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ON SEMIPERFECT RINGS DETERMINED BY ADJOINT GROUPS

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We characterize the semiperfect rings with an Engel adjoint group (correspondently with an adjoint group of finite abelian subgroup rank or adjoint FC-group).

1. Introduction. Let R be an associative ring. We denote by $\circ$ the so-called circle composition on R, defined by $a \circ b = a + b + ab$ for all a and b in R. It is well known that the set of all elements of R forms a semigroup with identity element $0 \in R$ under the circle composition. The group of all invertible elements of this semigroup is called the adjoint (or equivalently, quasi-regular) group of R denoted by R°. If R is a ring with identity $1 \neq 0$ then $\mathcal{U}(R)$ will always denote the unit group of R and moreover $R^\circ \cong \mathcal{U}(R)$.

Many authors have studied the rings with prescribed adjoint groups (or equivalently, groups of units) (see, for example, [1–19]).

In this paper we study the semiperfect rings with an Engel adjoint group (respectively adjoint group of finite abelian subgroup rank or adjoint FC-group) and give the several characterizations of those rings.

For convenience of readers we recall some notation.

R^+ always denote the additive group of R, $\mathcal{J}(R)$ the Jacobson radical of R, T^* is the multiplicative group of a division ring T, $charR$ is the characteristic of ring R, $1 + \mathcal{J}(R)$ is the unipotent subgroup of R.

Throughout the paper p is a prime. In the sequel we will use the following notation:

$K \rtimes T$ is a semidirect product of subgroups K and T with K is normal;
$GF(q)$ is the finite field of q elements, $q = p^m$;
$M_n(q)$ is the ring of all $n \times n$ matrices over $GF(q)$;
$GL_n(q)$ is the group of all invertible $n \times n$ matrices over $GF(q)$;
$Z(G)$ is the centre of G;
$C_G(H)$ is the centralizer of H in G;
τA is the torsion part of a group A;
$[x, y] = xyx^{-1}y^{-1}$ is the commutator of x and y in $\mathcal{U}(R)$;
$[[x, y]] = x \circ y \circ x^{-1} \circ y^{-1}$ is the commutator of x and y in R°.

Let us recall that a ring R is semilocal if the quotient $R/\mathcal{J}(R)$ is right Artinian, and semilocal ring R is semiperfect if all idempotents of $R/\mathcal{J}(R)$ can be lifted modulo $\mathcal{J}(R)$ to idempotents of R.

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All other notation and terminology can be found, for example, in [20–23].

2. In this section we study several properties of the semiperfect rings with the Engel adjoint groups. Let x and y be elements of group G and let n be a positive integer. The commutator $[x, {}_n y]$ is defined by the rule $[x, {}_{n+1} y] = [[x, {}_n y], y]$. Recall [22, p.40], an element x of G is called a right Engel (respectively a left Engel) element of G if for each g in G there is an integer $n = n(x, g) \geq 0$ such that $[x, {}_n g] = 1$ (respectively $[g, {}_n x] = 1$). The set of right Engel (respectively left Engel) elements of G is denoted by $R(G)$ (respectively $L(G)$). A group G is called an Engel group if every element x of G is left Engel (or equivalently, every element x of G is right Engel), that is $G = L(G)$ (or equivalently, $G = R(G)$). As it is known [22, Lemma 7.12] $L(G)$ contains the Hirsch-Plotkin radical $HP(G)$ of G and $R(G)$ contains the hypercentre $\bar{\zeta}(G)$ of G.

From [24] follows

Lemma 2.1. *Let T be a skew field. Then the following statements are equivalent:*
(i) T^* is locally nilpotent;
(ii) T^* is hypercentral;
(iii) T^* is nilpotent;
(iv) T^* is abelian;
(v) T^* is radical;
(vi) T^* is locally solvable;
(vii) T^* is solvable;
(viii) T^* is RN^*-group;
(ix) T^* is hypocentral.

As in [25], $\mathfrak{A}$ will denote the class of all fields which are the algebraic extensions of their simple subfields.

