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GENERALIZED EQUIVALENCE OF PAIRS OF MATRICES

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Pairs (A_1, B_1) and (A_2, B_2) of matrices over a commutative principal ideal domain R are called generalized eguivalent pairs provided $A_2 = UA_1V_1$, $B_2 = UB_1V_2$ for some invertible matrices U, V_1, V_2 over R. The generalized equivalence of pairs of matrices over R is investigated. In particular, necessary and sufficient conditions are found under which a pair of nonsingular matrices over R is generalized eguivalent to a pair of diagonal matrices. Some applications of these results are considered.

1. Let R be a commutative principal ideal domain and R_n the ring of $n \times n$ matrices over R. Matrices $A, B \in R_n$ are called *equivalent* (written $A \sim B$) if $A = UBV$ for some invertible matrices U and V of R_n, i.e. $U, V \in GL(n, R)$. Pairs of matrices (A_1, B_1) and (A_2, B_2) over R are called *equivalent* (written $(A_1, B_1) \sim (A_2, B_2)$) if $A_2 = UA_1V$, $B_2 = UB_1V$ for some matrices $U, V \in GL(n, R)$.

It is well known that every matrix $A \in R_n$ is equivalent to a canonical diagonal matrix D^A, i.e.

$$UAV = D^A = \operatorname{diag}(\mu_1^A, \dots, \mu_n^A), \quad \mu_i^A | \mu_{i+1}^A, \quad i = 1, \dots, n-1$$

for some matrices $U, V \in GL(n, R)$. A problem of equivalence of pairs of matrices is solved at most for pairs of matrices over fields [1]–[4]. This problem for pairs of matrices over rings is wild [5]. However many problems, in particular, the problems on the factorization and the divisibility of matrices, the multiplicativity of the canonical diagonal form of matrices, the similarity of pairs of matrices over fields, reduce to the investigation of the equivalence of pairs of matrices with the same left and different right transforming matrices [6]–[12].

Definition 1. Pairs of matrices (A_1, B_1) and (A_2, B_2) over R for which

$$A_2 = UA_1V_1, \qquad B_2 = UB_1V_2$$

for some matrices $U, V_1, V_2 \in GL(n, R)$ are called *generalized equivalent pairs* (written $(A_1, B_1) \approx (A_2, B_2)$).

In this paper the generalized eguivalence of pairs of matrices over R is investigated. A criterion of the generalized eguivalence of a pair of matrices to a pair of diagonal matrices is established. Some applications of these results are considered.

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2. Theorem 1. *Let $A, B \in R_n$ be nonsingular matrices. Then*

$$(A, B) \approx (D^A, D^B) \tag{1}$$

if and only if

$$(\operatorname{adj} B)A \sim (\operatorname{adj} D^B)D^A, \tag{2}$$

where $\operatorname{adj} B$ *denotes the adjoint matrix.*

Proof. Suppose that relation (1) holds, i.e.

$$UAV_1 = D^A, \quad UBV_2 = D^B \tag{3}$$

for some matrices $U, V_1, V_2 \in GL(n, R)$. Since $B(\operatorname{adj} B) = (\det B)I$, where I is the identity matrix, we can write

$$(\det B)A = B(\operatorname{adj} B)A. \tag{4}$$

Now, bearing in mind the relation (3), we obtain $(\det B)UAV_1 = UBV_2V_2^{-1}(\operatorname{adj} B)AV_1$, or

$$(\det B)D^A = D^B\Psi, \tag{5}$$

where $\Psi = V_2^{-1}(\operatorname{adj} B)AV_1$. It follows from (5) that $\Psi = (\operatorname{adj} D^B)D^A$. Thus, $(\operatorname{adj} B)A$ is equivalent to $(\operatorname{adj} D^B)D^A$.

