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THREE-DIMENSIONAL ELLIPTIC BOUNDARY VALUE PROBLEMS FOR AN OPEN LIPSCHITZ SURFACE

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We consider Dirichlet and Neumann boundary value problems for elliptic equation of the second order in $\mathbb{R}^3$ when the boundary conditions are given on the open Lipschitz surface. Using the structure of certain Hilbert spaces and properties of corresponding trace maps we get the existence and uniqueness of the solution of initial boundary value problems and the solutions of obtained boundary equations over an open surface S as well.

1. Introduction. Elliptic boundary value problems with the boundary conditions given on smooth open surface in $\mathbb{R}^3$ were considered in [4,10,11] and their analysis was based on the symbol calculus of pseudodifferential operators [5]. This approach requires the sufficient regularity of boundary surface and seems not to be useful for more general case. On the other hand, the method of Green's formula [2,3] works even for Lipschitz domain, i.e. when we have edges, corner points etc.

In this paper we present some modification of the method of Green's formula which uses the geometric structure of Hilbert spaces. Applying this approach we can consider the different types of boundary conditions and non-homogeneous equations as well.

In section 2 and 3 we discuss some questions connected with interior and exterior boundary value problems for a closed surface Σ. In section 4 we consider Dirichlet and Neumann problems for an open surface S. Reduction of the boundary value problems to equivalent integral and integro-differential equation over the Lipschitz surface S and properties of appropriate potentials are considered in section 5.

Here we consider different types of boundary value problems for the elliptic operator of the second order

$$Lu = -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + a_0 u,$$

where $a_{ij}, a_0 \in C^\infty(\mathbb{R}^3;\mathbb{R})$ are real functions bounded at infinity and $a_{ij} = a_{ji}$.

2. Interior boundary value problems. Let Ω_+ be a bounded Lipschitz domain. This means that its boundary Σ is locally the graph of a Lipschitz function [2,9]. Let us note that Σ can be piecewise smooth and have edges and corners.

In this section we consider the equation $Lu = f$ in Ω_+ with boundary conditions of Dirichlet or Neumann type.

As usual, denote by $C_0^\infty(\Omega_+)$ the class of infinitely differentiable functions with compact support in Ω_+. $C^\infty(\overline{\Omega}_+)$ denotes the space of functions which are C^∞ up

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to the boundary Σ, i.e. every derivative has a limit on the boundary. Since Ω_+ is a Lipschitz domain, every function $u \in C^\infty(\overline{\Omega}_+)$ has an extension $pu \in C^\infty(\mathbb{R}^3)$[9].

We use the following functional spaces on Ω_+ and Σ [7,9]:

$$\|u\|^2_{H^1(\Omega_+)} = \int_{\Omega_+} \{|\nabla u|^2 + u^2\}\, dx, \quad \|u\|^2_{H^1(\Omega_+,L)} = \|u\|^2_{H^1(\Omega_+)} + \|Lu\|^2_{L_2(\Omega_+)},$$

$$(u,v)_{H^1(\Omega_+,L)} = (u,v)_{H^1(\Omega_+)} + (Lu, Lv)_{L_2(\Omega_+)}.$$

$H^1_0(\Omega_+)$ is the closure of $C_0^\infty(\Omega_+)$ with respect to the norm $\|\cdot\|_{H^1(\Omega_+)}$. Then $H^1_0(\Omega_+) = \ker\gamma_0^+$. Here $\ker\gamma_0^+ = \{u \in H^1(\Omega_+) : \gamma_0^+ u = 0\}$, γ_0^+ is a trace map defined on $H^1(\Omega_+)$.

Futher, we essentialy use the following assumption on the Lipschitz domain Ω_+ [2]:

the trace map $\gamma_0^+ \colon H^1(\Omega_+) \to H^{1/2}(\Sigma)$ is continuous and surjective.

$$H^{-1}(\Omega_+) = (H^1_0(\Omega_+))', \quad H^{-1/2}(\Sigma) = (H^{1/2}(\Sigma))'.$$

Note that $C^\infty(\overline{\Omega}_+)$ is dense in $H^1(\Omega_+)$ [2,9] and in $H^1(\Omega_+, L)$ [2, Lemma 3.3].

