph on the even (or odd) size subsets in 2^Ω, in which two such subsets A and B are adjacent whenever $|A \triangle B| = 2$ ([1]). A connected graph forms a metric space by the natural distance, i.e., the length of the shortest path connecting its vertices.

Corollary 1. *Every n-point metric space admits an isomorphic embedding into the halved cube HQ_m, where m is defined by (1).*

Connected graphs are isomorphic iff they are isomorphic as metric spaces. Hence, the Frucht theorem [2] implies

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Corollary 2. *Every finite group is isomorphic to the isometry group of some subspace of a Hamming space.*

The results were announced in [3].

2. Let (X,ϱ) be a metric space. Define its *spectrum* SpX as the ordered set of distinct nonzero values taken by the distance function ϱ.

In [4] it is proved that every n-point metric space with $|SpX| = k$ is isomorphic to a space with the spectrum $\big(\binom{n}{2}, \binom{n}{2}+1, \dots, \binom{n}{2}+k-1\big)$.

Since every homothety is an isomorphism we have obtained

Lemma 1. *Every n-point metric space with $|SpX| = k$ is isomorphic to a space with the spectrum*

$$\left(2\binom{n}{2}, 2\binom{n}{2}+2, \dots, 2\binom{n}{2}+2k-2\right). \tag{2}$$

We say, that an n-point metric space (X,ϱ) and a distance function ϱ are *standard* if SpX has the form (2) for some $k \in \mathbb{N}$. Standard spaces are isomorphic iff they are isometric. Therefore, every class of pairwise isomorphic spaces contains a unique up to isometry standard space.

Evidently, that for an n-point metric space (X,ϱ) with the spectrum $(2l, 2l+2, \dots, 2l+2t)$ for some $l,t \in \mathbb{N}$ (so called *2-dense* spectrum) the function

$$d(x,y) = \begin{cases} \varrho(x,y) + n(n-1) - 2l, & \text{if } x \neq y, \\ 0, & \text{if } x = y \end{cases}$$

is the standard distance function on X.

Definition. We say, that a space (Y,σ) is *similar* to a space (X,ϱ) if these spaces are isomorphic to standard spaces (Y_1,σ_1), (X_1,ϱ_1) respectively and there exists a bijection $\tau\colon X_1 \to Y_1$ such that for a unique pair $\{x_0,y_0\} \subset X$, $\varrho_1(x_0,y_0)+2 = \sigma_1(\tau(x_0),\tau(y_0))$ and for all other $\{x,y\} \subset X$, $\varrho_1(x,y) = \sigma_1(\tau(x),\tau(y))$.

Lemma 2. *For every metric space (X,ϱ) there exist $r = r(X) \geq 0$ and a sequence*

$$(X_0,\varrho_0), (X_1,\varrho_1), \dots, (X_r,\varrho_r) \tag{3}$$

of spaces satisfying the conditions:
1) $|SpX_0| = 1$;
2) space X_i is similar to X_{i-1} for all $i = \overline{1,r}$;
3) spaces X and X_r are isomorphic.

Proof. If $|SpX| = 1$ then $r = 0$ and $(X_0,\varrho_0) = (X,\varrho)$.

Let (X,ϱ) be an n-point metric space and $SpX = (a_1, a_2, \dots, a_k)$, $k > 1$. Assume that the distance function ϱ takes l_1 times the value a_1, l_2 times the value a_2,..., l_k times the value a_k. Then $l_1 + l_2 + \dots + l_k = \binom{n}{2}$. Let $r = l_2 + 2l_3 + \dots + (k-1)l_k$. By Lemma 1 the space (X,ϱ) is isomorphic to some standard space. Denote it by (X_r,ϱ_r). Since $|SpX_r| = |SpX| = k > 1$ then there exist $x_0, y_0 \in X_r$ such that $\varrho_r(x_0,y_0) > 2\binom{n}{2}$. Define a space (X_{r-1},ϱ_{r-1}) setting $X_{r-1} = X_r$ and

$$\varrho_{r-1}(x,y) = \begin{cases} \varrho_r(x,y) - 2, & \text{if } \{x,y\} = \{x_0,y_0\}, \\ \varrho_r(x,y), & \text{if } \{x,y\} \neq \{x_0,y_0\}, \end{cases} \quad x,y \in X_{r-1}.$$