Now suppose that relation (2) holds. It follows from [6],[9],[10] that for matrices A, B there exist matrices $Q, S_1, S_2 \in GL(n, R)$ such that

$$T^A = QAS_1 = \left\| \begin{matrix} \mu_1^A & 0 & \dots & 0 \\ a_{21}\mu_1^A & \mu_2^A & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1}\mu_1^A & a_{n2}\mu_2^A & \dots & \mu_n^A \end{matrix} \right\|, \tag{6}$$

$$T^B = QBS_2 = \left\| \begin{matrix} \mu_1^B & 0 & \dots & 0 \\ b_{21}\mu_1^B & \mu_2^B & \dots & 0 \\ \dots & \dots & \dots & \dots \\ b_{n1}\mu_1^B & b_{n2}\mu_2^B & \dots & \mu_n^B \end{matrix} \right\|. \tag{7}$$

Then we rewrite equality (4) as follows: $(\det B)QAS_1 = QBS_2S_2^{-1}(\operatorname{adj} B)AS_1$, or

$$(\det B)T^A = T^BC, \tag{8}$$

where $C = S_2^{-1}(\operatorname{adj} B)AS_1$ is the lower triangular matrix with main diagonal

$$\Psi = (\operatorname{adj} D^B)D^A.$$

Since the triangular matrix C is equivalent to the diagonal matrix Ψ, it then follows from [13] that for the matrix C there exist lower unitriangular matrices $F = \|f_{i,j}\|_1^n$ and $H = \|h_{ij}\|_1^n$ ($f_{ij} = h_{ij} = 1$ for $i = j$ and $f_{ij} = h_{ij} = 0$ for $i > j$, $i, j = 1, \dots, n$) such that $FCH = \Psi$. Then from (8) we have

$$(\det B)T^AH = T^BF^{-1}FCH,$$

or

$$(\det B)T_1^A = T_1^B\Psi, \tag{9}$$

where $T_1^A = T^AH$ and $T_1^B = T^BF^{-1}$ are the lower triangular matrices of the form (6) and (7), respectively. We can write equality (9) as follows: $(\det B)W_1D^A = W_2D^B\Psi$, where W_1, W_2 are the lower unitriangular matrices. From this it follows that $W_1 = W_2 = W$. Thus, setting $U = W^{-1}Q$, $V_1 = S_1H$, $V_2 = S_2F^{-1}$ we obtain that $UAV_1 = D^A$, $UBV_2 = D^B$. This proves the theorem.

Corollary 1. *Let $A, B \in R_n$ be nonsingular matrices and $B|A$, i.e. $A = BC$ for some matrix $C \in R_n$. Then $(A, B) \approx (D^A, D^B)$ if and only if $B^{-1} \sim (D^B)^{-1} D^A$.*

If $(A, B) \approx (D^A, D^B)$, then we say that the pair of matrices (A, B) is *diagonalizable.*

Corollary 2. *Let pairs of matrices (A_i, B_i), $i = 1, 2$ over R be diagonalizable, i.e. $(A_i, B_i) \approx (D^{A_i}, D^{B_i})$, $i = 1, 2$. Then $(A_1, B_1) \approx (A_2, B_2)$ if and only if $A_1 \sim A_2$, $B_1 \sim B_2$, i.e. $D^{A_1} = D^{A_2}$, $D^{B_1} = D^{B_2}$.*

3. As has been stated above, a pair (A, B) of nonsingular matrices over R is generalized equivalent to a pair (T^A, T^B) of triangular matrices of the form (6) and (7), respectively. It is obvious that $(A_1, B_1) \approx (A_2, B_2)$, $A_i, B_i \in R_n$, $i = 1, 2$ if and only if $(T^{A_1}, T^{B_1}) \approx (T^{A_2}, T^{B_2})$. Therefore, it suffices to discuss the generalized equivalence of the pairs (T^A, T^B) of triangular matrices.

Theorem 2. *Let $A, B \in R_n$ be nonsingular matrices and $(A, B) \approx (T^A, T^B)$, where T^A and T^B are of the form (6) and (7), respectively. If $a_{ij} - b_{ij}$ is divisible by the greatest common divisor $\left(\frac{\mu_n^A}{\mu_j^A} \frac{\mu_n^B}{\mu_j^B}\right)$ for all $i, j = 1, \dots, n$, $i > j$, then*

$$(A, B) \approx (D^A, D^B).$$

Proof. Suppose that

$$UT^A = D^A V_1, \quad UT^B = D^B V_2, \tag{10}$$

where

$$U = \begin{Vmatrix} 1 & 0 & \dots & 0 \\ u_{21} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ u_{n1} & u_{n2} & \dots & 1 \end{Vmatrix},$$

and V_1, V_2 are some matrices from $GL(n, R)$. Then (10) implies that

$$\sum_{j=k-p}^{k} u_{kj} a_{j,k-p} \mu_{k-p}^A \equiv 0 (\operatorname{mod} \mu_k^A),$$
$$\sum_{j=k-p}^{k} u_{kj} b_{j,k-p} \mu_{k-p}^B \equiv 0 (\operatorname{mod} \mu_k^B),$$

