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ON A CONJECTURE OF B. HARTLEY AND SOME RELATED PROBLEMS

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In this note we are going to survey some questions and results related to a conjecture of Brian Hartley about identities for groups of units in group algebras.

1. Statement of questions

In this paper K will denote an infinite field, $\mathbb{Z}$ the ring of rational integers, $1 \neq G$ a periodic group, RG the group ring of G over a ring R, and $\mathrm{U}(R)$ the group of units of any ring R. F stands for a free group with two generators, and $\mathrm{GL}_n(R)$ for the full linear group over a given ring R.

Let p be a prime number. Then G is a *p-abelian group* if its commutator subgroup G' is a finite p-group, and G is *a p'-group* if it has no elements of order p. We will also apply some other standard notation and terminology, as for example in [B1,P1,S].

The result below is rather nontrivial, (see [T]), but very important for our considerations.

Lemma 1.1 (Tits' alternative). *Let $H \subseteq \mathrm{GL}_n(K)$ be a finitely generated group. Then either H contains a solvable subgroup of finite index or $F \subseteq H$.*

Several authors studied important identities for $\mathrm{U}(\mathbb{Z}G)$ and $\mathrm{U}(KG)$, (see for example [S,P1]). As arguments they used, among others, Dirichlet unit theorem, the structure of hamiltonian groups, and linearisation. Using, in addition, Tits' alternative Hartley and Pickel in [HPi] were able to prove the following result about units of integral group rings:

Theorem 1. *Let G be a locally finite group. Then the following conditions are equivalent:*
1. $\mathrm{U}(\mathbb{Z}G)$ *is nilpotent;*
2. $\mathrm{U}(\mathbb{Z}G)$ *is solvable;*
3. There exists a group identity satisfied by $\mathrm{U}(\mathbb{Z}G)$*;*
4. $F \not\subseteq \mathrm{U}(\mathbb{Z}G)$*;*
5. G is either abelian or a hamiltonian 2-group.

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Under any of these conditions $\mathbb{Z}G$ *satisfies many polynomial identities.*

Probably as a counterpart to the above theorem for group algebras Brian Hartley formulated the following conjecture:

Conjecture 1. *If* G *is a group such that* $\mathrm{U}(KG)$ *satisfies a group identity, then* KG *satisfies a polynomial identity, (is a PI-algebra).*

Applying the characterization of group algebras satisfying polynomial identities given by Passman and Isaacs, (see [P1]), we can reformulate the conjecture of Hartley as two, more group-theoretical ones:

Conjecture 2. *Let* K *be a field of characteristic* 0 *and let* G *be a group such that* $\mathrm{U}(KG)$ *satisfies a group identity. Then* G *has an abelian subgroup of finite index.*

Conjecture 3. *Let* K *be a field of characteristic* $p > 0$ *and let* G *be a group such that* $\mathrm{U}(KG)$ *satisfies a group identity. Then* G *contains a* p*-abelian subgroup of finite index.*

The question below is a natural extension of Hartley's conjecture

Problem 4. *Find necessary and sufficient conditions under which* $\mathrm{U}(KG)$ *satisfies a group identity.*

Following results of [HPi] one can also ask the following question:

Problem 5. *Find necessary and sufficient conditions under which* $F \subseteq \mathrm{U}(KG)$.

2. Finite case

Before discussing the above problems let us look on finite dimensional algebras. We begin with the following fact:

Lemma 2.1. *Let* C *be a commutative ring. If either* $\mathbb{Z} \subseteq C$ *or* C *contains an element transcendental over its minimal subring* C_0*, (generated by* 1*), then* $\mathrm{GL}_2(C)$ *contains a copy of* F.

Proof. If either $\mathbb{Z} \subseteq C$ or the order of C_0 has an odd prime divisor then one can apply elementary arguments, as in [W].

On the other hand, if C has an element x transcendental over C_0 then from [N] we know that $F \subseteq \mathrm{GL}_2(C_0[x])$. But we can provide an easier argument suggested by Z.S. Marciniak, and realized by A. Salwa.

Let $A = C[x]$. Consider subgroups $G, H \subset \mathrm{GL}_2(A)$, where $G = \langle \left[\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right] \rangle$ and $H = \langle \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] \rangle$. Then the group $\langle G, H\rangle$ acts in a natural way on the set $\Omega = A^2 \setminus (0,0)$. Let us put $\Omega_1 = \big\{(f,g) \mid \deg(f) > \deg(g)\big\}$ and $\Omega_2 = \Omega \setminus \Omega_1$. Now $G\Omega_1 \subseteq \Omega_2$ and $H\Omega_2 \subseteq \Omega_1$. Applying very simple Macbeath's lemma, (see Proposition 3.12.2 in [LS]), we obtain that the group $\langle G, H\rangle$ is in fact a free product of G and H. Now $F \subset \langle G, H\rangle$ because $|G| = 3$ and H is nontrivial.

Corollary 2.2. *Let* C *be an infinite domain. Then the group* $\mathrm{GL}_2(C)$ *satisfies no group identity.*

Proof. From the assumption we have, that the polynomial ring $C[x]$ is embeddable into a direct product of infinitely many copies of C, and the result follows.

Up to now Tits' alternative seems to be indispensable in the proof of the lemma below.

