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ON THE BRAUER GROUPS OF RATIONAL FUNCTION FIELDS OVER PSEUDOGLOBAL FIELDS

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Let k be an algebraic function field in one variable over a pseudofinite constant field (k is called a pseudoglobal field). The maximal divisible subgroup of the Brauer group $\operatorname{Br} k(t_1, \dots, t_n)$ of the rational function field $k(t_1, \dots, t_n)$ is described. With the purpose of description of $\operatorname{Br} k(t_1, \dots, t_n)$ for countable k, it is shown that the Ulm lengths of the p-primary components of the character group $\operatorname{Hom}_{\text{cont}}(\operatorname{Gal}(ks/k), \mathbb{Q}/\mathbb{Z})$ are $< \omega 2$. It is shown that there is only finitely many abelian extensions l/k of a given exponent which are unramified outside of a finite set of valuation.

By a pseudoglobal field k we mean an algebraic function field in one variable over a pseudofinite [1] constant field $\varkappa$. Recall that a field $\varkappa$ is called pseudofinite if $\varkappa$ satisfies the following three properties:

1) $\varkappa$ is perfect;
2) $\varkappa$ has a unique extension of each degree;
3) $\varkappa$ is pseudo-algebraically closed, i.e. every irreducible variety over $\varkappa$ has a $\varkappa$-rational point.

It is known [1] that every field $\varkappa$ of non-zero characteristic p, algebraic over the prime subfield $\mathbb{Z}/p\mathbb{Z}$ and having finite q-primary degree for all prime q (i.e. $[\varkappa : \mathbb{Z}/p\mathbb{Z}] = \prod_q q^{v_q}$ with $v_q < \infty$ for all prime q) is pseudofinite. On the other hand, M. Jarden [2] has shown that if $\varkappa$ is finitely generated over $\mathbb{Q}$ then for almost all $\sigma \in \operatorname{Gal}(\varkappa_s/\varkappa)$ the fixed field $\varkappa_s(\sigma)$ of σ is pseudofinite.

B. Fein, M. Schacher and J. Sonn [3,4] have completely described the Brauer group $\operatorname{Br} k(t_1, \dots, t_n)$ in the case of rational function field $k(t_1, \dots, t_n)$ over a global field k. Our aim is to try, using theirs methods, to describe the Brauer groups $\operatorname{Br} k(t_1, \dots, t_n)$ in the case of countable pseudoglobal field k (nevertheless, we remark that the presented below results do not involve any assumption about cardinality of k). This attempt is motivated by the fact that an important part of the global class field theory can be extended to the pseudoglobal fields.

Unfortunately, we are unable now to get a description of $\operatorname{Br} k(t_1, \dots, t_n)$ for pseudoglobal k as complete as that given by B. Fein, M. Schacher and J. Sonn for global k. Thus, this note presents only the first attempt in determining $\operatorname{Br} k(t_1, \dots, t_n)$ for a pseudoglobal field k.

We follow the notation and terminology of [3,4]; for convenience, recall some notations and notions. In what follows, p will denote a prime number, $p \neq 2$. We

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denote by $A_{(p)}$ the p-primary component of an abelian torsion group A, $A_m = \{a \in A \mid ma = 0\}$ is the kernel of multiplication by m. The Ulm subgroups of $A_{(p)}$ are defined inductively for any ordinal λ by: $A_{(p)}(0) = A_{(p)}$, $A_{(p)}(\lambda+1) = pA_{(p)}(\lambda)$, and for a limit ordinal λ we have $A_{(p)}(\lambda) = \bigcap_{\alpha<\lambda} A_{(p)}(\lambda)$. The least ordinal λ with $A_{(p)}(\lambda) = pA_{(p)}(\lambda)$ is denoted by $l_p(A)$ and is called the Ulm length of $A_{(p)}$. $\mathcal{D}A_{(p)}$ is the maximal divisible subgroup of $A_{(p)}$, that is the intersection of all $A_{(p)}(\lambda)$. One has $A_{(p)} = \mathcal{D}A_{(p)} \oplus RA_{(p)}$ for some subgroup $RA_{(p)}$ of $A_{(p)}$, which is called the reduced subgroup of $A_{(p)}$. The group $\mathcal{D}A_{(p)}$ is isomorphic to the direct sum of $r_p(A)$ copies of $\mathbb{Z}(p^\infty)$. If the cardinal $r_p(A)$ is determined, then the description of the countable group $A_{(p)}$ is reduced to finding the Ulm invariants of $RA_{(p)}$.