$p = 1, \dots, n$; $k = p + 1, \dots, n$, where $a_{ii} = b_{ii} = u_{ii} = 1$, or

$$\begin{aligned} u_{k,k-p} &\equiv \sum_{j=k-p+1}^{k} u_{kj} a_{j,k-p} (\operatorname{mod} \frac{\mu_k^A}{\mu^A k - p}), \\ u_{k,k-p} &\equiv \sum_{j=k-p+1}^{k} u_{kj} b_{j,k-p} (\operatorname{mod} \frac{\mu_k^B}{\mu^B k - p}), \end{aligned} \tag{11}$$

$p = 1, \dots, n$; $k = p + 1, \dots, n$.

It is not difficult to prove that, if the conditions of Theorem 2 hold, then the system of congruences (11) is solvable. Then the matrix U whose entries u_{ij} for $i > j$ are the solutions of the system of congruences (11), $u_{ij} = 1$ for $i = j$ and $u_{ij} = 0$ for $i < j$, $i, j = 1, \dots, n$ satisfies equalities (10). This concludes the proof of Theorem 2.

Corollary 3. *Let $A, B \in R_n$ and* $(\det A, \det B) = 1$. *Then* $(A, B) \approx (D^A, D^B)$.

Corollary 4. *Let $A_i, B_i \in R_n$ and* $(\det A_i, \det B_i) = 1$, $i = 1, 2$. *Then* $(A_i, B_i) \approx (A_2, B_2)$ *if and only if* $A_1 \sim A_2$, $B_1 \sim B_2$, *i.e.* $D^{A_1} = D^{A_1}$, $D^{B_1} = D^{B_2}$.

4. Let $A, B \in R_n$. It is known [14],[15] that if $B|A$, then $D^B|D^A$.

It is therefore reasonable to ask when $D^B|D^A$ will give $B|A$?

This problem is connected with diagonalizability of the pair of matrices (A, B). In fact, if $D^B|D^A$ and $(A, B) \approx (D^A, D^B)$, i.e. $D^A = D^B\Psi$ and $UAV_1 = D^A$, $UBV_2 = D^A$ for some diagonal matrix Ψ and invertible matrices U, V_1, V_2 of R_n, then it is easy to make sure that B is a left divisor of A. In many cases this condition is necessary and sufficient for divisibility of matrices.

Theorem 3. *Let $A, B \in R_n$ be nonsingular matrices and $D^B|D^A$, i.e.*

$$D^A = \operatorname{diag}(\mu_1^B, \ldots, \mu_n^B) \operatorname{diag}(\psi_1, \ldots, \psi_n).$$

If

$$\left(\frac{\mu_i^B}{\mu_j^B}, (\psi_i, \psi_j)\right) = 1, \quad i, j = 1, \ldots, n, i > j, \tag{12}$$

then B is a left divisor of A if and only if

$$(A, B) \approx (D^A, D^B), \text{ i.e. } B^{-1}A \sim (D^B)^{-1}D^A.$$

Proof. If $D^B|D^A$ and the pair of matrices (A, B) is diagonalizable, then from what has been said above it follows that B is a left divisor of A.

Now suppose that $B|A$, i.e. $A = BC$ for some matrix $C \in R_n$. Since $QAS_1 = T^A$ and $QBS_2 = T^B$, for some matrices $Q, S_1, S_2 \in GL(n, R)$, are of the form (6) and (7), respectively, from the latter equality we have

$$T^A = T^B C_1, \tag{13}$$

where $C_1 = S_2^{-1}CS_1 = \|c_{ij}\|_1^n$ is a lower triangular matrix, moreover, $c_{ij} = \psi_i$ for $i = j$, $i, j = 1, \ldots, n$. It suffices to show that $C_1 \sim \Psi$. It follows from Roth's Theorem [16] and its extension [13] that the triangular matrix C_1 is equivalent to the diagonal matrix Ψ if and only if the system of equations

$$c_{ii}x_{ij} - \sum_{k=j}^{i-1} c_{kj}y_{ik} = c_{ij}, \quad i, j = 1, \ldots, n, \quad i > j \tag{14}$$

is solvable.