Then ϱ_{r-1} is a distance function and the space (X_r,ϱ_r) is similar to (X_{r-1},ϱ_{r-1}). Repeating this procedure r times we obtain a sought chain $(X_0,\varrho_0), (X_1,\varrho_1), \dots, (X_r,\varrho_r)$.

Note, that the number $r(X)$ is not greater then $\frac{1}{2}\binom{n}{2}\big(\binom{n}{2}-1\big) - 1$ and we have the equality whenever all nonzero values of distance function are different.

Lemma 3. *Let (X,ϱ), (Y,σ) be n-point metric spaces and Y is similar to X. If there exists an isomorphic embedding φ of X into the Hamming space H_m and the spectrum of $\varphi(X)$ is 2-dense then there exists an isomorphic embedding ψ of the space (Y,σ) into the Hamming space H_k, where $k = m + n^2 - 2n + 7$ and the spectrum of $\psi(X)$ is 2-dense.*

Proof. According to Lemma 1 every metric space is isomorphic to some standard space and it is sufficient to embed into the Hamming space standard spaces only. Therefore, without loss of generality, we may assume that the spaces X and Y are standard. Let $X = \{x_1, x_2, \dots, x_n\}$, $Y = \{y_1, y_2, \dots, y_n\}$. Denote $z_i = \varphi(x_i)$, $i = \overline{1,n}$. By the condition of the lemma $Sp(\varphi(X))$ is 2-dense. Hence, there exists $C \in R$ such that $d(z_i, z_j) + C = \varrho(x_i, x_j)$ for $i, j = \overline{1,n}, i \neq j$ (we denote by d the distance function in the Hamming space H_m for all $m \in N$). Since Y is similar to X, there exists a map $\tau: X \to Y$ such that $\tau(x_i) = y_i$, $i = \overline{1,n}$. Assume that $\varrho(x_{n-1}, x_n) + 2 = \sigma(y_{n-1}, y_n)$ and $\varrho(x_i, x_j) = \sigma(y_i, y_j)$ for $\{i,j\} \neq \{n-1, n\}$.

Consider continuations $\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n \in H_k$ of vectors $z_1, z_2, \dots, z_n$ ($k = m + n^2 - 2n + 7$):

$$\tilde{z}_1 = z_1 * (\underbrace{111\dots1}_{n}\underbrace{000\dots0}_{n}\underbrace{000\dots0}_{n}\dots\underbrace{000\dots0}_{n}\underbrace{000\dots0}_{n}0000000),$$
$$\tilde{z}_2 = z_2 * (\underbrace{000\dots0}_{n}\underbrace{111\dots1}_{n}\underbrace{000\dots0}_{n}\dots\underbrace{000\dots0}_{n}\underbrace{000\dots0}_{n}0000000),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\tilde{z}_{n-2} = z_{n-2} * (\underbrace{000\dots0}_{n}\underbrace{000\dots0}_{n}\underbrace{000\dots0}_{n}\dots\underbrace{000\dots0}_{n}\underbrace{111\dots1}_{n}0000000)$$
$$\tilde{z}_{n-1} = z_{n-1} * (\underbrace{100\dots0}_{n}\underbrace{100\dots0}_{n}\underbrace{100\dots0}_{n}\dots\underbrace{100\dots0}_{n}\underbrace{100\dots0}_{n}1111000)$$
$$\tilde{z}_n = z_n * (\underbrace{010\dots0}_{n}\underbrace{010\dots0}_{n}\underbrace{010\dots0}_{n}\dots\underbrace{010\dots0}_{n}\underbrace{010\dots0}_{n}0001111).$$

Then the map $\psi: Y \to H_k$ such that $\psi(y_i) = \tilde{z}_i$ ($i = \overline{1,n}$) is an isomorphic embedding of (Y,σ) into the Hamming space H_k.