For $u \in H^1(\Omega_+, L)$, $v \in H^1(\Omega_+)$ the first Green formula holds [2]

$$\int\limits_{\Omega_+} \Big\{ \sum_{i,j=1}^{3} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 uv \Big\}\, dx = \int\limits_{\Omega_+} Lu \cdot v dx + \langle \gamma_1^+ u, \gamma_0^+ v \rangle \tag{2.1}$$

and for $u, v \in H^1(\Omega_+, L)$ the second Green formula holds

$$\int\limits_{\Omega_+} (Lu \cdot v - Lv \cdot u)\, dx = \langle \gamma_1^+ v, \gamma_0^+ u \rangle - \langle \gamma_1^+ u, \gamma_0^+ v \rangle. \tag{2.2}$$

Here $\langle \cdot, \cdot \rangle$ is the relation of duality between $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$. $\gamma_1^+ u \in H^{-1/2}(\Sigma)$ coinsides with $\frac{\partial u}{\partial n_x}$ for $u \in C^\infty(\overline{\Omega}_+)$, where $\frac{\partial}{\partial n_x} = \sum_{i,j=1}^3 \cos(\vec{n}, \vec{x}_i) a_{ij} \frac{\partial}{\partial x_j}$ is a conormal derivative, $\cos(\vec{n}, x_i)$ are the coordinates of the almost everywhere defined outward pointing vector of normal $\vec{n}$ to Σ. Note that $\gamma_1^+ \colon H^1(\Omega_+, L) \to H^{-1/2}(\Sigma)$ is a continuous trace map [2].

If $\sum_{i,j=1}^3 a_{ij}(x) t_i t_j > 0$ and $a_0(x) > 0$ for arbitrary $x \in \Omega_+$, $t = (t_1, t_2, t_3) \in \mathbb{R}^3$, we can connect with the operator L in Ω_+ the following norm and inner product

$$\|u\|^2_L = \int\limits_{\Omega_+} \Big\{ \sum_{i,j=1}^{3} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + a_0 u^2 \Big\}\, dx, \quad (u,v)_L = \int\limits_{\Omega_+} \Big\{ \sum_{i,j=1}^{3} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 uv \Big\}\, dx.$$

Lemma 2.1. *Suppose the functions $a_{ij}(x)$, $a_0(x)$ satisfy the following conditions for $x \in \Omega_+$:*

$$\sum_{i,j=1}^{3} a_{ij} t_i t_j \geq c_1 \sum_{i=1}^{3} t_i^2, \qquad t \in \mathbb{R}^3,\ c_1 > 0, \tag{2.3}$$

$$a_0(x) \geq c_2 > 0. \tag{2.4}$$

Then the norms $\|\cdot\|_{H^1(\Omega_+)}$ and $\|\cdot\|_L$ are equivalent.

Proof. We have to show that there exist constants $\alpha, \beta > 0$ such that for every $u \in H^1(\Omega_+)$

$$\alpha \|u\|_{H^1(\Omega_+)} \leq \|u\|_L \leq \beta \|u\|_{H^1(\Omega_+)}$$

Applying (2.3) and (2.4) we can get

$$\|u\|_L^2 \geq c_1 \int\limits_{\Omega_+} \sum_{i=1}^{3} \Big(\frac{\partial u}{\partial x_i}\Big)^2 dx + c_2 \int\limits_{\Omega_+} u^2 dx \geq \alpha^2 \|u\|^2_{H^1(\Omega_+)},$$

where $\alpha^2 = \min\{c_1, c_2\}$. Since a_{ij}, a_0 are bound, we have

$$\|u\|_L^2 \leq M \int\limits_{\Omega_+} \sum_{i,j=1}^{3} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dx + m \int\limits_{\Omega_+} u^2 dx$$

$$\leq 3M \int\limits_{\Omega_+} \sum_{i=1}^{3} \Big(\frac{\partial u}{\partial x_i}\Big)^2 dx + m \int\limits_{\Omega_+} u^2 dx \leq \beta^2 \|u\|^2_{H^1(\Omega_+)},$$

where $M = \max a_{ij}(x)$, $m = \max a_0(x)$, $\beta = \max\{3M, m\}$.□

Condition (2.3) means that L is an elliptic operator in Ω_+.

By virtue of Lemma 2.1 we can treat the space $H^1(\Omega_+)$ with the norm $\|\cdot\|_L$ and inner product $(\cdot,\cdot)_L$. Hence we can rewrite the first Green formula (2.1) for $u \in H^1(\Omega_+, L)$, $v \in H^1(\Omega_+)$ as

$$(u, v)_L = (Lu, v)_{L_2(\Omega_+)} + \langle \gamma_1^+ u, \gamma_0^+ v\rangle. \tag{2.5}$$

Let us show that the first Green formula (2.5) is valid for $u \in H^1(\Omega_+)$, $v \in H_0^1(\Omega_+)$, i.e.

$$(u, v)_L = \langle Lu, v\rangle, \tag{2.6}$$

where the relation of duality $\langle\cdot,\cdot\rangle$ between $H_0^1(\Omega_+)$ and $H^{-1}(\Omega_+)$ is considered as an extension of $(\cdot,\cdot)_{L_2(\Omega_+)}$ because $H_0^1(\Omega_+)$ is dense in $L_2(\Omega_+)$. According to the definition of derivative in the sence of distributions, we have for $u \in H^1(\Omega_+)$ and $\varphi \in C_0^\infty(\Omega_+)$: $\langle Lu, \varphi\rangle = (u, \varphi)_L$. So far as $C_0^\infty(\Omega_+)$ is dense in $H_0^1(\Omega_+)$ and $(u, v)_L$ is a continuous bilinear form on $H^1(\Omega_+) \times H^1(\Omega_+)$, we can extend equality $\langle Lu, \varphi\rangle = (u, \varphi)_L$ by continuity on $H^1(\Omega_+) \times H_0^1(\Omega_+)$ and obtain (2.6). Here $L : H^1(\Omega_+) \to H^{-1}(\Omega)$ is a continuous operator.