Lemma 2.2. *Let R be a local ring such that $R = \mathcal{J}(R) + B$ for some field B. Then $\mathcal{U}(R) = (1 + \mathcal{J}(R)) \rtimes B^*$.*

Proof. Let z is an arbitrary element of $\mathcal{U}(R)$. Then $z = j + b$ with $j \in \mathcal{J}(R)$ and $b \in B^*$ and as consequence

$$z = (jb^{-1} + 1)b \in (1 + \mathcal{J}(R))B^*.$$

If $x \in (1 + \mathcal{J}(R)) \cap B^*$ then $x = b_1 = 1 + j_1$ for some elements $b_1 \in B^*$ and $j_1 \in \mathcal{J}(R)$. From this $j_1 = b_1 - 1 \in B$, a contradiction. Thus $(1 + \mathcal{J}(R)) \cap B^* = 1$. Moreover, $(1 + \mathcal{J}(R)) \lhd \mathcal{U}(R)$ and so $\mathcal{U}(R) = (1 + \mathcal{J}(R)) \rtimes B^*$, as desired.

Lemma 2.3. *Let R be a local ring with a locally nilpotent unit group $\mathcal{U}(R)$ and the Jacobson radical $\mathcal{J}(R)$ is a nil ideal.*

(i) If $\operatorname{char} R = \operatorname{char}(R/\mathcal{J}(R))$ then R contains a subfield B such that an extension $(\mathcal{J}(R) + B)/\mathcal{J}(R) \subseteq R/\mathcal{J}(R)$ is algebraic.

(ii) If $\operatorname{char} R/\mathcal{J}(R) = 0$ (respectively $\operatorname{char} R/\mathcal{J}(R) = p$) and $R/\mathcal{J}(R) \notin \mathfrak{A}$ then the unit group $\mathcal{U}(R)$ has a normal series

$$1 \leq K \times T \leq \mathcal{U}(R),$$

where $K = 1 + \mathcal{J}(R)$ is a locally nilpotent torsion-free group for $\operatorname{char} R = 0$ (respectively a locally nilpotent p-group for $\operatorname{char} R = p$), T is a locally cyclic p'-group and a quotient $\mathcal{U}(R)/(K \times T)$ is an abelian torsion free group.

(iii) If $\operatorname{char}(R/\mathcal{J}(R)) = p$ and $R/\mathcal{J}(R) \in \mathfrak{A}$ then $\mathcal{U}(R) = (1 + \mathcal{J}(R)) \times T$, where $1 + \mathcal{J}(R)$ is a locally nilpotent p-group for $\operatorname{char} R = p^m$ (respectively a locally

nilpotent torsion-free group for $\operatorname{char} R = 0$*) and* $T \cong (R/\mathcal{J}(R))^*$ *is a locally cyclic abelian* p'*-group.*

Proof. (i) Let Φ is a family of all subfields of R ordering by inclusion. By Lemma of Zorn Φ has a maximal element B. If the quotient $R/\mathcal{J}(R)$ contains an element $x+\mathcal{J}(R)$ which is transcendental over $(B+\mathcal{J}(R))/\mathcal{J}(R)$ then a subring B_0 generated by the set $B \cup \{x\}$ (where x is a preimage of a coset $x+\mathcal{J}(R)$) has a trivial intersection with the Jacobson radical $\mathcal{J}(R)$ and so it is a field, a contradiction.
(ii) If $\operatorname{char}(R/\mathcal{J}(R)) = 0$ (respectively $\operatorname{char}(R/\mathcal{J}(R)) = p$) then $\mathcal{J}(R)^+$ is torsion-free (respectively p-group) and by Lemma 2.4 of [17] $\mathcal{J}(R)^\circ$ is also torsion-free (respectively p-group). Thus the unit group $\mathcal{U}(R)$ has a normal series $1 \leq K \times T \leq \mathcal{U}(R)$ such that $K = 1+\mathcal{J}(R)$, T is a locally cyclic p'-group (see [26]) and T is isomorphic to the periodic part $\tau(R/\mathcal{J}(R))^*$ of multiplicative group $(R/\mathcal{J}(R))^*$ and $\mathcal{U}(R)/(K\times T)$ is abelian torsion-free.
(iii) If $R/\mathcal{J}(R) \in \mathfrak{A}$ and $\operatorname{char}(R/\mathcal{J}(R)) = p$ then $(R/\mathcal{J}(R))^*$ is a locally cyclic p'-group (see [26]) and so $\mathcal{U}(R) = (1+\mathcal{J}(R)) \times F$, where $F \cong (R/\mathcal{J}(R))^*$ and $1+\mathcal{J}(R)$ is a locally nilpotent torsion-free (respectively p-group) for $\operatorname{char} R = 0$ (respectively for $\operatorname{char} R = p^m$). The lemma is proved.