Lemma 2.3 ([Go]). *Let D be a central, finite dimensional division K-algebra. If $D \neq K$ then $F \subset \mathrm{U}(D)$.*

Our field K is infinite. Hence, as a consequence we obtain

Theorem 2. *Let A be a semiprime, finite dimensional k-algebra. If $\mathrm{U}(A)$ satisfies a group identity then A is commutative.*

Now as consequences for group algebras we have:

Corollary 2.4 ([Go]). *Let K be of characteristic 0 and let G be a (locally) finite group. Then $F \not\subseteq \mathrm{U}(KG)$ if and only if G is abelian.*

In positive characteristic the situation is not so clear. We present it as three separate facts.

Corollary 2.5 ([Go]). *Let K be of characteristic $p > 0$ and let G be a finite group. Then $F \not\subseteq \mathrm{U}(KG)$ if and only if either G is p-abelian or K is algebraic over its prime subfield.*

Corollary 2.6 ([Go]). *Let K be a field of characteristic $p > 0$ which is algebraic over its prime subfield. If G is any finite nonabelian p'-group then $\mathrm{U}(KG)$ does not satisfy a nontrivial group identity, but $\mathrm{U}(KG)$ is locally finite.*

Proposition 2.7. *Let K be a field of characteristic $p > 0$ not algebraic over its prime subfield and let G be locally finite. Then the following conditions are equivalent:*
1. $F \not\subseteq \mathrm{U}(KG)$;
2. G' is a p-group.

3. Main results

Results related to finite dimensional algebras do not help in the case of infinite and not locally finite groups. Suitable argument was initiated in [GJV] and with additions from [GSV] is as follows:

Lemma 3.1 ([GJV,GSV]). *Let A be a semiprime K-algebra such that $\mathrm{U}(A)$ satisfies a group identity.*
1. If $a, b \in A$ are such that $ab = 0$ then $acb = 0$ for any nilpotent element $c \in A$;
2. Any idempotent of A is central.

For group algebras over fields of characteristic 0 we can also apply the following, not very difficult result:

Lemma 3.2 ([MS]). *If u is a nontrivial bicyclic unit of $\mathbb{Z}G$ then the subgroup $\langle u, u^* \rangle \subset \mathrm{U}(\mathbb{Z}G)$ is isomorphic to F.*

Further $(x, y) = x^{-1}y^{-1}xy$ and $[x, y] = xy - yx$. Under this notation, from the above lemmas we immediately obtain

Theorem 3 ([GJV,MS]). *Let K be a field of characteristic 0. Then the following conditions are equivalent:*
1. There exists a nontrivial group identity satisfied by $\mathrm{U}(KG)$;
2. $F \not\subseteq \mathrm{U}(KG)$;
3. G is abelian;

4. $\mathrm{U}(KG)$ *satisfies the identity* $(x, y) \equiv 1$*;*
5. KG *satisfies the identity* $[x, y] \equiv 0$*.*

In positive characteristic we have not so strong result.

Theorem 4 ([GSV,P]). *Let* K *be a field of characteristic* $p > 0$*. Then the following conditions are equivalent:*
1. There exists a nontrivial group identity satisfied by $\mathrm{U}(KG)$*;*
2. G *contains a* p*-abelian subgroup of finite index and the commutator subgroup* G' *of* G *is a* p*-group of bounded exponent;*
3. $\mathrm{U}(KG)$ *satisfies the identity* $(x, y)^{p^n} \equiv 1$ *for some* $n \geq 0$*;*
4. KG *satisfies the identity* $[x, y]^{p^m} \equiv 0$ *for some* $m \geq 0$*.*

The main ingredients of the proof are:

Lemma 3.3 ([GSV]). *Let* K *be of characteristic* $p > 0$ *and let* $\mathrm{U}(KG)$ *satisfy a nontrivial identity.*
1. If KG *is semiprime then* G *is abelian;*
2. If there exists a nilpotent ideal I *of* KG *such that the factor algebra* KG/I *is semiprime then* G *is* p*-abelian;*
3. In any case G *contains a* p*-abelian subgroup of finite index and the commutator subgroup* G' *of* G *is a* p*-group. In particular* G *is locally finite.*

Corollary 3.4 ([P]). *If* $\mathrm{U}(KG)$ *satisfies an identity then* $\mathrm{U}(K[G/N])$ *satisfies the same identity for any normal subgroup* N *of* G*.*

Lemma 3.5 ([P]). *Let* $G = \langle A, b\rangle$*, where* A *is normal abelian subgroup of* G*. If* $\mathrm{U}(KG)$ *satisfies an identity, then the commutator subgroup* G' *of* G *is of finite exponent.*

As an illustration of the above theorem let us consider the following example.

Example 1. Let A be an abelian p-group, which is not of bounded exponent, and let G be a semidirect product of A with a copy of C_2 generated by b, where $a^b = a^{-1}$ for any $a \in A$. Then G has an abelian subgroup of index 2, but the commutator subgroup G' of G is a p-group of unbounded exponent.

As an extension of some results from the previous section we have

Theorem 5 ([GJV,GoP]). *Let* K *be a field of characteristic* $p > 0$ *not algebraic over its prime subfield and let* G *be a* p'*-group. Then the following conditions are equivalent:*
1. There exists a nontrivial group identity satisfied by $\mathrm{U}(KG)$*;*
2. $F \not\subseteq \mathrm{U}(KG)$*;*
3. G *is abelian.*

Remarks. 1. The above theorem can certainly be proved under not so strong restrictions on the group G.
2. All results were formulated here only for periodic groups. If one is going to extend them to arbitrary groups then some additional techniques are needed and formulation of results is less readable (see for exaple [S,HPi,B2]).
3. The results of papers [GJV,GSV,P] and [GoP] discussed in this note inspired also investigation of units of some PI-algebras which are not group algebras (see [BRT]).

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