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ON NORMAL SUBGROUPS OF JONKIER GROUP

Ya. Lavreniuk

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The quotient group of the Jonkier group over a prime finite field $\mathbb{F}_p$ by the kernel of its action on an affine space of dimension n over $\mathbb{F}_p$ is considered. The criterion of normality of subgroups of this quotient groups is established. Therefore, all normal subgroups of J_n that include this kernel are described.

The group of automorphisms of the affine space $\mathbb{A}^n$ of dimension n over a fixed field $\mathbb{K}$, which is called the affine group of Cremona Cr_n, can be characterized as the group of inversed corteges of polynomials from $\mathbb{K}[x_1,\dots,x_n]$ of the form

$$\langle a_1(x_1,\dots,x_n),\dots,a_n(x_1,\dots,x_n)\rangle \tag{1}$$

with respect to the superposition operation. For infinite fields it can be also characterised as the group of automorphisms of the ring of polynomials $\mathbb{K}[x_1,\dots,x_n]$. In the case of a finite field $\mathbb{K}$ different sets of the form (1) are different automorphisms of $\mathbb{K}[x_1,\dots,x_n]$, but they can cause equal automorphisms of $\mathbb{A}^n$. In this case for the group of automorphisms of ring of polynomials $\mathbb{K}[x_1,\dots,x_n]$ the established term affine group of Cremona is used. Two main counterparts of Cr_n can be naturally distinguished — the group $GL_n(\mathbb{K})$ of linear transformations and the group J_n of triangular transformations, or the Jonkier group.

The Jonkier group J_n over a field $\mathbb{K}$ is determined as the group of corteges of polynomials of the form $\big(\alpha_1x_1+a_1,\dots,\alpha_nx_n+a_n(x_1,\dots,x_{n-1})\big)$ with $x_i\in\mathbb{K}$, $\alpha_i\in\mathbb{K}^*$, $a_i(x_1,\dots,x_{i-1})\in\mathbb{K}[x_1,\dots,x_{i-1}]$, $1\le i\le n$. If the characteristic of $\mathbb{K}$ is equal to 0, then J_n acts on the affine space $\mathbb{A}^n$ over the field $\mathbb{K}$ faithfully. In this case the normal structure of J_n was described by Ivanenko [1].

The group J_n acts unfaithfully on the Affine space $\mathbb{A}^n$ over a prime finite field $\mathbb{F}_p$ and the kernel of this action is the normal subgroup I_nCJ_n, which is determined as the group of corteges of polynomials of the form

$$\Big(x_1, x_2+(x_1^p-x_1)f_{21}(x),\dots,x_n+\sum_{i=1}^{n-1}(x_i^p-x_i)f_{ni}(x_1,\dots,x_{n-1})\Big),$$

where $f_{ij}(x_1,\dots,x_{i-1})\in\mathbb{F}_p[x_1,\dots,x_{i-1}$, $1\le j\le i-1$, $1\le i\le n$.

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In this paper the criterion of subgroups normality of group $R_n = J_n/I_n$ over the field $\mathbb{F}_p$, p is prime, $p \neq 2$ is established. Therefore, all the normal subgroups of J_n that include I_n are described.

We shall use the definitions and notation from [2]. The group R_n over $\mathbb{F}_p$ is determined as the group of corteges of polynomials of the form

$$\big(\alpha_1 x_1 + a_1, \ldots, \alpha_n x_n + a_n(x_1, \ldots, x_{n-1})\big),$$

where $x_i \in \mathbb{F}_p$, $\alpha_i \in \mathbb{F}_p^*$, $a_i(x_1, \ldots, x_{i-1}) \in \mathbb{F}_p[x_1, \ldots, x_n]/(x_1^p - x_1, \ldots, x_{i-1}^p - x_{i-1})$, $1 \leq i \leq n$.

The group R_n is a semidirect product of $\bigoplus_n \mathbb{F}_p^*$ and P_n, where P_n is the Sylov p-subgroup of the symmetric group of degree p^n. Therefore, each element of R_n is determined by the pair $(\widetilde{u}, \overline{u})$, where $\widetilde{u} = (\alpha_1, \ldots, \alpha_n) \in \bigoplus_n \mathbb{F}_p^*$ and $\overline{u} = [a_1, a_2(x_1), \ldots, a_n(x_1, \ldots, x_{n-1})]$ is the table of reduced polynomials [2].