We write k_s for the separable closure of a field k. The group $X(k)$ of all continuous homomorphisms from $\mathrm{Gal}(k_s/k)$ into the discrete group $\mathbb{Q}/\mathbb{Z}$ is called the character group of k.

We denote by $\mathrm{Br}\, k$ the Brauer group of a field k. For $p \neq \mathrm{char}\, k$, $\mu(p^n)$ stands for the group of all roots of unity over the field k, $\mu(p^\infty)$ is the union of all $\mu(p^n)$.

Now we can formulate our main results.

Theorem 1. *Let $K = k(t_1, \ldots, t_n)$ be the rational function field in n variables $t_1, \ldots, t_n$ over a pseudoglobal field k. Then*
a) $(\mathrm{Br}\, K)_{(p)} = (\mathcal{D}\,\mathrm{Br})_{(p)}$ *if either $p = \mathrm{char}\, k$ or $\mu(p^\infty) \subset k$.*
b) $(\mathcal{D}\,\mathrm{Br}\, K)_{(p)} = \bigoplus_{|k|} \mathbb{Z}(p^\infty)$.

Theorem 2. *Let k be a pseudoglobal field, and let V_k be the set of all discrete valuations of k. Suppose that S is a finite subset of V_k, and n is a natural number, $(n, \mathrm{char}\, k) = 1$. Then there exist only finitely many abelian extensions l/k of a given exponent n which are unramified at all valuations of $V_k \setminus S$.*

Remark. We consider only the valuations which are trivial on the constant field.

As usual, ω stands for the first infinite ordinal number.

Theorem 3. $l_p(X(k)) < \omega 2$ *for any pseudoglobal field k.*

For the convenience of the reader we recall the relevant results which will be used in our proofs. We formulate these results as the lettered theorems.

Theorem A. *Let k be a pseudoglobal field. Then there is an exact sequence*

$$0 \to \mathrm{Br}\, k \to \bigoplus_v \mathrm{Br}\, k_v \xrightarrow{\mathrm{inv}_k} \mathbb{Q}/\mathbb{Z} \to 0,$$

where the sum is taken over all discrete valuations of k (which are trivial on the constant field of k), k_v is the completion of k at the valuation v.

We sketch the proof of Theorem A; the details can be found in [5] (see also [6] or [7, Ch.I, Appendix A]). Let $\varkappa$ be the pseudofinite constant field of k, $\varkappa_s$ stands for the separable closure of $\varkappa$. Let C be a smooth, absolutely irreducible curve over $\varkappa$ with the function field k, $G_\varkappa = \mathrm{Gal}(\varkappa_s/\varkappa)$. Let $\overline{C} = C \otimes_\varkappa \varkappa_s$ be C considered as a curve over $\varkappa_s$, and let $\mathrm{Pic}\, \overline{C}$ be the divisor class group of $\overline{C}$.

D.S. Rim and G. Whaples have shown in [6] that there is an exact sequence

$$0 \to H^1(G_\varkappa, \mathrm{Pic}\, \overline{C}) \to \mathrm{Br}\, k \to \bigoplus_v \mathrm{Br}\, k_v \xrightarrow{\mathrm{inv}_k} \mathbb{Q}/\mathbb{Z} \to 0,$$

where k is an algebraic function field in one variable over a quasifinite constant field $\varkappa$.

But if $\varkappa$ is pseudofinite, we have $H^1(G_\varkappa, \mathrm{Pic}\, \overline{C}) = 0$ and this gives the required assertion.

