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DECOMPOSITION THEOREMS FOR SEMI-PERFECT RINGS

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We propose a general method for constructing two-sided Pierce decomposition of a semi-perfect ring using the structure of its quivers. As a sequence, we obtain the decomposition theorems for various classes of semi-perfect rings.

The basic facts about the semi-perfect rings, introduced by Bass [1], can be found in [2], [3], [4], [5].

The goal of this paper is to prove the decomposition theorems for semi-perfect rings associated with the decomposition of corresponding quivers into connected components.

1. In this section we shall give the necessary information about quivers (see [6]).

Following Gabriel a finite oriented graph will be called a quiver. A simply-laced quiver (no multiple arrows including loops) will be called simple. Denote by $1, \dots, s$ the vertices (points) of a quiver Q and assume that we have t_{ij} arrows between the points i and j. Let $[Q]$ denote the incidence matrix of the quiver Q:

$$[Q] = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1s} \\ \cdots & \cdots & \cdots & \cdots \\ t_{s1} & t_{s2} & \cdots & t_{ss} \end{pmatrix}.$$

The results below can be found in [7] and [8].

A real matrix $A = (a_{ij})$ is called non-negative if so are all elements a_{ij}.

Denote by $M_n(R)$ the set of all real matrices of order n.

Let $\tau(1), \dots, \tau(n)$ be a permutation of numbers $1, 2, \dots, n$ and let $P_\tau = \sum_{i=1}^n e_{i\tau(i)}$ be the permutation matrix, where e_{ij} are corresponding matrix units. Clearly, $P_\tau^T P_\tau = E$.

Definition 1. A matrix $B \in M_n(R)$ is called permutational reducible if there exists a permutation matrix P_τ such that $P_\tau^T B P_\tau = \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix}$, where B_1 and B_2 are square matrices of order less than n. Otherwise, the matrix is permutational irreducible.

Definition 2. A finite oriented graph is called strongly connected if there is an oriented path between any two of its vertices.

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Proposition 1 [8, Ch.9]. *A quiver Q is strongly connected if and only if the matrix $[Q]$ is permutational irreducible.*

Note that a renumeration of vertices of quiver Q transforms the matrix $[Q]$ into a matrix $P_\tau^T[Q]P$.

Proposition 2. *There exists a permutation matrix P with* $P^T[Q]P = \begin{pmatrix} B_1 & B_{12} & \cdots & B_{1t} \\ 0 & B_2 & \cdots & B_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{mt} \end{pmatrix}$, *where matrices $B_1, B_2, \ldots, B_m$ are permutational irreducible.*

Proof is obvious.

A maximal (with respect to inclusion) strongly connected subquiver of Q is called strongly connected component of Q. The proposition 2 immediately implies the following well-known fact about the existence of the partition of the set of vertices of a quiver Q into non-intersecting subsets such that the subgraphs corresponding to those subsets are strongly connected graphs (strongly connected components of quiver Q).

Definition 3 [9]. Let $Q_1, \ldots, Q_m$ be all strongly connected components of a quiver Q. The condensation Q^* of quiver Q is a quiver whose vertices $q_1, \ldots, q_m$ correspond to strongly connected components $Q_1, \ldots, Q_2$ and there is an arrow between q_i and q_j if and only if Q has an arrow between Q_i and Q_j.

Definition 4 [10]. A quiver without oriented cycles is called acyclic quiver.

Proposition 3 [9, §63]. *The condensation of any quiver is a simple acyclic quiver without loops.*

Proof follows from Propositions 1 and 2.

For convenience, we will consider any quiver consisting of one point as a strongly connected quiver.

Proposition 4. *A strongly connected acyclic quiver is a point.*

It follows from this proposition that strongly connected quivers and acyclic quivers are polar types of quivers.

Proposition 5. *A quiver Q is acyclic if and only if there exists a permutational matrix P such that the matrix $P^T[Q]P$ is strictly upper triangular.*

Proof follows from Propositions 1 and 2.

2. In [10, §6.4] we have the following definition of the quiver $\Gamma(A)$ of an artinian algebra A (such an algebra is always a right artinian ring).