Equality (13) implies that

$$b_{ij}\psi_j\mu_j^B + \sum_{k=j+1}^{i} b_{ik}c_{kj}\mu_k^B = 0, \quad i, j = 1, \ldots, n, \quad i > j$$

or

$$b_{ij}\psi_j + \sum_{k=j+1}^{i} b_{ik}c_{kj}\frac{\mu_k^B}{\mu_j^B} = 0, \quad i, j = 1, \ldots, n, \quad i > j, \tag{15}$$

where $b_{ik} = 1$, for $k = i$. It then follows from (12) and (15) that $(\psi_j, \psi_k)|c_{ij}$, $i, j = 1, \ldots, n$, $i > j$ for all $k = i + 1, \ldots, n$. This means that the system of equations (14) is solvable for x_{ij}, y_{ij}, $i, j = 1, \ldots, n$, $i > j$. Therefore, $C_1 \sim \Psi$ and hence the matrix $C = B^{-1}A$ is equivalent to the matrix $\Psi = (D^B)^{-1}D^A$. This proves the theorem.

Corollary 5. *Let $A, B \in R_n$ be nonsingular matrices and $D^B | D^A$, i.e. $D^A = D^B \Psi$ for some diagonal matrix Ψ of R. If at least one of the following conditions holds:*

(i) $(\det D^B, \det \Psi) = 1$,

(ii) $(\det D^B, \det \Psi) = \delta$, $(\delta, \mu^A_{n-1}) = 1$,

(iii) $(\mu^A_i, \mu^B_n) = \mu^B_i$, *for all* $i = 1, \dots, n-1$,

(iv) all elementary divisors of A are prime,

then B is a left divisor of A if and only if $(A, B) \approx (D^A, D^B)$.

5. Using the above results we formulate the following assertions on the multiplicative property of canonical diagonal form of matrices.

Lemma. *Let $B, C \in R_n$ be nonsingular matrices and $A = BC$. Then*

$$D^A = D^B D^C$$

if and only if the pair of matrices (A, B) is diagonalizable, i.e.

$$UAV_1 = D^A, \quad UBV_2 = D^B \tag{16}$$

for some matrices $U, V_1, V_2 \in GL(n, R)$ and the matrix $\Psi = V_2^{-1} C V_1$ is a d-matrix, i.e.

$$\Psi = \operatorname{diag}(\psi_1, \dots, \psi_n), \quad \psi_i | \psi_{i+1}, \quad i = 1, \dots, n-1,$$

for any matrices $V_1, V_2 \in GL(n, R)$ which satisfy conditions (16).

Proof of Lemma follows from Corollary 1.

Corollary 6 [15],[17],[10]. *Let $B, C \in R_n$ be nonsingular matrices and d^{BC}_{n-1} the greatest common divisor of the minors of order $n-1$ of the matrix BC. If at least one of the following conditions holds:*

$$(\det B, \det C) = 1, \tag{17}$$

$$(\det B, \det C) = \delta, \quad (\delta, d^{BC}_{n-1}) = 1, \tag{18}$$

then $D^{BC} = D^B D^C$.

Corollary 7 [10],[8]. *Let $A, B \in R_n$ be nonsingular matrices and $B | A$, i.e. $A = BC$ and $D^A = D^B \Psi$ for some matrix C and diagonal matrix Ψ of R. If at least one of the following conditions holds:*

$$\left(\det B, \frac{d^A_{n-1}}{d^B_{n-1}}\right) = 1, \tag{19}$$

$$(\mu^A_i, \mu^B_n) = \mu^B_i, \quad i = 1, \dots, n-1, \tag{20}$$

then $D^C = \Psi$, i.e. $D^A = D^B D^C$.

Indeed, under conditions (17)–(20), by Corollary 5, the pair of matrices (A, B), where $A = BC$, is diagonalizable, i.e. relation (16) holds and it can be easily verified that $\Psi = V_2^{-1} C V_1$ is a d-matrix.

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