Dirichlet problem. *We look for a function $u \in H^1(\Omega_+)$ such that $Lu = f$, $\gamma_0^+ u = g$, where $f \in H^{-1}(\Omega_+)$, $g \in H^{1/2}(\Sigma)$. As usual we can split this problem into two independent ones: $Lv = 0$, $\gamma_0^+ v = g$ and $Lw = f$, $\gamma_0^+ w = 0$. Then $u = v + w$.*

Let H be a Hilbert space with inner product $(\cdot,\cdot)_H$, $M \subset H$. We denote $M^\perp = \{u \in H : (u, v)_H = 0, v \in M\}$, $\ker L = \{u \in H^1(\Omega_+) : \ Lu = 0\}$.

Lemma 2.2. *$H_0^1(\Omega_+)^\perp = \ker L$.*

Proof. $H^1(\Omega_+) = H_0^1(\Omega_+) \oplus H_0^1(\Omega_+)^\perp$, because $H_0^1(\Omega_+)$ is a subspace of $H^1(\Omega_+)$. If $u \in \ker L$, $v \in H_0^1(\Omega_+)$, we have from (2.5) $(u, v)_L = \langle Lu, v\rangle + \langle\gamma_1^+ u, \gamma_0^+ v\rangle = 0$ or $u \in H_0^1(\Omega_+)^\perp$. It yields that $\ker L \subset H_0^1(\Omega_+)^\perp$. Obviously, $\ker L \cap H_0^1(\Omega_+) = \varnothing$. Let $u \in H_0^1(\Omega_+)^\perp, v \in H_0^1(\Omega_+)$. Then by (2.6), $\langle Lu, v\rangle = 0$ for every $v \in H_0^1(\Omega_+)$. Since $L\colon H^1(\Omega_+) \to H^{-1}(\Omega_+)$, we have $Lu = 0$ and finally $H_0^1(\Omega_+)^\perp = \ker L$.□

Corollary 2.1. *$H^1(\Omega_+) = H_0^1(\Omega_+) \oplus \ker L$.*

Corollary 2.1 implies that the homogeneous problem $Lu = 0, \gamma_0^+ u = 0$ has only trivial solution.

Lemma 2.3. *The operator* $\gamma_0^+\colon \ker L \to H^{1/2}(\Sigma)$ *is an isomorphism.*

Proof. Lemma follows from the representation $H^1(\Omega_+) = H_0^1(\Omega_+) \oplus \ker L$, because the trace map $\gamma_0^+\colon \ker L \to H^{1/2}(\Sigma)$ is surjective and continuous.□

Corollary 2.2. *The Dirichlet problem* $Lu = 0$, $\gamma_0^+ u = g$ *has a unique solution for every* $g \in H^{1/2}(\Sigma)$.

Lemma 2.4. *The operator* $L : H_0^1(\Omega_+) \to H^{-1}(\Omega_+)$ *is an isomorphism.*

Proof. For $u \in H_0^1(\Omega_+)$ from (2.6) we have $\langle Lu, u\rangle = \|u\|_L^2$, i.e. L is positively defined on $H_0^1(\Omega_+)$. It means that L is an isomorphism: injectivity is obvious and surjectivity follows from [1, Theorem 2.1.16].□

Corollary 2.3. *The Dirichlet problem* $Lu = f, \gamma_0^+ u = 0$ *has a unique solution for every* $f \in H^{-1}(\Omega_+)$.

Neumann problem. First we define a trace map $\gamma_1^+\colon H^1(\Omega_+, L) \to H^{-1/2}(\Sigma)$ [2]. For an arbitrary $g \in H^{1/2}(\Sigma)$ and $u \in H^1(\Omega_+, L)$ let

$$\langle \gamma_1^+ u, g\rangle = (u, (\gamma_0^+)^{-1} g)_L - (Lu, (\gamma_0^+)^{-1} g)_{L_2(\Omega_+)}, \tag{2.7}$$

where $(\gamma_0^+)^{-1} : H^{1/2}(\Sigma) \to \ker L$ is inverse of γ_0^+.

In this section we look for a function $u \in H^1(\Omega_+, L)$ such that $Lu = f$, $\gamma_1^+ u = g$, where $f \in L_2(\Omega_+)$, $g \in H^{-1/2}(\Sigma)$. We have a decomposition $u = v + w$: $Lv = 0$, $\gamma_1^+ v = g$ and $Lw = f$, $\gamma_1^+ w = 0$.