Let A be a right artinian algebra. Suppose that $e_1, \ldots, e_r$ are primitive (or local) idempotents of algebra A such that the right ideals $P_1 = e_1A, \ldots, P_r = e_rA$ represent the isomorphism classes of principal (indecomposable) right A-modules. The quiver of the algebra A is an oriented graph $\Gamma(A) = (V, E)$ whose set of vertices is $V = \{e_1, \ldots, e_r\}$ and the set of edges is $E = \{(e_i, e_j) : e_iRe_j \neq 0\}$.

Analogously, the quiver $\Gamma(A)$ can be defined for any semi-perfect ring A. Clearly, the quiver $\Gamma(A)$ will be the same for the rings Morita equivalent to A. Recall that a finite-dimensional algebra A over field k is called an algebra of finite type if it has finitely many nonequivalent indecomposable representations. Note that if A is an algebra of finite type with zero square of the radical then its Gabriel quiver $Q(A)$ coincides with $\Gamma(A)$ [10, Ch.8].

Denote by M^n the direct sum of n copies of a module $M, M^0 = 0$.

Let $A = P_1^{n_1} \oplus \cdots \oplus P_s^{n_s}$ be a decomposition of semi-perfect ring A into a sum of non-isomorphic indecomposable projective A-modules and let $1 = f_1 + \cdots + f_s$

be the corresponding decomposition of $1 \in A$ into a sum of mutually orthogonal idempotents, i.e. $f_iA = P_i^{n_i}$, $(i = 1, \dots, s)$.

We will need the following proposition [11, §3.7].

Proposition 6. *Let A be a semi-perfect ring and $1 = e_1 + \cdots + e_m = f_1 + \cdots + f_n$ are two decompositions of $1 \in A$ into a sum of mutually orthogonal local idempotents. Then $m = n$ and there exists an inverse element a and a permutation $i \mapsto \sigma(i)$ such that $e_i = af_{\sigma(i)}a^{-1}$, $(i = 1, \dots, n)$.*

Reformulate the definition of quivers $\Gamma(A)$ of a semi-perfect ring A in terms of indecomposable projective A-modules $P_1, \dots, P_s$. Let R be the Jacobson radical of a ring A.

Using Proposition 6 and the isomorphism $\mathrm{Hom}(eA, fA) \simeq fAe$ for idempotents e and f one can easily see that the quiver $\Gamma(A)$ can be defined by vertices $1, \dots, s$ corresponding to modules $P_1, \dots, P_s$ (or to idempotents $f_1, \dots, f_s$), an arrow from i to j $(i \neq j)$ if and only if $\mathrm{Hom}(P_j, P_i) \neq 0$ and a loop at a point i if and only if $\mathrm{Hom}(P_i, P_iR) \neq 0$.

Let J be an ideal of a ring A in the Jacobson radical R of A such that the idempotents can be lifted modulo J.

Consider the quotient ring $\bar{A} = A/J = \bar{A}_1 \times \cdots \times \bar{A}_t$, where all rings $\bar{A}_1, \dots, \bar{A}_t$ are indecomposable and decompose $\bar{1} \in \bar{A}$ into a sum of mutually orthogonal idempotents. Let $W = J/J^2$ and substitute the idempotents $\bar{f}_1, \dots, \bar{f}_t$ by the corresponding points $1, \dots, t$. We connect points i and j by an arrow if and only if $\bar{f}_iW\bar{f}_j \neq 0$. The obtained finite oriented graph $Q(A, J)$ is called the quiver associated with the ideal J.

Since the prime radical I of a ring A is a nil-ideal, it is contained in the Jacobson radical R of A. Using the fact that the idempotents can be lifted modulo any nil-ideal [11] one can consider the quiver $Q(A, \Gamma)$ associated with the prime radical I.

Definition 5. The quiver $Q(A, I)$ of a semi-perfect ring A is called prime. Further, we will denote it by $PQ(A)$.

The quiver $Q(A, R)$ of a right noetherian semi-perfect ring A can be obtained from the quiver $Q(A)$ [12], [6] by substituting all the arrows from one point to another (possibly, the same point) by one arrow.