Theorem B (Auslander-Brumer-Faddeev). *Let k be an infinite field, t transcedental over k, and let $p \neq \operatorname{char} k$. Then*

$$\big(\operatorname{Br}(k(t))\big)_{(p)} \cong \big(\operatorname{Br} k \oplus (\bigoplus_{|k|}(\bigoplus_{l} X(l)))\big)_{(p)},$$

where l ranges over all finite extensions of k in k_s.

For the proof of Theorem B we refer to [4] or [8].

We will also need the Merkur'ev-Suslin theorem. To state it, recall some necessary definitions. Let k be a field. By definition,

$$K_2(k) = k^* \otimes_{\mathbb{Z}} k^* / \langle a \otimes (1-a) \mid a \in k^* \rangle.$$

Let ξ_n be a primitive n-th root of unity and let $A = (u, v, n, \xi_n, k)$ be the central simple algebra over k generated by $\alpha, \beta \in A$ such that $\alpha^n = u$, $\beta^n = v$, $\alpha\beta = \xi_n \beta\alpha$. Let $[A]$ be the class of A in $\operatorname{Br} k$, and let $[u, v]$ be the class of $u \otimes v$ in $K_2(k)$. Then the assignement $[u, v] \to [(u, v, n, \xi_n, k)]$ induces [9] the abstract norm residue homomorphism

$$R_{n,k} \colon K_2(k)/nK_2(k) \to (\operatorname{Br} k)_n \otimes_{\mathbb{Z}} \mu(n),$$

which does not depend on the choice of the primitive root ξ_n of unity.

Theorem C. (Merkur'ev-Suslin). *Let k be an arbitrary field. Suppose that n is a natural number, $(n, \operatorname{char} k) = 1$. Then the map*

$$R'_{n,k} \colon K_2(k)/nK_2(k) \to H^2\big(k, \mu(n) \otimes \mu(n)\big)$$

is an isomorphism. Here $R'_{n,k}$ is induced by the assignement $[u, v] \to du \cup dv$, where

$$d \colon k^* = H^0(k, k_s^*) \to H^1(k, \mu(n))$$

is the homomorphism from the corresponding cohomology sequence and $\cup$ stands for the cup-product.

For the proof of Theorem C we refer to [10].

We begin the proof of Theorem 1 with a preliminary lemma.

Lemma 1. *Let k be a field, $(p, \operatorname{char} k) = 1$. Suppose that $\mu(p^\infty) \subset k$. Then $(\operatorname{Br} k)_{(p)}$ is divisible.*

Proof. Note that if $\mu(n) \subset k$, then there exist canonical isomorphisms

$$H^2\big(k, \mu(n) \otimes \mu(n)\big) \cong H^2(k, \mu(n)) \otimes \mu(n) \cong (\operatorname{Br} k)_n \otimes \mu(n),$$

and $R_{n,k}$ coincides with $R'_{n,k}$ modulo these isomorphisms. It follows from the Merkur'ev-Suslin theorem that there exist isomorphisms depending on the choice of a primitive root ξ_{p^n} of unity

$$\widetilde{R}_{p^n,k} \colon K_2(k)/p^n K_2(k) \to (\operatorname{Br} K)_{p^n},$$

where $\widetilde{R}_{p^n,k}([u, v]) = [(u_1 v, p^n, \xi_{p^n}, k)]$.