Lemma 2.5. *The operator* $\gamma_1^+\colon \ker L \to H^{-1/2}(\Sigma)$ *is an isomorphism.*

Proof. From Lemma 2.3 we see that the transposition operator $(\gamma_0^+)'\colon H^{-1/2}(\Sigma) \to (\ker L)'$ is an isomorphism. Denote by J the canonical isometry from $\ker L$ onto $(\ker L)'$. Then from (2.7) for $u, v \in \ker L$ we have $\langle \gamma_1^+ u, \gamma_0^+ v\rangle = (u, v)_L = \langle Ju, v\rangle$. By transposition we can obtain $\langle \gamma_1^+ u, \gamma_0^+ v\rangle = \langle (\gamma_0^+)' \gamma_1^+ u, v\rangle$ or $\langle (\gamma_0^+)' \gamma_1^+ u - Ju, v\rangle = 0$ for $v \in \ker L$. It means that $(\gamma_0^+)' \gamma_1^+ = J$ or $\gamma_1^+ = [(\gamma_0^+)']^{-1} J$. Thus γ_1^+ is an isomorphism.□

Corollary 2.4. *The Neumann problem* $Lu = 0$, $\gamma_1^+ u = g$ *has a unique solution for all* $g \in H^{-1/2}(\Sigma)$.

As it was mentioned above for the Lipschitz domain Ω_+ the operator γ_1^+ defined by (2.7) is continuous from $H^1(\Omega_+, L)$ into $H^{-1/2}(\Sigma)$ [2].

Lemma 2.6. *The trace map* $\gamma_1^+\colon H^1(\Omega_+, L) \to H^{-1/2}(\Sigma)$ *is surjective.*

Proof. So far as $\ker L$ is a closed subspace of $H^1(\Omega_+, L)$, the surjectivity of γ_1^+ follows from Lemma 2.5.□

Put $\ker \tilde\gamma_0^+ = \{u \in H^1(\Omega_+, L) : \gamma_0^+ u = 0\}$, $\ker \gamma_1^+ = \{u \in H^1(\Omega_+, L) : \gamma_1^+ u = 0\}$. Since γ_0^+ and γ_1^+ are continuous on $H^1(\Omega_+, L)$, we see that $\ker \tilde\gamma_0^+$ and $\ker \gamma_1^+$ are closed subspaces of $H^1(\Omega_+, L)$.

Lemma 2.7. *The operator* $L\colon \ker \tilde\gamma_0^+ \to L_2(\Omega_+)$ *is an isomorphism.*

Proof. So far as $L_2(\Omega_+) \subset H^{-1}(\Omega_+)$ for every $f \in L_2(\Omega_+)$, there exists a unique $u \in H_0^1(\Omega_+)$ such that $Lu = f$ (see Lemma 2.4). This implies that $u \in \ker \tilde\gamma_0^+$, because $Lu \in L_2(\Omega_+)$.□

We can show that $H^1(\Omega_+, L) = \ker \tilde{\gamma}_0^+ \oplus \ker L$. Indeed, let $u \in \ker L$, $v \in \ker \tilde{\gamma}_0^+$. Then from (2.5) we have $(u,v)_{H^1(\Omega_+,L)} = (u,v)_L = 0$. Thus $\ker \tilde{\gamma}_0^+ \subset (\ker L)^\perp$. For every $u \in H^1(\Omega_+, L)$ we have a unique decomposition $u = u_1 + u_2$, where $u_1 \in \ker L$, $u_2 \in (\ker L)^\perp$. Since $(u_1,u_2)_{H^1(\Omega_+,L)} = (u_1,u_2)_L = 0$ and $Lu_2 \in L_2(\Omega_+)$, we obtain that $u_2 \in \ker \tilde{\gamma}_0^+$. So we can infer that $(\ker L)^\perp \subset \ker \tilde{\gamma}_0^+$ and hence $(\ker L)^\perp = \ker \tilde{\gamma}_0^+$.

Lemma 2.8. *The operator* $L\colon \ker \gamma_1^+ \to L_2(\Omega_+)$ *is an isomorphism.*

Proof. If $u = v \in \ker \gamma_1^+$ from the Green formula (2.5) we have $\|u\|_L^2 = (Lu,u)_{L_2(\Omega_+)}$. Thus the operator L is injective on $\ker \gamma_1^+$. Let $f \in L_2(\Omega_+)$. Then there exists a unique $u \in \ker \tilde{\gamma}_0^+$ and $Lu = f$ (Lemma 2.7). From Lemma 2.6 we have $\gamma_1^+ u = 0$ or $\gamma_1^+ u = g \in H^{-1/2}(\Sigma)$. In the second case Lemma 2.5 gives us a unique function $u_0 \in \ker L$ such that $\gamma_1^+ u_0 = g$. Hence we have a unique function $v = u - u_0 \in \ker \gamma_1^+$ and $Lv = f$.□

Corollary 2.5. *The Neumann problem* $Lu = f$, $\gamma_1^+ u = 0$, $u \in H^1(\Omega_+, L)$, *has a unique solution for arbitrary* $f \in L_2(\Omega_+)$.