Denote by Q_u the simple quiver obtained from a quiver Q using this procedure. $Q_u(A) = Q(A, R)$. Clearly, the quivers Q and Q_u are connected simultaneously.

3. In the following theorem the incidence matrix $[\Gamma(A)]$ of the quiver $\Gamma(A)$ is as in Proposition 2, where the matrices $B_1, \dots, B_t$ are permutational irreducible.

Theorem 1. *Let A be a semi-perfect ring. There exists a decomposition of $1 \in A$ into a sum of mutually orthogonal idempotents: $1 = g_1 + \cdots + g_t$ such that $A = \bigoplus_{i,j=1}^{t} g_iAg_j$ is the two-sided Pierce decomposition, where $g_iAg_j = 0$ $(j < i)$ and the incidence matrices of the quivers $\Gamma(A_i)$ of the rings $A_i = g_iAg_i$ $(i = 1, \dots, t)$ coincide with B_i.*

Proof. Let $Q_1, \dots, Q_t$ be strongly connected components of quiver $\Gamma(A)$ the incidence matrices of which are $B_1, \dots, B_t$. Let g_i be the sum of idempotents from the decomposition $1 = f_1 + \cdots + f_t$ corresponding to the points of Q_i, $i = 1, \dots, t$. It follows immediately that two-sided Pierce decomposition $A = \bigoplus_{i,j=1}^{t} g_iAg_i$ satisfies the conditions of the theorem.

Corollary 1 [12]. *Any semi-perfect ring A can be uniquely decomposed into a finite direct product of indecomposable rings $A_1, \dots, A_m$ with connected quivers $\Gamma(A_i)$, $i = 1, \dots, m$.*

Next we will consider the noetherian (two-sided) semi-perfect rings.

Theorem 2 [4]. *Let A be an arbitrary ring, $e^2 = e \in A$, $1 = e+f$, $eAf = X$, $fAe = Y$ and let $A = \begin{pmatrix} eAe & X \\ Y & fAf \end{pmatrix}$ be the corresponding two-sided Pierce decomposition. If the ring A is right noetherian (artinian) then the rings eAe and fAf are right noetherian (artinian) and the fAf-module X and the eAe-module Y are finitely generated. Conversely, if these conditions hold for some idempotents e and f in A such that $1 = e + f$ then A is right noetherian (artinian).*

Theorem 3. *Let A be a noetherian semi-perfect ring and the matrix $[Q]$ is block upper triangular where the diagonal matrices $B_1, \dots, B_t$ are permutational irreducible. Then these exists a decomposition of $1 \in A$ into a sum of mutually orthogonal idempotents: $1 = g_1 + \dots + g_t$ such that $A = \bigoplus_{i,j=1}^{t} g_i A g_j$ is a two-sided Pierce decomposition with $g_i A g_j = 0$ $(j < i)$ and the incidence matrices of the quivers $Q(A_i)$ of rings $A_i = g_i A g_i$ coincide with B_i, $i = 1, \dots, t$.*

Proof. Let $Q_1, \dots, Q_t$ be the strongly connected components of the quivers $Q(A)$ corresponding to the matrices $B_1, \dots, B_t$ on the main diagonal in the incidence matrix $[Q(A)]$ (see Proposition 2). We will prove the theorem by induction on t. The case $t = 1$ is trivial. Denote by $g_1 = e$ the sum of idempotents $f_1, \dots, f_m$ from the set $f_1, \dots, f_s$ that correspond to the component Q_1, $f_1 = 1 - e$.

Set $A_1 = eAe$, $A_2 = fAf$, $eAf = X$, $fAe = Y$. Due to the decomposition in [4, 5] we have the following presentation for the Jacobson radical R of A, $R = \begin{bmatrix} R_1 & X \\ Y & R_2 \end{bmatrix}$, where R_i is the Jacobson radical of the ring A_i $(i = 1, 2)$.

Obviously, $R^2 = \begin{pmatrix} R_1^2 + XY & R_1X + XR_2 \\ YR_1 + R_2Y & R_2^2 + YX \end{pmatrix}$.