Lemma 2.4. *Let* R *be a ring such that* $\bigcap_{n=1}^{\infty} \mathcal{J}(R)^n = 0$ *and the quotient* $R/\mathcal{J}(R) = K_1 \oplus \cdots \oplus K_l$ *is a direct sum of fields* K_i $(i = 1,\dots,l)$ *and the unit group* $\mathcal{U}(R)$ *is an Engel group. If all fields* K_i *are algebraic over their simple subfields then*
(i) $[\mathcal{U}(R),\mathcal{U}(R)] \leq 1+\mathcal{J}(R)$,
(ii) $[\mathcal{U}(R), 1+\mathcal{J}(R)^m] \leq 1+\mathcal{J}(R)^{m+1}$,
(iii) $[1+\mathcal{J}(R), {}_m\mathcal{U}(R)] \leq 1+\mathcal{J}(R)^{m+1}$ *for all positive integers* m.

Proof. Since $(R/\mathcal{J}(R))^* \cong K_1^* \times \cdots \times K_l^*$ is an abelian group then (i) holds.
For (ii), let $\mathcal{J}(R)^m \neq 0$ and $\mathcal{J}(R)^{m+1} = 0$ for some positive integer m. Suppose that

$$ai - ia \neq 0 \tag{2.1}$$

for some $a \in \mathcal{U}(R)$ and $i \in \mathcal{J}(R)^m$. Let i_n is an element of $\mathcal{J}(R)$ defined by the rule $[1+i_n, a] = [1+i, {}_n a]$ for every positive integer n. Then there is a positive integer s such that $[1+i_s, a] = 1+i_s - a i_s a^{-1} = i_{s+1} \neq 1$, but $[1+i_{s+1}, a] = 1+i_{s+1} - a i_{s+1} a^{-1} = 1$. Hence, $a i_{s+1} = i_{s+1} a$, $a i_s - i_s a = -i_{s+1} a \neq 0$.

Let P_w is the simple subfield of $K_w, \alpha = a+\mathcal{J}(R)$ is an arbitrary element of $R/\mathcal{J}(R)$. Then $\alpha = a_1 a_2 \dots a_u + \mathcal{J}(R)$, where $\alpha_w = a_w + \mathcal{J}(R) \in K_w$ $(w = 1,\dots,u)$. From (2.1) follows that

$$a_w i - i a_w \neq 0 \tag{2.2}$$

for some $w (1 \leq w \leq u)$. Then as before

$$[1+i_z, a_w] = 1 + i_z - a_w i_z a_w^{-1} = i_{z+1} \neq 1, \tag{2.3}$$

and $[1+i_{z+1}, a_w] = 1+i_{z+1} - a_w i_{z+1} a_w^{-1} = 1$. Hence,

$$a_w i_{z+1} = i_{z+1} a_w \quad , \tag{2.4}$$

and

$$a_w i_z - i_z a_w = -i_{z+1} a_w \neq 0 \tag{2.5}$$

Let $f_w(X) = X^k + b_1 X^{k-1} + \cdots + b_k$ is the minimum polynomial of $\alpha_w = a_w + \mathcal{J}(R)$ over P_w. A map $D\colon P_w(\alpha_w) \to \mathcal{J}(R)^m$, given by the rule $D(g) = g i_z - i_z g$, $g \in$