Lemma 2.9. $\ker \gamma_1^+$ *is dense in* $H^1(\Omega_+)$.

Proof. Let $f \in (H^1(\Omega_+))'$ be such that $\langle f, u\rangle = 0$, $u \in \ker \gamma_1^+$. By Riesz's theorem we have a unique $v \in H^1(\Omega_+)$ such that $\langle f,u\rangle = (u,v)_L = (Lu,v)_{L_2(\Omega_+)} + \langle \gamma_1^+ u, \gamma_0^+ v\rangle = (Lu,v)_{L_2(\Omega_+)} = 0$, according to (2.5). Thus we get $v \in H^1(\Omega_+)$ such that for an arbitrary $u \in \ker \gamma_1^+$ it follows $(Lu,v)_{L_2(\Omega_+)} = 0$ or using Lemma 2.8 $(g,v)_{L_2(\Omega_+)} = 0$ for an arbitrary $Lu = g \in L_2(\Omega_+)$. It means that $v = 0$ or $f = 0$. Thus $\langle f, w\rangle = 0$ for an abitrary $w \in H^1(\Omega_+)$ and we get what was to be proved.□

3. Exterior boundary value problems. In this section we consider boundary value problems for the operator L in unbounded domain $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega}_+$. Similiary to the case of interior problems we introduce the following functional spaces in Ω_-:

$$\|u\|^2_{H^1(\Omega_-)} = \int\limits_{\Omega_-} \{|\nabla u|^2 + u^2\}\, dx, \quad \|u\|^2_{H^1(\Omega_-,L)} = \|u\|^2_{H^1(\Omega_-)} + \|Lu\|^2_{L_2(\Omega_-)},$$

$$(u,v)_{H^1(\Omega_-,L)} = (u,v)_{H^1(\Omega_-)} + (Lu,Lv)_{L_2(\Omega_-)}.$$

Let $C_0^\infty(\Omega_-)$ be the class of infinitely differentiable functions with compact support in Ω_-. $C_0^\infty(\overline{\Omega}_-)$ denotes the space of functions from $C_0^\infty(\mathbb{R}^3)$ restricted to $\overline{\Omega}_-$. Then $C_0^\infty(\overline{\Omega}_-)$ is dense in $H^1(\Omega_-)$ and $H^1(\Omega_-, L)$ [8].

$H_0^1(\Omega_-)$ is the closure of $C_0^\infty(\Omega_-)$ in the norm $\|\cdot\|_{H^1(\Omega_-)}$. Then $H_0^1(\Omega_-) = \ker \gamma_0^+ = \{u \in H^1(\Omega_-) : \gamma_0^- u = 0\}$, where $\gamma_0^- : H^1(\Omega_-) \to H^{1/2}(\Sigma)$ is a continuous and surjective trace map. $H^{-1}(\Omega_-) = (H_0^1(\Omega_-))'$.

Put

$$\|u\|_L^2 = \int\limits_{\Omega_-} \Big\{ \sum_{i,j=1}^3 a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + a_0 u^2 \Big\}\, dx, \quad (u,v)_L = \int\limits_{\Omega_-} \Big\{ \sum_{i,j=1}^3 a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0 uv \Big\}\, dx,$$

where $u, v \in H^1(\Omega_-)$.

Lemma 3.1. *Suppose the functions $a_{ij}(x), a_0(x)$ are smooth and bounded in Ω_- and satisfy the conditions (2.3), (2.4) in Ω_-. Then the norms $\|\cdot\|_{H^1(\Omega_-)}$ and $\|\cdot\|_L$ are equivalent.*

The proof is analogous to that of Lemma 2.1.

Thus we can use the space $H^1(\Omega_-)$ with the norm $\|\cdot\|_L$ and inner product $(\cdot,\cdot)_L$. Analogously as in Ω_+ for all $u, v \in C_0^\infty(\overline{\Omega}_-)$ we have

$$(u,v)_L = (Lu,v)_{L_2(\Omega_-)} - \int\limits_{\Sigma} \frac{\partial u}{\partial n_y} v ds_y \tag{3.1}$$

and since $C_0^\infty(\overline{\Omega}_-)$ is dense in $H^1(\Omega_-)$ and $H^1(\Omega_-, L)$, we can extend (3.1) by continuity. Then we have for $u \in H^1(\Omega_-, L)$, $v \in H^1(\Omega_-)$

$$(u,v)_L = (Lu,v)_{L_2(\Omega_-)} - \langle \gamma_1^- u, \gamma_0^- v\rangle, \tag{3.2}$$

where $\gamma_1^- \colon H^1(\Omega_-, L) \to H^{-1/2}(\Sigma)$.

Analogously we can show the validity of the first Green formula $(u,v) = \langle Lu, v\rangle$ for $u \in H^1(\Omega_-)$, $v \in H_0^1(\Omega_-)$ (see also (2.6)).