Indeed, from the definition of $\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n$ we obtain

a) $d(\tilde{z}_i, \tilde{z}_j) = d(z_i, z_j) + 2n$, $i, j = \overline{1, n-2}$;

b) $d(\tilde{z}_i, \tilde{z}_{n-1}) = d(z_i, z_{n-1}) + n - 3 + n - 1 + 4 = d(z_i, z_(n-1)) + 2n$, $i = \overline{1, n-2}$;

c) $d(\tilde{z}_i, \tilde{z}_n) = d(z_i, z_n) + n - 3 + n - 1 + 4 = d(z_i, z_n) + 2n$, $i = \overline{1, n-2}$;

d) $d(\tilde{z}_{n-1}, \tilde{z}_n) = d(z_{n-1}, z_n) + 2(n-2) + 6 = d(z_(n-1), z_n) + 2n + 2$.

So, a)–c) imply $d(\tilde{z}_i, \tilde{z}_j) - 2n + C = d(z_i, z_j) + 2n - 2n + C = \varrho(x_i, x_j) = \sigma(y_i, y_j)$ for $\{i,j\} \neq \{n-1, n\}$ and d) implies $d(\tilde{z}_{n-1}, \tilde{z}_n) - 2n + C = d(z_{n-1}, z_n) + 2n + 2 - 2n + C = \varrho(x_{n-1}, x_n) + 2 = \sigma(y_{n-1}, y_n)$.

Hence, ψ is an isomorphic embedding.

Note, that the space $\psi(Y)$ has the 2-dense spectrum too.

3. *Proof of the theorem.* Let X be an n-point metric space. Consider the number $r(X)$.

For $r(X) = 0$ we have the n-point space X in which all nonzero distances are equal. Such a space is isomorphic to the space of unit vectors of H_n.

Let $r(X) \geq 1$. We will prove that X is isomorphic to a subspace of the Hamming space H_m with the 2-dense spectrum, where $m = r(X)(n^2 - 2n + 7)$. Since $r(X) + 1 \leq \frac{1}{2}\binom{n}{2}\left(\binom{n}{2} - 1\right)$, we obtain the required bound. We use the induction on $r(X)$.

If $r(X) = 1$ then as in Lemma 3 we can show that X is isomorphic to the subspace of the Hamming space H_m ($m = n^2 - 2n + 7$) with the 2-dense spectrum.

Let $r(X) > 1$. Then the space X is similar to some space Y with $r(Y) = r(X) - 1$. By the assumption of induction the space Y is isomorphic to some subspace of $H_{m'}$ with the 2-dense spectrum ($m' = (r(X) - 1)(n^2 - 2n + 7)$). According to Lemma 3 there exists an isomorphic embedding of X into the Hamming space H_m, where $m = m' + n^2 - 2n + 7 = r(X)(n^2 - 2n + 7)$, and the subspace of H_m, which is isomorphic to X, has the 2-dense spectrum.

Remark. It easy to prove that an introduced notion of isomorphism for metric spaces coincides with a wellknown notion of isomorphism by metric transform ([5],[6]). Namely, let $F(t)$ be a continuous monotone increasing function of $t > 0$ with $F(0) = 0$ (such functions are called scales [5]) and let (X, ϱ) be any metric space. If we replace the distance $\varrho(x, y)$ in X by $F(\varrho(x, y))$ then we obtain a new space which is said to be a *metric transform* of X by F and denote it by $F(X)$. Two spaces X, Y are isomorphic if there exists a scale F such that $F(X)$ and Y are isometric. Isomorphic embeddings of metric spaces by metric transforms into different "standard" spaces have been studied by many authors (cf. [6–8]). Using [8] one can show that every n-point metric space admits an isomorphical embedding by metric transform into the Hamming space H_m, $m = O(n^5)$. However our embedding is of order $O(n^6)$, we indicate the obvious construction and our proof is more simple.

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