$P_w(\alpha_w)$, defines a homomorphism of P_w-modules $P_w(\alpha_w)$ and $\mathcal{J}(R)^m$. Combining (2.4) and (2.5), we obtain

$$\alpha_w D(\alpha_w) = -\alpha_w i_{z+1}\alpha_w = -i_{z+1}\alpha_w^2 = D(\alpha_w)\alpha_w$$

and consequently

$$0 = D(f_w(\alpha_w)) = (k\alpha_w^{k-1} + \cdots + b_{k-1})D(\alpha_w). \tag{2.6}$$

Since an element $k\alpha_w^{k-1} + \cdots + b_{k-1}$ of $P_w(\alpha_w)$ is nontrivial, from (2.6) follows that $D(\alpha_w) = 0$, and so $i_z - a_w i_z a_w^{-1} = 0$, a contradiction with (2.3). Thus $a_w i - i a_w = 0$ and hence (ii) holds. The (iii) follows from (ii).

Corollary 2.5. *Let R be a ring and the quotient $R/\mathcal{J}(R) = K_1 \oplus \cdots \oplus K_u$ is a direct sum of fields. If the unit group $\mathcal{U}(R)$ is an Engel group and all fields are algebraic over their simple subfields then the quotient $R/\mathcal{J}(R)^2$ is commutative.*

For any ring R, as in [27], $R^{(2)}$ is the ideal generated by all elements $rs - sr$ with r, s in R, and, inductively, $R^{(n)}$ is the ideal generated by all $ab - ba$ with $a \in R^{(n-1)}$, $b \in R$. The ring R is said to be strongly Lie nilpotent if $R^{(m)} = 0$ for some m.

From the Lemma 2.4 also follows

Proposition 2.6. *Let R be a ring with the nilpotent Jacobson radical $\mathcal{J}(R)$ and an Engel group $\mathcal{U}(R)$. If the quotient $R/\mathcal{J}(R) = K_1 \oplus \cdots \oplus K_n$ is a direct sum of fields K_i $(i = 1, \ldots, u)$, which are algebraic over their simple subfields, then $\mathcal{U}(R)$ is nilpotent and R is a strongly Lie nilpotent ring.*

Proposition 2.7. *Let R be a semiperfect ring and the quotient $R/\mathcal{J}(R) = K_1 \oplus \cdots \oplus K_l$ is a direct sum of fields K_j $(j = 1, \ldots, l)$. If all elements of $\mathcal{J}(R)$ are right Engel then*

$$R = S \oplus T,$$

where either $S = 0$ or $S = \oplus_{i=1}^k S_i$ is a direct sum of local rings S_i, and either $T = \mathcal{J}(R)$ or T is a semiperfect ring with the quotient $T/\mathcal{J}(T) \cong GF(2) \oplus \cdots \oplus GF(2)$.

Proof. We prove this proposition in the same manner as the Theorem 2.2 of [25].

If $K_i \cong GF(2)$ for all $i (i = 1, \ldots, m)$ then $\mathcal{U}(R) = 1 + \mathcal{J}(R)$ and $S = 0$.

Then let, for example, $F_1 \neq GF(2)$. Put $B = eRe$ where e is nontrivial idempotent of B such that $\pi(e)$ is an identity of K_1 where $\pi\colon R \to R/\mathcal{J}(R)$ is a canonical epimorphism. Then e is an identity of B and, moreover, $B/\mathcal{J}(B) \cong \pi(B) = K_1$ and B is a local ring. Since $B/\mathcal{J}(B) \neq GF(2)$ there exists an element a of $\mathcal{U}(B)$ such that $a - e \in \mathcal{U}(B)$. Further, if b is an inverse of a in $\mathcal{U}(B)$ then $(a-e)\circ(b-e) = (b-e)\circ(a-e) = 0$, and so $a - e \in R^\circ$. Let $R^+ = B^+ \oplus X \oplus Y \oplus C$ be a two-sided Pierce decomposition of additive group R^+, i.a. for all $x \in X$, $y \in Y$ and $c \in C$ holds $ex = x$, $ey = 0$, $ye = y$, $xe = 0$, $ec = ce = 0$ and $X^2 = 0$, $Y^2 = 0$.