Using (3.2) for $u, v \in H^1(\Omega_-, L)$ we have the second Green formula in Ω_-:

$$(Lu,v)_{L_2(\Omega_-)} - (u,Lv)_{L_2(\Omega_-)} = \langle \gamma_1^- u, \gamma_0^- v\rangle - \langle \gamma_1^- v, \gamma_0^- u\rangle. \tag{3.3}$$

Using the properties of the trace map $\gamma_0^- \colon H^1(\Omega_-) \to H^{1/2}(\Sigma)$ and formula (3.2) we can prove the following lemmas and corollaries analogously as in the section 2.

Dirichlet problem. We consider in Ω_- the following boundary value problem: $Lu = f$, $\gamma_0^- u = g$, where $f \in H^{-1}(\Omega_-)$, $g \in H^{-1/2}(\Sigma)$. Then $u = v + w$, $Lv = 0$, $\gamma_0^- v = g$ and $Lw = f$, $\gamma_0^- w = 0$.

Lemma 3.2. $H_0^1(\Omega_-)^\perp = \ker L$.

Corollary 3.1. $H^1(\Omega_-) = H_0^1(\Omega_-)^\perp \oplus \ker L$.

Lemma 3.3. *The operator $\gamma_0^- \colon \ker L \to H^{1/2}(\Sigma)$ is an isomorphism.*

Corollary 3.2. *The Dirichlet problem $Lu = 0$, $\gamma_0^- u = g$, $u \in H^1(\Omega_-)$ has a unique solution for every $g \in H^{1/2}(\Sigma)$.*

Lemma 3.4. *The operator $L \colon H_0^1(\Omega_-) \to H^{-1}(\Omega_-)$ is an isomorphism.*

Corollary 3.3. *The Dirichlet problem $Lu = f$, $\gamma_0^- u = 0$ has a unique solution for every $f \in H^{-1}(\Omega_-)$.*

Neumann problem. We define the trace map $\gamma_1^- \colon H^1(\Omega_-, L) \to H^{-1/2}(\Sigma)$ for an arbitrary $g \in H^{1/2}(\Sigma)$ and $u \in H^1(\Omega_-, L)$ as

$$\langle \gamma_1^- u, g\rangle = -(u, (\gamma_0^-)^{-1} g)_L + (Lu, (\gamma_0^-)^{-1} g)_{L_2(\Omega_-)},$$

where $(\gamma_0^-)^{-1} \colon H^{1/2}(\Sigma) \to \ker L$.

The Neumann problem in Ω_- is posed as follows: find $u \in H^1(\Omega_-, L)$ such that $Lu = f$, $\gamma_1^- u = g$, where $f \in L_2(\Omega_-)$, $g \in H^{-1/2}(\Sigma)$. We have $u = v + w$ such that $Lv = 0$, $\gamma_1^- v = g$ and $Lw = f$, $\gamma_1^- w = 0$.

Lemma 3.5. *The trace map* $\gamma_1^-\colon \ker L \to H^{-1/2}(\Sigma)$ *is an isomorphism.*

Corollary 3.4. *The Neumann problem* $Lu = 0$, $\gamma_1^+ u = g$ *has a unique solution for every* $g \in H^{-1/2}(\Sigma)$.

Lemma 3.6. *The trace map* $\gamma_1^-\colon H^1(\Omega_-, L) \to H^{-1/2}(\Sigma)$ *is continuous and surjective.*

Put $\ker\gamma_1^- = \{u \in H^1(\Omega_-, L)\colon \gamma_1^- u = 0\}$.

Lemma 3.7. *The operator* $L\colon \ker\gamma_1^- \to L_2(\Omega_-)$ *is an isomorphism.*

Corollary 3.5. *The Neumann problem* $Lu = f$, $\gamma_1^- u = 0$, $u \in H^1(\Omega_-, L)$, *has a unique solution for an arbitrary* $f \in L_2(\Omega_-)$.

4. Open boundary. The boundary value problems when the boundary conditions are given on an open surface $S \subset \mathbb{R}^3$ are the most interesting for us. We consider here this type of problems for the open Lipschitz surface by using the approach developed in the previous sections.

We suppose that an open Lipschitz surface S is a part of some closed Lipschitz surface Σ. According to our notation $\Sigma = \bar{S} \cup S_0$, $\Sigma = \partial\Omega_+$, Ω_+ and $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega}_+$ are inner bounded and outer unbounded domains. S is considered as a doublesided surface and orientation of S is fixed in such a way that $\vec{n}$ is pointed into Ω_-.