Fix the roots ξ_{p^n} so that $(\xi_{p^n})^{p^m} = \xi_{p^{n-m}}$ for $m < n$. It is clear that $(\mathrm{Br}\, k)_{p^m} \subset (\mathrm{Br}\, k)_{p^n}$, and $(\mathrm{Br}\, k)_{(p)} = \bigcup_{n=1}^{\infty} (\mathrm{Br}\, k)_{p^n}$. Therefore, every element $[A] \in (\mathrm{Br}\, k)_{(p)}$ is of the form $[A] = \sum_i r_i[(u_i, v_i, p^{n_i}, \xi_{p^{n_i}}, k)]$ for suitable $u_i, v_i \in k$, $r_i \in \mathbb{Z}$ and $n_i > 0$. But it is known that

$$[(u_i, v_i, p^{n_i}, \xi_{p^{n_i}}, k)] = p^r[(u_i, v_i, p^{n_i+r}, \xi_{p^{n_i+r}}, k)]$$

(see e.g. [9, §15] or [11, p.80, Lemma 6]). So, we have $[A] = p^r \sum_i r_i[(u_i, v_i, p^{n_i+r}, \xi_{p^{n_i+r}}, k)]$ for all $r \geq 0$, and $(\mathrm{Br}\, k)_{(p)}$ is divisible as required.

Proof of Theorem 1. Witt's theorem (for $p = \mathrm{char}\, k$, see e.g. [4, p.47] or [11, p.110]) and Lemma 1 (for $p \neq \mathrm{char}\, k$) imply the assertion a).

In order to prove the assertion b) we consider the exact sequence from Theorem A

$$0 \to \mathrm{Br}\, k \to \bigoplus_v \mathrm{Br}\, k_v \to \mathbb{Q}/\mathbb{Z} \to 0.$$

It is known [12] that $\mathrm{Br}\, k_v = \mathbb{Q}/\mathbb{Z}$, therefore, we have the exact sequence

$$0 \to \mathrm{Br}\, k \to \bigoplus_{|k|} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Using this sequence and applying Theorem B we conclude that $\mathcal{D}(\mathrm{Br}\, k)_{(p)} \cong \bigoplus_{|k|} \mathbb{Z}(p^\infty)$ as was to be proved.

Corollary. *Suppose that $K = k(t_1, \ldots, t_n)$ and $L = (t_1, \ldots, t_m)$ are the rational function fields over a pseudoglobal field k. Then $\mathcal{D}\,\mathrm{Br}\, K = \mathcal{D}\,\mathrm{Br}\, L \cong \bigoplus_{|k|} \mathbb{Q}/\mathbb{Z}$.*

Proof. It is sufficient to use the reasonings from the beginning of the proof of Theorem 16 in [13, p.525].

Proof of Theorem 2. First we remark that this theorem extends a well known result about abelian extensions of global fields or algebraic function fields with algebraically closed constant fields (see [14, Ch.6, §1]) to the case of algebraic function fields with pseudofinite constant fields. Our reasonings are essentially the same as in [14]. Therefore, we restrict ourselves to present only a sketch of the proof. For more details we refer the reader to [14].

First we suppose that $\mu(n) \subset k$ and $S = \varnothing$. Then there is an exact sequence

$$0 \to \varkappa^*/\varkappa^{*n} \to B_{un}/\varkappa^{*n} \to (\mathrm{Pic}\, k)_n \to 0,$$

where B_{un} is the subgroup of k^* corresponding to the maximal unramified abelian extension l/k of a given exponent n, $\varkappa$ is the constant field of k, and $\mathrm{Pic}\, k$ is the divisor class group of k. Since $\varkappa$ has precisely one extension of each degree, $\varkappa^*/\varkappa^{*n}$ is isomorphic to $\mu(n)$ by Kummer theory, hence it is finite. Now, let C be a smooth, absolutely irreducible curve with the function field k, and let $\overline{C}$ be C considered as a curve over $\varkappa_s$. Denote by J_C the Jacobian variety of C, and let $G_\varkappa = \mathrm{Gal}(\varkappa_s/\varkappa)$. It is known [15, p.128, Lemma 1] that $J_C(\varkappa_s)$ is $G_\varkappa$-isomorphic to $\mathrm{Pic}_0\,\overline{C} = \mathrm{Pic}_o(k\varkappa_s)$. Since $\varkappa$ is pseudofinite, we have $H^1(\varkappa, J_C(\varkappa_s)) = 0$, therefore, the exact sequence of $G_\varkappa$-modules

$$0 \to J_C(\varkappa_s) \to \mathrm{Pic}\,\overline{C} \to \mathbb{Z} \to 0$$

yields the exact cohomology sequence $0 \to J_C(\varkappa) \to H^0(G, \mathrm{Pic}\,\overline{C}) \to \mathbb{Z} \to 0$. This implies that $(J_C(\varkappa))_n \cong \left(H^0(G_\varkappa, \mathrm{Pic}\,\overline{C})\right)_n$. But $(J_C(\varkappa))_n \subset (J_C(\varkappa_s))_n \cong$