Without loss of generality and for simplifying the proof suppose that $\mathcal{J}(R)^m \neq 0$ and $\mathcal{J}(R)^{m+1} = 0$ for some positive integer m. Obviously, $x \circ (-x) = (-x) \circ x = 0$ and so $x \in R^\circ$. Moreover, the commutator

$$[[x, a - e]] = x \circ (a - e) \circ (-x) \circ (b - e) = (e - a)x$$

and $[[x, {}_n a - e]] = (e - a)^n x$ for every positive integer n.

Since $\mathcal{J}(R)^\circ \subseteq R(R^\circ)$, we have $(e - a)^k x = 0$ for some k. But $(e - a)^k \in \mathcal{U}(B)$ and therefore $ex = x$. Thus $X^2 = 0$ and $Y^2 = 0$. Hence $R = B \oplus C$, and the proposition is proved.

The following theorem extended the Theorem of 2.2 [25]. Its proof is similarly to proof of 2.2 of [25] and follows also from Proposition 2.7.

Theorem 2.8. *Let R be a semiperfect ring. Then the following statements are equivalent:*

(i) *R° is an Engel (respectively n-Engel, locally nilpotent, with normalizer condition, hypercentral, nilpotent, Baer or Grunberg) group.*

(ii) *$R = S \oplus T$ where either $S = 0$ or $S = \oplus_{i=1}^{k} S_i$ is a direct sum of local rings S_i, either $T = \mathcal{J}(R)$ or T is a semiperfect ring with the quotient $T/\mathcal{J}(T) \cong GF(2) \oplus \cdots \oplus GF(2)$ and further, S_i° and T° are an Engel (respectively n-Engel, locally nilpotent, with normalizer condition, hypercentral, nilpotent, Baer or Grunberg) group.*

We extend Proposition 3.5 of [25] to

Lemma 2.9. *Let R be a right Artinian local ring. If the unit group $\mathcal{U}(R)$ is an Engel group then it is nilpotent.*

Proof. Show that for every positive integer k there exists a positive integer $n(k)$ such that $[1 + \mathcal{J}(R), {}_{n(k)}\mathcal{U}(R)] \leq 1 + \mathcal{J}(R)^{k+1}$. Without loss of generality and for simplifing the proof let $\mathcal{J}(R)^m \neq 0$ and $\mathcal{J}(R)^{m+1} = 0$. Let $V = \mathcal{J}(R)^m$. Then V is a finite dimensional K-space, where $K = R/\mathcal{J}(R)$. Suppose that $\dim_K V = t$. Prove this lemma by induction on t. Let $i \in V, a \in \mathcal{U}(R)$. Put $\sigma_a(i) = -aia^{-1} + i$. Then $[1 + i_n, a] = 1 + \sigma_a^n(i)$ for some positive integer $s = s(a, i)$. Clearly $\sigma_a^n(i) = 0$ for some n and the centralizer $C_V(a)$ of a in V is nontrivial. Let $a_1, \dots, a_s$ are arbitrary elements of V and put $a_0 = a_1 a_2 \dots a_s$. Since $C_V(a_0)$ is nontrivial, we have that $\bigcap_{i=1}^{s} C_V(a_i)$ is nontrivial and as consequently $I = \bigcap_{a \in \mathcal{U}(R)} C_V(a) \neq 0$. Put $\bar{R} = R/I$. Obviously, $\bar{R}/\mathcal{J}(\bar{R}) = F \cong K$. From $\dim_F(\mathcal{J}(\bar{R})^m) < t$ follows $[\bar{1} + \mathcal{J}(\bar{R})^m, {}_{n(m)}\mathcal{U}(\bar{R})] = \bar{1}]$ for some integer $n(m)$, and therefore $[1 + \mathcal{J}(R)^m, {}_{n(m)}\mathcal{U}(R)] \leq 1 + \mathcal{J}(R)$. Hence, $[1 + \mathcal{J}(R)^m, {}_{n(m)+1}\mathcal{U}(R)] = 1$, as desired.

