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SOME PROBLEMS IN INFINITE-DIMENSIONAL TOPOLOGY

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On universality of infinite products

It is well known that the Hilbert cube $Q = [0,1]^\omega$ contains a closed topological copy of any metrizable compactum whereas its pseudointerior $s = (0,1)^\omega$ contains a closed topological copy of each Polish (= complete-metrizable separable) space. By the other words, the Hilbert cube is $\mathcal{M}_0$-universal and its pseudointerior s is $\mathcal{M}_1$-universal.

Let us recall that a space X is defined to be $\mathcal{C}$*-universal*, where $\mathcal{C}$ is a class of spaces, if for every space $C \in \mathcal{C}$ there exists a closed embedding $e : C \to X$. Further for a countable ordinal α by $\mathcal{M}_\alpha$, $\mathcal{A}_\alpha$ we denote respectively the multiplicative and additive Borel class corresponding to the ordinal α. In particular, for initial ordinals α we have: $\mathcal{M}_0$ is the class of all compacta, $\mathcal{M}_1$ the class of all Polish spaces (equivalently, absolute G_δ-sets), $\mathcal{A}_1$ the class of all σ-compacta (equivalently, absolute F_σ-sets), and $\mathcal{A}_2$ is the class of all absolute $G_{\delta\sigma}$-sets. (All spaces considered in this note are metrizable and separable, all maps are continuous).

Recalling the fact that the countable power X^ω of any compact nondegenerate (resp. Polish noncompact) absolute retract X is homeomorphic to the Hilbert cube Q (resp. to its pseudointerior s) [To_1], [To_2] we see that the countable power of any compact nondegenerate AR is $\mathcal{M}_0$-universal and the countable power of any Polish noncompact AR is $\mathcal{M}_1$-universal. Here a natural question arises: is this fact true for higher Borel classes, namely, is the countable power X^ω $\mathcal{M}_\alpha$-universal for any absolute retract $X \in \mathcal{M}_\alpha \setminus \bigcup_{\xi<\alpha} \mathcal{M}_\xi$ (cf. [DM, Question 6.3])?

This question was answered in negative in [BR], where a two-dimensional absolute retract X of arbitrary high Borel complexity was constructed such that its countable power X^ω was not $\mathcal{A}_1$-universal. Another example was given by R.Cauty [Ca] who had shown that the power X^ω is not $\mathcal{A}_2$-universal, provided $X = span(A)$ is the linear span of any subset A of a linearly independent Cantor set in a Banach space. These two counterexamples give ground to the following conjectures.

1. Conjecture. *The countable power X^ω of any finite-dimensional space X is not $\mathcal{A}_1$-universal.*

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2. Conjecture. *The countable power X^ω of any strongly countable-dimensional space X is not $\mathcal{A}_2$-universal.*

A space X is called strongly countable-dimensional if X is a countable union of its closed finite-dimensional subsets.

On universal pairs and universal spaces

To move further we need to introduce the conception of a universal pair. Writing (X,Y) we always assume that Y is a subspace of X. Given two classes of spaces $\mathcal{K}$ and $\mathcal{C}$, we let $(\mathcal{K},\mathcal{C})$ to denote the class of pairs (K,C) such that $\mathcal{C} \ni C \subset K \in \mathcal{K}$. For a class of spaces $\mathcal{C}$ and a nonnegative integer n let $\mathcal{C}[n] = \{X \in \mathcal{C} \mid \dim X \le n\}$.

A pair (X,Y) is defined to be $(\mathcal{K},\mathcal{C})$-*universal* if for every pair $(K,C) \in (\mathcal{K},\mathcal{C})$ there exists a closed embedding $e : K \to X$ such that $e^{-1}(Y) = C$.

Evidently, if a pair (X,Y) is $(\mathcal{M}_0,\mathcal{C})$-universal, then the space Y is $\mathcal{C}$-universal. In some cases the converse is also true: if X is a Polish space and $\mathcal{C}$ is a sufficiently high Borel class, then $\mathcal{C}$-universality of the space Y implies the $(\mathcal{M}_0,\mathcal{C})$-universality of the pair (X,Y) (see [BRZ, 3.1.1]). Can this result be generalized?