Let u_k be the cortege of the initial k coordinates of $u \in R_n$, that is

$$u_k = \big(\alpha_1 x_1 + a_1, \ldots, \alpha_k x_k + a_k(x_1, \ldots, x_{k-1})\big)$$

and

$$X_k^u = \big(\alpha_1 x_1 + a_1, \ldots, \alpha_k x_k + a_k(x_1, \ldots, x_{k-1})\big).$$

Definition 1. The number $h[cx_1^{i_1} \ldots x_s^{i_s}] = 1 + i_1 + i_2 p + \cdots + i_s p^{s-1}$ is called the height of monomial $cx_1^{i_1} \ldots x_s^{i_s}$.

The maximal height of monomials of polynomial is called its height (we assume additionally that the height of 0 is 0).

The monomial of maximal height is called the main term of the polynomial.

Definition 2. The greatest number r such that $u_r = \mathrm{Id}$ is called the depth of element $u \in R_n$. If $u_1 \neq \mathrm{Id}$, then the depth of u is 0.

Any subgroup of the group R_n with minimal elements depth equal to r is called a subgroup of depth r.

Lemma 1. *1. For an arbitrary polynomial $f(X_k)$ and an arbitrary $u \in R_n$ of depth r the following inequality holds:*

$$h[f(X_k) - f(X_k^u)] \leq p^k - p^r, \tag{2}$$

moreover, there exists a polynomial $f(X_k)$ such that

$$h[f(X_k) - f(X_k^u)] = p^k - p^r. \tag{3}$$

2. For an arbitrary $u \in R_n$ of depth r, such that $\alpha_{k+1} \neq 1$ there exists a polynomial $f(X_k)$ such that

$$h[\alpha_{k+1} f(X_k) - f(X_k^u)] = p^k. \tag{4}$$

Proof. 1. The coefficients $f(X_k) - f(X_k^u)$ of the monomials of the form

$$(x_k \ldots x_{r+1})^{p-1} x_r^{s_r} \ldots x_1^{s_1}$$

for an arbitrary $0 \le s_i \le p-1$ are equal to 0, because u is of depth r. This implies that (2) holds.

Let $x_r^{s_r} \dots x_1^{s_1}$ be the main part of the polynomial $a_{r+1}(X_r)$. Then for (3) we have

$$f(X_k) = (x_k \dots x_{r+1})^{p-1} x_r^{p-1-s_r} \dots x_1^{p-1-s_1}.$$

2. It is easy to see that for an arbitrary $u \in R_n$ that satisfies condition 2 of Lemma 1, (4) holds for $f(X_k) = (x_k \dots x_1)^{p-1}$. □

Define a partial preorder on R_n. Let $u \prec v$ if there exists m such that $\widetilde{u} = \widetilde{v}^m$ and $h([\bar{u}]_i) \le h([\bar{u}]_i)$, $1 \le i \le n$.

Definition 3. A subgroup K of the group R_n is called parallelotopic if for any $u, v \in R_n$ from $u \prec v$ and $v \in K$ it follows that $u \in K$.

Remark 1. Each parallelotopic subgroup K is a semidirect product $\widetilde{K} \leftthreetimes \overline{K}$, where $\widetilde{K}$ is some subgroup $\bigoplus_n \mathbb{F}_p^*$ and $\overline{K}$ is some parallelotopic (in the sense of [2]) subgroup of P_n ($\overline{K} = K \cap P_n$), is determined uniquely by the vector

$$l(K) = (k_1, \dots, k_n), 0 \le k_i = \max_{g \in \overline{K}} \{h([g]_i)\} \le p^{i-1}.$$

Lemma 2. *Each normal subgroup of the group R_n is parallelotopic.*

Proof. Let NCR_n, $v \in N$, $u \prec v$. Then there exists an element $t = (\tau_1 x_1, \dots, \tau_n x_n)$ such that for $w = v^t v^{-1} \in N \cap P_n$ the equality $h([\overline{w}]_i) = h([\overline{v}]_i)$ holds. The subgroup $N \cap P_n$ is a normal parallelotopic subgroup of the group P_n, because $N \cap P_n$ is invariant under inner automorphisms of R_n (Th.3, p.31, [2]). Therefore, $\overline{u} \in N \cap P_n$. Taking into account that $\overline{v} \in N \cap P_n$, we obtain also that $\widetilde{v} \in N$. Since $u \prec v$, we have $\widetilde{u} \in N$. Thus, $u = \widetilde{u}\overline{u} \in N$. □

Let $C < R_n$. We shall construct a subgroup $\widetilde{C} < \bigoplus_n \mathbb{F}_p^*$ such that $\widetilde{u} \in \widetilde{C}$ if and only if $u \in C$. Let $\pi_j \colon \bigoplus_n \mathbb{F}_p^* \to \mathbb{F}_p^*$ be the projector on the jth coordinate $(\pi_j(\alpha_1, \dots, \alpha_n) = \alpha_j)$.