Corollary 2.10. *Let R be a right Artinian ring. Then the following statements are equivalent:*

(i) *R° is locally nilpotent;*

(ii) *R° is nilpotent;*

(iii) *R° is hypercentral;*

(iv) *R° is an Engel group;*

(v) *R° is a n-Engel group;*

(vi) *$R = S \oplus T$ where either $S = 0$ or $S = \oplus_{i=1}^{n} S_i$ is direct sum of right Artinian local rings S_i, either $T = \mathcal{J}(R)$ or T is right Artinian ring with the quotient $T/\mathcal{J}(T) \cong GF(2) \oplus \cdots \oplus GF(2)$ and, further, S_i° and T° are nilpotent.*

Corollary 2.11. *Let R be a semilocal ring R and its unit group $\mathcal{U}(R)$ is an Engel group. Then the following statements are equivalent:*

(i) *R is a right Artinian ring;*

(ii) *R is a left Artinian ring;*

(iii) *R satisfies the minimal condition for two-sided ideals.*

Lemma 2.12. *Let R be a local ring such that $\bigcap_{n=1}^{\infty} \mathcal{J}(R)^n = 0$ and the quotient $R/\mathcal{J}(R)$ is a field of characteristic p algebraic over its simple subfield. If the unit group $\mathcal{U}(R)$ is hypocentral then*

(i) *$[\mathcal{U}(R), \mathcal{U}(R)] \leq 1 + \mathcal{J}(R)$,*

(ii) *$[\mathcal{U}(R), 1 + \mathcal{J}(R)^m] \leq 1 + \mathcal{J}(R)^{m+1}$ for all positive integers m.*

Proof. (i) is obviously.

To prove (ii), assume that $\mathcal{J}(R)^m \neq 0$ and $\mathcal{J}(R)^{m+1} = 0$ for some positive integer m. Moreover, $\mathcal{U}(R) = (1 + \mathcal{J}(R)) \rtimes T$, where $T \cong (R/\mathcal{J}(R))^*$ is an abelian p'-group and $1 + \mathcal{J}(R)$ is a nilpotent p-group. Since $\mathcal{U}(R)$ is locally finite, we have that $\mathcal{U}(R)$ is locally nilpotent by [21, p.399] and so $\mathcal{U}(R)$ is nilpotent. Thus $\mathcal{J}(R)^m \leq Z(R)$, as desired.

3. In this section we characterize the semiperfect rings whose adjoint group has finite abelian subgroup rank.

Recall that a group G has finite torsion-free rank if it has a finite series whose factors are either periodic or infinite cyclic. For example, the additive group of rational numbers $\mathbb{Q}^+$ has torsion-free rank 1 since $\mathbb{Q}/\mathbb{Z}$ is periodic. A group G has finite abelian subgroup rank if each abelian subgroup of G has finite torsion-free rank and each abelian p-subgroup of G has finite Prüfer rank for every prime p.

The following result is due to the Lemma 3.2 of [18].

Lemma 3.1. *If A is a radical ring with the torsion-free additive group A^+ and the adjoint group A° has finite torsion-free rank n then $A^{n+1} = 0$.*

Proof. A is a nil-ring by Theorem B of [17]. By Zorn's lemma there exists a maximal locally nilpotent subring S of R, which is, in fact, nilpotent by Lemma 3.1 of [17]. Assume that $S \neq R$. By the result of Szasz [28] S properly contained in its idealizer

$$I = \{r \in R | rS + Sr \subseteq S\}\,.$$

Let a be an arbitrary element of I which is not in S. The subring S_1 generated by set $S \cup \{a\}$ is contained in the idealizer I, and therefore S is an ideal of S_1. Since A is nil, we have that the quotient ring S_1/S is nilpotent, and S_1 is also nilpotent, a contradiction. Thus $R = S$. The lemma is proved.