3. Problem. *Let (X,Y) be a pair, n a non-negative integer, and $\alpha \ge 2$ a countable ordinal.*

a) Suppose $X \in \mathcal{M}_\alpha$ and Y is an $\mathcal{A}_\alpha[n]$-universal space. Is the pair (X,Y) $(\mathcal{M}_0[n],\mathcal{A}_\alpha)$-universal?

b) Suppose $X \in \mathcal{A}_\alpha$ and Y is an $\mathcal{M}_\alpha[n]$-universal space. Is the pair (X,Y) $(\mathcal{M}_0[n],\mathcal{M}_\alpha)$-universal?

For $n = 0$ and $\alpha \ge 3$ the answer to b) is "yes" (see [Ke, 28.19]). This answer is obtained by the technique of game theory which is essentially zero-dimensional and can not be applied in higher dimensions.

On σZ_n-spaces

Let n be a nonnegative integer. A subset A of a space X is called a Z_n-*set* in X if $A \subset X$ is closed and every map $f : I^n \to X$ of the n-dimensional cube can be uniformly approximated be maps missing the set A. A subset $A \subset X$ is a Z_∞-*set* in X, provided A is a Z_n-set in X for every $n \ge 0$.

A space X is defined to be a σZ_n-*space* if $X = \bigcup_{i=1}^\infty X_i$, where each X_i is a Z_n-set in X.

It is worth to remark that Z_0-sets are exactly closed nowhere dense subsets whereas σZ_0-spaces are spaces of the first Baire category. It is known since S. Banach [B] that every Borel non-complete metric group is of the first Baire category (equivalently, is a σZ_0-space). Having this in mind, T.Dobrowolski and J.Mogilski asked in [DM, Question 4.4] if every Borel non-complete pre-Hilbert space is a σZ_∞-space. This question has a negative solution: a suitable counterexample is the linear span of the Erdös set $E = \{(x_i) \in l^2 \mid (x_i) \in \mathbb{Q}^\omega\}$ in the Hilbert space l^2. In [Ba] it is shown that *span*(E) is not a σZ_∞-space whereas it is a σZ_n-space for every $0 \le n < \infty$. Moreover, the space *span*(E) is countable-dimensional (i.e. is a countable union of finite-dimensional subsets) and is linearly homeomorphic to its own square. (In fact, each absolute neighborhood retract X of the first Baire category, homeomorphic to its square $X \times X$, is a σZ_n-space for each $n \in \mathbb{N}$. This follows from the result of [BT] stating that the product $X \times Y$ of a σZ_n-space $X \in$ ANR and a σZ_m-space $Y \in$ ANR is a σZ_{n+m+1}-space).

4. Problem. *Let H be a (Borel) pre-Hilbert space of the first Baire category. Is H a σZ_n-space for every $n \in \mathbb{N}$?*

As it is shown by the example of [Ba], there are countable dimensional Borel pre-Hilbert spaces which are not σZ_∞-spaces. What about strongly countable-dimensional spaces?

5. Problem. *Let H be a strongly countable-dimensional infinite-dimensional pre-Hilbert space. Is H a σZ_∞-space?*

Remark that according to [Do] each strongly countable-dimensional infinite-dimensional linear metric space is a σZ_1-space. To answer Problem 5 in positive it suffices to prove that H is a σZ_2-space, see [Do], [Kr]. The latter will follow, provided the answer to one of the following questions is in positive (to get this just apply [BT]).

6. Problem. *Let H be a strongly countable-dimensional infinite-dimensional pre-Hilbert space.*

a) Is H homeomorphic to a dense linear subspace of the product $X \times Y$, where X and Y are strongly countable-dimensional infinite-dimensional linear metric spaces?

b) Is H homeomorphic to a dense linear subspace of the product $X \times Y \times Z$, where X, Y, Z are linear metric spaces of the first Baire category?

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