It follows from the matrix $[Q(A)]$ that the quiver $Q(A)$ contains no arrows from points $m+1, \dots, s$ to the points $1, \dots, m$. Now the two-sided Pierce decomposition for A and 1 for R implies that $Y = YR_1 + R_2Y$. Applying Theorem 2 we conclude that Y is finitely generated left A_2-module and finitely generated right A_1-module. It follows from the Nakayama Lemma that $Y = 0$, i.e. $A = \begin{bmatrix} A_1 & X \\ 0 & A_2 \end{bmatrix}$. Clearly,

$$[Q(A_2)] = \begin{pmatrix} B_2 & B_{23} & \dots & B_{2t} \\ 0 & B_3 & \dots & B_{3t} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & B_t \end{pmatrix}$$

and we can apply the induction to the ring A_2 which completes the proof of the theorem.

Corollary 2 [12]. *Any semi-perfect noetherian ring A can be uniquely decomposed into a finite direct product of indecomposable rings $A_1, \dots, A_m$ with connected quivers $Q(A_i)$, $i = 1, \dots, m$.*

A semi-perfect ring A is called reduced if the quotient ring A/R is a direct product of skew-fields.

Corollary 3. *Any reduced noetherian semi-perfect ring A with acyclic quiver $Q(A)$ is artinian and there exists a decomposition of $1 \in A$ into a sum of mutually orthogonal local idempotents: $1 = e_1 + \dots + e_s$ such that $e_i A e_j = 0$ if $j < i$ and the rings $e_i A e_i$ are skew-fields, $i, j = 1, \dots, s$.*

Suppose now that I is the prime radical of A. Let $\bar{A} = A/I = \bar{A}_1 \times \dots \times \bar{A}_t$ be a decomposition of semi-prime semi-perfect ring $\bar{A}$ into a direct product of indecomposable rings $\bar{A}_1, \dots, \bar{A}_t$ and let $1 = \bar{f}_1 + \dots + \bar{f}_t$ be the corresponding decomposition of $\bar{1} \in \bar{A}$ into a sum of mutually orthogonal idempotents.

By [11] the idempotents can be lifted modulo I preserving the orthogonality. It is clear that in this case $1 = f_1 + \dots + f_t$, where $f_i f_j = \delta_{ij} f_j$ $(i, j = 1, \dots, t)$ and $\bar{f}_i = f_i + I$.

Let $A_{ij} = f_i A f_j$, $(i, j = 1, \ldots, t)$ and $I_k = f_k A f_k$ $(k = 1, \ldots, t)$.

Then we have the following two-sided Pierce decomposition of I: $I = \begin{bmatrix} I_1 & A_{12} & \ldots & A_{1t} \\ A_{21} & I_2 & \ldots & A_{2t} \\ \ldots & \ldots & \ldots & \ldots \\ A_{t1} & A_{t2} & \ldots & I_t \end{bmatrix}$.

Proposition 7 [13] (see also [14]). *Let I be the prime radical of a ring A, $e^2 = e \in A$, and $e \neq 0$. Then eIe coincides with the prime radical of ring eAe.*

The last proposition immediately implies that I_k is the prime radical of the ring A_{kk}, $(k = 1, \ldots, t)$.

Theorem 4. *Let A be a semi-perfect ring with nilpotent prime radical I and let the matrix $[PQ(A)]$ be upper triangular with permutational irreducible diagonal matrices $B_1, \ldots, B_t$. Then these exists a decomposition of $1 \in A$ into a sum of mutually orthogonal idempotents $1 = g_1 + \cdots + g_t$ such that $A = \bigoplus_{i,j=1}^{t} g_i A g_j$, is a two-sided Pierce decomposition of the ring A, where $g_i A g_j = 0$ $(j < i)$ and the incidence matrices of quivers $Q(A_i)$ of the rings $A_i = g_i A g_i$ coincide with B_i, $(i = 1, \ldots, t)$.*

The proof of this theorem is absolutely similar to that of Theorem 3.

Corollary 4 [12]. *Any semi-perfect ring A with nilpotent prime radical can be uniquely decomposed into a finite direct product of indecomposable rings $A_1, \ldots, A_m$ with connected prime quivers $PQ(A_i)$, $i = 1, \ldots, m$.*

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