Lemma 3.2. *If the multiplicative group T^* of skew field T has finite torsion-free rank then T is field.*

Proof. Clearly, T^* has a subnormal subgroup T_1 such that $Z(T^*) \leq T_1$ and the quotient $T_1/Z(T^*)$ is periodic or infinite cyclic. If $T_1/Z(T^*) \cong \mathbb{Z}$ then the multiplicative group F^* of subfield F which is generated by T_1 is subnormal in T^*, a contradiction with Theorem 2 of [29]. Then the quotient $T_1/Z(T_1^*)$ is periodic. If it is nonabelian then T_1 contains a noncyclic free group W by Proposition 3.9 of [30]. Since W has also finite torsion-free rank and W is torsion-free, it has a subnormal cyclic subgroup V such that $C_W(V) = V$. Let X is a subnormal subgroup of W such that $V \triangleleft X$. Then the isolator

$$I_X(V) = \{x \in X | x^n \in V \text{ for some positive integer } n = n(x)\}$$

of V in X is a cyclic group (see [21, p.413]), and therefore the quotient X/V is torsion-free. Moreover, by Theorem of Nielsen and Schreier [31, p.33] X is a free group, a contradiction with Theorem of Greenberg [31, p.35]. Thus the quotient $T_1/Z(T^*)$ is abelian, a contradiction with Theorem 2 of [29]. This means that T is field, as desired.

Lemma 3.3. *Let R be a local ring. If its adjoint group R° has finite torsion-free rank then the following holds:*

(i) $\operatorname{char} R = p^s$, *$jr$ is a nil ideal, R° has a normal series*

$$1 \leq A_1 \leq A_2 \leq R^\circ \tag{3.1}$$

such that $A_1 = \mathcal{J}(R)^\circ$ is a p-group, A_2/A_1 is a locally cyclic abelian p'-group and R°/A_2 is subgroup of the direct product of finitely many copies of the additive group of rational numbers ;

(ii) R is a left and right Noetherian ring of characteristic 0, $\mathcal{J}(R)^m = 0$ ($m \in \mathbb{N}$), R° has a normal series

$$1 \leq B_1 \leq B_2 \leq R^\circ \tag{3.2}$$

such that $B_1 = \mathcal{J}(R)^\circ$ is a nilpotent torsion-free group of finite Prüfer rank, B_2/B_1 is a locally cyclic abelian periodic group and R°/B_2 is a subgroup of the direct product of finitely many copies of the additive group of rational numbers.

Proof. (i)⇒(ii). Since the adjoint group $\mathcal{J}(R)^\circ$ of Jacobson radical $\mathcal{J}(R)$ has finite torsion-free rank we conclude that $\mathcal{J}(R)$ is nil by Theorem 3(a) of [17]. Put $T = R/\mathcal{J}(R)$. By Lemma 3.2 T is a field and by Lemma 3.4 of [18] $T^*/\tau T^*$ has finite Prüfer rank.
1) Suppose that $\operatorname{char} T = p$. Then $pR \leq \mathcal{J}(R)$ and consequently $\operatorname{char} R = p^s$ for some positive integer s. Hence $\mathcal{J}(R)^\circ$ and $\mathcal{J}(R)^+$ are p-groups and $\mathcal{J}(R)^\circ$ has the normal series (3.1).
2) Let $\operatorname{char} T = 0$. Then by Lemma 3.2 of [17] the group $\mathcal{J}(R)^+$ is torsion-free and by Lemma 3.1 $\mathcal{J}(R)^m = 0$ for some positive integer m. Moreover, $\mathcal{J}(R)^+$ is divisible and by Lemma 3.4 of [18] $\mathcal{J}(R)^+$ has finite Prüfer rank. Hence R is right and left Noetherian ring. Finally, by Theorem A(c) of [18] $\mathcal{J}(R)^\circ$ is a nilpotent torsion-free group of finite Prüfer rank. Moreover, the periodic part τT^* of T^* is a countable locally cyclic abelian periodic group and so R° has a normal series (3.2). The lemma is proved.

Theorem 3.4. *Let R be a semilocal ring. Then the following statements are equivalent:*
(i) R° has finite abelian subgroup rank;
(ii) R is a finite ring.

Proof. (ii)⇒(i) is immediate.
(i)⇒(ii). The Theorem 3(a) of [17] implies that $\mathcal{J}(R)$ is nil. Moreover, $GL_n(T)$ has finite abelian subgroup rank if and only if $T \cong GF(p^n)$ for some prime p and positive integer n. Hence the quotient $R/\mathcal{J}(R)$ is finite and $\operatorname{char}(R/\mathcal{J}(R)) = m$ for some positive integer m. So $mR \leq \mathcal{J}(R)$. By Theorem B(c) of [17] $\mathcal{J}(R)^+$ has finite abelian subgroup rank and so its a Černikov group. Thus R is a left and right Artinian ring and $\mathcal{J}(R)^s = 0$ for some positive integer s. By Lemma 122.5 of [33] R^+ is a Černikov group. Since R^+ has finite exponent m, it is finite. The theorem is proved.