First we introduce a suitable norm in the domain $\Omega = \mathbb{R}^3 \setminus \bar{S}$. We assume that $u \in H^1(\Omega)$ if $u \in L_2(\Omega)$ and $|\nabla u| \in L_2(\Omega)$. We use the notation $u_\pm(x)$ for the restrictions of $u(x)$ to $\Omega_\pm$ respectively. Then $H^1(\Omega)$ is a Hilbert space of functions $u(x), x \in \Omega$, with the norm

$$\|u\|^2_{H^1(\Omega)} = \|u_+\|^2_{H^1(\Omega_+)} + \|u_-\|^2_{H^1(\Omega_-)} \tag{4.1}$$

and the inner product

$$(u, v)_{H^1(\Omega)} = (u_+, v_+)_{H^1(\Omega_+)} + (u_-, v_-)_{H^1(\Omega_-)},$$

where $\|\cdot\|_{H^1(\Omega_\pm)}$ and $(\cdot,\cdot)_{H^1(\Omega_\pm)}$ are defined in section 2 and section 3. Obviously, $\|\cdot\|_{H^1(\Omega)}$ doesn't depend on the choice of S_0.

Analogously we the introduce space $H^1(\Omega, L) = \{u \in H^1(\Omega) : Lu \in L_2(\Omega)\}$:

$$\|u\|^2_{H^1(\Omega,L)} = \|u_+\|^2_{H^1(\Omega_+,L)} + \|u_-\|^2_{H^1(\Omega_-,L)}.$$

Let $C_0^\infty(\Omega)$ be a class of infinitely differentiable functions with compact support in Ω. Then we denote by $H_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the norm (4.1). $H^{-1}(\Omega) = (H_0^1(\Omega))'$.

In this section we need some functional spaces and notation connected with S and S_0. Denote by $H^{1/2}(S)$ the restriction of functions from $H^{1/2}(\Sigma)$ onto S, i.e. $f \in H^{1/2}(S)$ admits an extension pf onto Σ that belongs to $H^{1/2}(\Sigma)$. The norm in $H^{1/2}(S)$ is defined by

$$\|f\|_{H^{1/2}(S)} = \inf_{pf} \|pf\|_{H^{1/2}(\Sigma)}, \tag{4.2}$$

where the infimum is taken over all extensions of f that belong to $H^{1/2}(\Sigma)$. Then we can define the continuous and surjective operator $r_S\colon H^{1/2}(\Sigma) \to H^{1/2}(S)$.

Put $H^{1/2}_{\bar S}(\Sigma) = \{u \in H^{1/2}(\Sigma) : \operatorname{supp} u \in \bar S\}$ and show that $H^{1/2}_{\bar S}(\Sigma)$ is a closed subspace of $H^{1/2}(\Sigma)$. Let $\{v_n\} \subset H^{1/2}_{\bar S}(\Sigma)$ and $v_n \to v \in H^{1/2}(\Sigma)$. Since we have a dense and continuous inclusion $H^{1/2}(\Sigma) \subset L_2(\Sigma)$ for every $w \in H^{1/2}_{\bar S_0}(\Sigma)$,

$$|(w, v_n - v)_{L_2(\Sigma)}| \le c\|w\|_{L_2(\Sigma)}\|v_n - v\|_{H^{1/2}(\Sigma)} \to 0, \quad n \to \infty.$$

So far as $(w, v_n)_{L_2(\Sigma)} = 0$, we obtain $(w, v)_{L_2(\Sigma)} = 0$ for all $w \in H^{1/2}_{\bar S_0}(\Sigma)$. It means that $v \in H^{1/2}_{\bar S}(\Sigma)$ or $H^{1/2}_{\bar S}(\Sigma)$ is a closed subspace of $H^{1/2}(\Sigma)$.

We denote by $H^{1/2}_{00}(S)$ the restriction of functions from $H^{1/2}_{\bar S}(\Sigma)$ on S, i.e. if $f \in H^{1/2}_{00}(S)$ then there exists an extension $p_0 f$ by zero on Σ that belongs to $H^{1/2}_{\bar S}(\Sigma)$. We define the norm in $H^{1/2}_{00}(S)$ as $\|f\|_{H^{1/2}_{00}(S)} = \|p_0 f\|_{H^{1/2}(\Sigma)}$.

Note that $H^{1/2}_{00}(S)$ is essentially different from $H^{1/2}(S)$ and for smooth S we can define equivalent norm in $H^{1/2}_{00}(S)$ as [7]

$$\|u\|^2_{H^{1/2}_{00}(S)} = \|u\|^2_{H^{1/2}(S)} + \|\rho^{-1/2}u\|^2_{L_2(S)},$$

where $\rho(x)$ is the distance from $x \in S$ to the smooth edge ∂S.

For every $f \in H^{-1/2}(\Sigma)$ we can consider a functional g defined on $H^{1/2}_{00}(S)$ and given for all $v \in H^{1/2}_{00}(S)$ by

$$\langle g, v\rangle = \langle f, p_0 v\rangle,$$

where $p_0 v \in H^{1/2}_{\bar S}(\Sigma)$. Since $\|v\|_{H^{1/2}_{00}(S)} = \|p_0 v\|_{H^{1/2}_{\bar S}(\Sigma)}$, we see that $g \in H^{-1/2}(S) = (H^{1/2}_{00}(S))'$ and $\|g\|_{H^{-1/2}(S)} \le \|f\|_{H^{-1/2}(\Sigma)}$. The functional g is called the restriction of f on S.