$(\mathbb{Z}/n\mathbb{Z})^{2g}$ [16, Ch.II] and we obtain that $\left(H^0\big(G_\varkappa, \operatorname{Pic}\overline{C}\big)\right)_n$ is a finite group, therefore, its subgroup $(\operatorname{Pic} C)_n = (\operatorname{Pic} k)_n$ is finite as well. Then the exact sequence with the middle term $B_{un}/\varkappa^{*n}$ shows that the group $B_{un}/\varkappa^{*n}$ is finite. Therefore, the Galois group $\operatorname{Gal}(l/k)$ is finite by Kummer theory, and the Theorem holds for $S = \varnothing$ and $\mu(n) \subset k$.

It remains to prove that the general case can be reduced to the considered one, but this follows from [14, Ch.6, §1, Lemma 1.6], which finishes the proof.

In order to prove Theorem 3 we need some preliminary results.

Lemma 2. *Let $\varkappa'$ be a subfield of the algebraic closure $\overline{\mathbb{Q}}$ of the rational number field. Suppose that $\varkappa'$ has at most one extension of each degree. It $\varkappa'$ contains no primitive p^n-th root of unity and λ' is any p-extension of $\varkappa'$ of dimension p^m, then λ' does not contain the primitive p^{n+m}-th roots of unity.*

Proof. Consider the following diagram of field extensions. The degrees of these extensions are indicated in the diagram

$$\begin{array}{ccccccc}
 & & \varkappa' & \xrightarrow{\;p^{n-r}\;} & \varkappa'(\xi_{p^n}) & \xrightarrow{\;p^{m}\;} & \varkappa'(\xi_{p^{n+m}}) \\
 & & | & & | & & | \\
\mathbb{Q} & \xrightarrow{\;p^{r-1}(p-1)\;} & \mathbb{Q}(\xi_{p^r}) & \xrightarrow{\;p^{n-r}\;} & \mathbb{Q}(\xi_{p^n}) & \xrightarrow{\;p^{m}\;} & \mathbb{Q}(\xi_{p^{n+m}}). \\
 & & \| & & & & \\
 & & \varkappa' \cap \mathbb{Q}(\xi_{p^n}) & & & &
\end{array}$$

Here ξ_{p^k} denotes a primitive p^k-th root of unity. By the Galois theory, $\varkappa'(\xi_{p^n})/\varkappa'$ and $\varkappa'(\xi_{p^{n+m}})/\varkappa'(\xi_{p^n})$ are the Galois extensions of degrees p^{n-r} and p^m, respectively [17, p.224, Theorem 4]. Since $\varkappa'$ has at most one extension of each degree and $[\lambda' : \varkappa'] = p^m$, we see that, under assumptions $\xi_{p^n} \notin \varkappa'$ and $\xi_{p^{n+m}} \in \lambda'$, we have $n - r > 0$ and $m + n - r \le m$. This contradiction shows that $\xi_{p^{n+m}} \notin \lambda'$ as required.

Lemma 3. *Let $\mathbb{F}_q$ be a finite field of characteristic l, and $\mathbb{F}_{q^{p^m}}$ its extension of degree p^m. Suppose that $\xi_{p^n} \notin \mathbb{F}_q$. Then there exists a natural number $a = a(m,n)$ depending only on m, n and l, such that $\xi_{p^a} \notin \mathbb{F}_{q^{p^m}}$.*

Proof. Let $\mathbb{F}_l$ be the prime subfield of $\mathbb{F}_q$ and let p^k be the maximal power of p dividing $[\mathbb{F}_l(\xi_{p^n}) \cap \mathbb{F}_q : \mathbb{F}_l]$.