4. Recall [34, p.100], a group G is said to be a FC-group if $|G : C_G(x)|$ is finite for every element x of G.

Lemma 4.1. *Let R be a local ring. Then the unit group $\mathcal{U}(R)$ is a FC-group if and only if either R is a finite ring or R is a commutative ring.*

Proof. Let R be an infinite ring with an FC-group of units $\mathcal{U}(R)$. Since $(R/\mathcal{J}(R))^*$ is also a FC-group we have that the quotient $R/\mathcal{J}(R)$ is a field.

If $x \in \mathcal{U}(R)$ then $x^n \in Z(\mathcal{U}(R))$ for some integer n (see [20]) and so $x^n \in Z(R)$. If x is not in $\mathcal{U}(R)$ then $x \in \mathcal{J}(R)$ and therefore $x^m \in Z(R^\circ)$ for some integer m. Since the map $\varphi\colon R^\circ \to \mathcal{U}(R)$ defined by the rule

$$\varphi(x) = 1 + x \quad \text{with } x \in R^\circ$$

is an isomorphism and $\varphi(Z(R^\circ)) = Z(\mathcal{U}(R))$, we conclude $1 + x^m \in Z(\mathcal{U}(R))$ and therefore $x^m \in Z(R)$.

Suppose that R is not commutative. Then by Lemma 6 of [35] R^+ is torsion-free. Moreover, $\mathcal{U}(R)' \leq 1+\mathcal{J}(R)$ and by Theorem from [35] R contains the commutator ideal I which is nil and $\mathcal{U}(R)' \leq 1 + I$.

From Lemma 2.4 of [17] follows that $1+I$ is torsion-free. But $\mathcal{U}(R)' \leq \tau(1+I)$ and therefore $\mathcal{U}(R)$ is abelian. Thus R is a commutative ring and this completes the proof.

Theorem 4.2. *Let R be a semilocal ring. Then the following statements are equivalent.*
(1) $\mathcal{U}(R)$ is a FC-group.
(2) Either R is a finite ring or R is a commutative ring.
(3) $\mathcal{U}(R)$ is a central-by-finite group.

Proof. (1)⇒(2). Let R be an infinite ring. As consequence of Lemma 2 of [34] $R/\mathcal{J}(R) = \oplus_{i=1}^{s} T_i$ is a direct sum of fields T_i.
(a) If the unipotent group $1+\mathcal{J}(R)$ is torsion-free then $(1+\mathcal{J}(R)) \cap \mathcal{U}(R)' = 1$ (see [20]) and so R is a commutative ring.
(b) Suppose that unipotent group $1+\mathcal{J}(R)$ is mixed and R is noncommutative. Then the commutator ideal I of R is nil. Moreover, $\mathcal{U}(R)' \leq 1+I$. From [20] follows that $\mathcal{U}(R)' \leq \tau(1+\mathcal{J}(R))$. If the intersection $(1+I) \cap \tau(1+\mathcal{J}(R))$ is nontrivial then the additive group $\mathcal{J}(R)^+$ has a nontrivial element of finite order, a contradiction with Lemma 6 of [35]. Therefore, $\mathcal{U}(R)' = 1$. By Proposition 2.7 $R = S \oplus T$, where either $S = 0$ or $S = \oplus_{i=1}^{k} S_i$ is a direct sum of local rings S_i, and T is a semiperfect ring with the quotient $T/\mathcal{J}(R) \cong GF(2) \oplus \cdots \oplus GF(2)$. Then S is commutative ring by Lemma 4.1. Since $2T \leq \mathcal{J}(T)$, we have $4(ab-ba) = 0$ for all elements a and b of T and by Lemma 6 of [35] T is a commutative ring.
(2)⇒(3) and (3)⇒(1) are immediate. The theorem is proved.

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