Theorem 1. *A subgroup K of the group R_n of depth r is normal if and only if K is parallelotopic and such that:*
if $\pi_i(\widetilde{K}) = \{1\} (i > r)$, then $k_i \ge p^{i-1} - p^r$, (∗)
and if $\pi_i(\widetilde{K}) \ne \{1\}$, then $k_i = p^{i-1}$. (∗∗)

Proof. Let K be a normal subgroup of R_n. Then, by Lemma 2, it is paralellotopic. If $\pi_i(\widetilde{K}) \ne \{1\}$, then there exists $u \in K$ such that $\pi_i(\widetilde{u}) \ne 1$. For an arbitrary $t \in R_n$ $v = t^{-1} u t \in K$ holds. Let

$$v = \big(\beta_1 x_1 + b_1, \dots, \beta_n x_n + b_n(x_1, \dots, x_{n-1})\big).$$

Let us write down the condition of conjugacy of u and v for i-th coordinate:

$$\alpha_i = \beta_i \text{ and } b_i(X_{i-1}) = \frac{1}{\tau_1} \big[a_i(X_{i-1}^t) - t_i(X_{i-1}^v) + \alpha_i t_i(X_{i-1})\big] \tag{5}$$

We may suppose that the depth u to be equal to r. From (5) and Lemma 1 it follows that among the conjugated with u elements there exists an element v such that $h([\overline{v}]_i) = p^{i-1}$. Thus $k_{i-1} \ge p^{i-1}$. Since $k_{i-1} \le p^{i-1}$, we have $k_i = p^{i-1}$.

If $\pi_i(\widetilde{K}) = \{1\}$, then for an arbitrary $u \in K$: $\pi_i(\widetilde{u}) = 1$, besides that there exists $u \in K$ of depth r. By (5) and Lemma 1 we obtain that among the elements conjugated with u there exists v such that $h([\overline{v}]_i) = \max\{h([\overline{u}]_i), p^{i-1} - p^r\} \geq p^{i-1} - p^r$. Thus $k_i \geq p^{i-1} - p^r$.

Suppose further that K is a parallelotopic subgroup of the group R_n and $(*)$, $(**)$ hold. Prove it by induction.

If $\pi_{r+1}(\widetilde{K}) \neq \{1\}$ then the K_{r+1} is characteristic in P_{r+1}, and therefore K_{r+1} is normal in R_{r+1}. If $\pi_i(\widetilde{K}) = \{1\}$, then $r+1$th coordinate of the elements from K_{r+1} contains all transformations. Thus, in any case K_{r+1} is a normal subgroup in R_{r+1}.

Suppose that K_{i-1} is normal in $R_{i-1} (r+1 < i < n)$. Let us prove that K_i is normal in R_i. Let $u \in K$.

If $\pi_i(\widetilde{K}) \neq \{1\}$, then $k_i = p^{i-1}$. Thus, for arbitrary $v \in R_i$ the inequality $h([\overline{v}]_i) \leq k_i$ holds. Taking into consideration that K_{i-1} is normal in R_{i-1} we obtain that all conjugated with a_i elements lays in K_i. Hence, in this case K_i is normal in R_i.

If $\pi_i(\widetilde{K}) = \{1\} (i > r)$, then $k_i \geq p^{i-1} - p^r$. By Lemma 1 and (5) we obtain that for an arbitrary v conjugated with u $h([\overline{v}]_i) = \max\left\{h([\overline{u}]_i), p^{i-1} - p^r\right\}$ holds. Since $u \in K$, we have $h([\overline{u}]_i) \leq k_i$ and $p^{i-1} - p^r \leq k_i$ by the condition, and therefore $h([\overline{v}]_i) \leq k_i$. This implies that in this case all conjugated with a_i elements lay in K_i. Hence, K_i is normal in R_i and inductively, K is normal in R_n. □

REFERENCES

[1] N. Ivanenko, *Normal structure of Jonkier group over a field of the characteristic 0*, Ukrainian math. journal **46** (1994), no. 6, 692-698. (Ukrainian)

[2] L.A. Kalujnin, *Selected topics of groups theory*, KGU, Kyiv, 1979. (Russian)

Kyiv University, Department of Mech. and Math.

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