If $\xi_{p^n} \notin \mathbb{F}_{q^{p^m}}$, we are done. In the other case, consider the field $\mathbb{F}_l(\xi_{p^n}) \subset \mathbb{F}_q(\xi_{p^n})$. $[\mathbb{F}_l(\xi_{p^n}) : \mathbb{F}_l] = d_1 = p^{n_1} h_1$, where d_1 is the minimal positive natural number with the property $l^{d_1} \equiv 1 (\operatorname{mod} p^n)$, $n_1 \le n-1$, $h_1 | p-1$. Choose an extension $\mathbb{F}_l(\xi_{p^a})$ such that $[\mathbb{F}_l(\xi_{p^a}) : \mathbb{F}_l] = d_2 = p^{n_2} h_2$, where $n_2 - n_1 \ge m$. Assuming $\mathbb{F}_l(\xi_{p^a}) \subset \mathbb{F}_{q^{p^m}}$ and comparing the degrees $[\mathbb{F}_l(\xi_{p^n}) : \mathbb{F}_l]$, $[\mathbb{F}_q(\xi_{p^n}) : \mathbb{F}_q]$, $[\mathbb{F}_l(\xi_{p^a}) : \mathbb{F}_l]$ and $[\mathbb{F}_q(\xi_{p^a}) : \mathbb{F}_q]$, we obtain that $n_1 \ge k+1$ and $n_2 \le k+m$, so $n_2 - n_1 \le m-1$. This contradiction shows that $\mathbb{F}_l(\xi_{p^a}) \not\subset \mathbb{F}_{q^{p^m}}$ and completes the proof.

Lemma 4. *Let k be a pseudoglobal field, $(p, \operatorname{char} k) = 1$. Suppose that there exists an $n \in \mathbb{N}$ such that $\mu(p^n) \not\subset k$. Let $\sigma \in (X(k)(\omega))_{(p)}$ be a character of infinite height. Suppose that $k_\sigma = k_s^{\operatorname{Ker}\sigma}$ is the corresponding fixed field. Then the extension k_σ/k is unramifield at all valuations of the field k.*

Proof. Let $\varkappa$ be the constant field of k, and let l be any finite p-extension of k, λ being its constant field.

Denote by $\varkappa' = \mathrm{Abs}(\varkappa) = \{\alpha \in \varkappa \mid \alpha$ is algebraic over the prime subfield of $\varkappa\}$ the field of absolute numbers of the field $\varkappa$. Since $\varkappa$ is pseudofinite, $\varkappa'$ has at most one extension of each degree.

If $\mathrm{char}\,\varkappa = 0$, then, by Lemma 2, $\mu(p^\infty) \not\subset \mathrm{Abs}\,\lambda$ and therefore $\mu(p^\infty) \not\subset \lambda$.

Suppose that $\mathrm{char}\,\varkappa = l > 0$. If $\lambda/\varkappa$ is any extension of degree p^m, then λ is a finite algebra over $\varkappa$. Choosing a basis $e_1, \dots, e_{p^m}$ of $\lambda/\varkappa$, we can identify every element $\overline{x} = x_1e_1 + \cdots + x_{p^m}e_{p^m}$ with p^m-tuple $\overline{x} = (x_1, \dots, x_{p^m})$; an element $x \in \varkappa$ is identified with $(x, 0, \dots, 0)$. The multiplication in λ is determined by p^{3m} structure constants

$$y_{ij}^k : \big(\sum x_ie_i\big)\big(\sum x_je_j\big) = \sum y_{ij}^k e_k.$$

It follows (see [18, p.319] for more details) that there exists a predicat $\mathcal{D}_{p^m}(\overline{y})$ in p^{3m} variables y_{ij}^k such that the formula $\forall y \mathcal{D}_{p^m}(\overline{y})$ is a device for saying: "for every extension of degree p^m".

Denote by $P(n,m)$ the following formula

$$\forall x \quad (x^{p^n} = 1 \Rightarrow x^{p^{n-1}} = 1) \Rightarrow \forall \overline{x} \quad (\overline{x}^{p^a} = 1 \Rightarrow \overline{x}^{p^{a-1}} = 1),$$

where $a = a(m,n)$ is some natural number depending on m and n. Then the elementary statement

$$R(m,n) = \forall \overline{y}\big(\mathcal{D}_{p^n}(\overline{y}) \wedge P(m,n)\big)$$

translates to saying: for every extension $\lambda/\varkappa$ of degree p^m, $\mu(p^n) \not\subset \varkappa$ implies $\mu(p^a) \not\subset \lambda$. But $R(m,n)$ holds for all finite fields of characteristic l by Lemma 3, so by [1, Th.8$'$, p.262] it holds for all pseudofinite fields of characteristic l.

So, if this pseudofinite field $\varkappa$ does not contain $\mu(p^\infty)$, then every finite p-extension $\lambda/\varkappa$ has that property as well.

Now the proof can be finished by the same method as in the global ground field case [4]. We reproduce briefly the necessary reasonings.

Suppose that $[k_\sigma : k] = p^m$. Since σ is a character of infinite height, for every N there exists an extension k_τ with $[k_\tau : k_\sigma] = p^N$, k_τ/k_σ is cyclic and k_τ/k is still cyclic. Let v be any valuation of k. Choose the valuations of k_σ and k_τ which extend v, and denote them again by v; k_v, $k_{\sigma,v}$ and $k_{\tau,v}$ being the corresponding completions. Suppose that the maximal unramified extension $\overline{k}_v$ of k_v in $k_{\sigma,v}$ does not belong to $k_{\sigma,v}$. We deal with cyclic extensions, so the intermediate fields are linearly ordered. Since $(p, \mathrm{char}\,k) = 1$, the Hilbert theory [19, p.118–126] (see also [20, Ch.I, §§6–8]) implies that $k_{\tau,v}/\overline{k}_v$ is totally and tamely ramified of degree p^{N+s}, $s > 0$. Tamely ramified extensions of degree p^{N+s} are obtained by adjoining a p^{N+s}-root of prime element; they are cyclic if and only if $\xi_{p^{s+N}} \in \overline{k}_v$. But we have proven that $\overline{k}_v$ does not contain $\xi_{p^{s+N}}$ for sufficiently large N, because its residue class field is a finite extension of $\varkappa$ of degree dividing p^m. The obtained contradiction concludes the proof.

Now, we are ready to prove Theorem 3. Having disposed of Lemma 4, we can reason similarly as in the case of global field k, using the arguments of B. Fein and M. Shacher [4, Theorem 7.2, p.63].

Proof of Theorem 3. Let $\sigma \in (X(k)(\omega 2))_{(p)}$ be a reduced character, and let k_σ be the corresponding cyclic extension of k. For every $n > 0$ there exists a $\tau \in$

$\big(X(k)(\omega)\big)_{(p)}$ with $p^n\tau=\sigma$; k_τ being the corresponding cyclic extension of k_σ. By Lemma 4, k_τ is everywhere unramified. By Theorem 2 there is only finitely many possibilities for k_τ. Consider the directed graph Γ whose vertices are the reduced elements of $\big(X(k)(\omega2)\big)_{(p)}$, and two vertices σ_1 and σ_2 are connected if $p\sigma_2=\sigma_1$. Applying Theorem 2 and Lemma 4, we obtain that the graph Γ is locally finite. By "Konig infinity lemma" [21, p.53], Γ is finite. Consequently, there are only finitely many reduced characters σ of height $<\omega2$.

Corollary. *$\big(X(k)(\omega)\big)_{(p)}$ is isomorphic to the direct sum of a finite group and a finite number of copies of $\mathbb{Z}(p^\infty)$.*

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