Since the operator $r_S\colon H^{1/2}_{\bar S}(\Sigma) \to H^{1/2}_{00}(S)$ is an isometry, $r'_S\colon H^{-1/2}(S) \to (H^{1/2}_{\bar S}(\Sigma))'$ is also isometry and we can identify $H^{-1/2}(S)$ and $(H^{1/2}_{\bar S}(\Sigma))'$. $H^{1/2}_{\bar S}(\Sigma)$ is a closed subspace of $H^{1/2}(\Sigma)$. Hence by the Hahn-Banach theorem every $g \in (H^{1/2}_{\bar S}(\Sigma))'$ can be extended by continuity (but not uniquely) to $f \in H^{-1/2}(\Sigma)$. Finally we get that operator of restriction $\tilde r_S\colon H^{-1/2}(\Sigma) \to H^{-1/2}(S)$ is continuous and surjective.

Let $f \in H^{-1/2}(\Sigma)$ and C be an arbitrary open set of Σ. We will say that $f = 0$ in C if the restriction of f onto C is a null functional on $H^{1/2}_{00}(C)$, i.e. $\langle f, v\rangle = 0$ for all $v \in H^{1/2}_{\bar C}(\Sigma)$. Let $C_{\max}$ be the largest open set on which $f = 0$. The complement $\Sigma \setminus C_{\max}$ is denoted by $\operatorname{supp} f$ and is called the support of the functional f.

Put $H^{-1/2}_{\bar S}(\Sigma) = \{f \in H^{-1/2}(\Sigma) : \operatorname{supp} f \subset \bar S\}$. From definition we have $\langle f, v\rangle = 0$ for all $f \in H^{-1/2}_{\bar S}(\Sigma)$, $v \in H^{1/2}_{\bar S_0}(\Sigma)$. Let $H^{-1/2}_{00}(S) = (H^{1/2}(S))'$. Every $f \in H^{-1/2}_{\bar S}(\Sigma)$ determines the functional $g \in H^{-1/2}_{00}(S)$:

$$\langle g, v\rangle = \langle f, pv\rangle, \tag{4.3}$$

where $pv \in H^{1/2}(\Sigma)$ is an arbitrary extension of $v \in H^{1/2}(S)$. If p^*v is another extension then $pv - p^*v \in H^{1/2}_{\bar S_0}(\Sigma)$ and we see that (4.3) doesn't depend on the choice of pv because $\langle f, pv - p^*v\rangle = 0$.

An inverse question whether every $g \in H^{-1/2}_{00}(S)$ determines some functional f from $H^{-1/2}_{\bar S}(\Sigma)$ is considered in the following proposition.

Proposition 4.1. *$H_{00}^{-1/2}(S)$ is isometric to $H_{\bar{S}}^{-1/2}(\Sigma)$.*

Proof. We see that $r_S\colon H^{1/2}(\Sigma)/H_{\bar{S}_0}^{1/2}(\Sigma) \to H^{1/2}(S)$ is an isometry, because $\ker r_S = H_{\bar{S}_0}^{1/2}(\Sigma)$. Then $r'_S\colon (H^{1/2}(S))' \to (H^{1/2}(\Sigma)/H_{\bar{S}_0}^{1/2}(\Sigma))'$ is also an isometry. By [1, Theorem 2.1.6] we see that $(H^{1/2}(S)/H_{\bar{S}_0}^{1/2}(\Sigma))'$ is isometric to the space $H_{\bar{S}_0}^{1/2}(\Sigma)^{\oplus} = \{f \in H^{-1/2}(\Sigma) : \langle f, v\rangle = 0 \text{ for all } v \in H_{\bar{S}_0}^{1/2}(\Sigma)\}$. Since $H_{\bar{S}_0}^{1/2}(\Sigma)^{\oplus}$ is $H_{\bar{S}}^{-1/2}(\Sigma)$ by definition we get that $H_{00}^{-1/2}(S)$ is isometric to $H_{\bar{S}}^{-1/2}(\Sigma)$ and we can define $\langle g, r_S v\rangle = \langle r'_S g, v\rangle$, $\|g\|_{H_{00}^{-1/2}(S)} = \|r'_S g\|_{H_{\bar{S}}^{-1/2}(\Sigma)}$, where $g \in H_{00}^{-1/2}(S)$, $v \in H^{1/2}(\Sigma)$.□

Throughout this section we denote by $\gamma_0^{\pm}$, $\gamma_1^{\pm}$ the trace maps onto S from $\Omega_{\pm}$ and by $\gamma_{0,\Sigma}^{\pm}$, $\gamma_{1,\Sigma}^{\pm}$ the trace maps onto Σ respectively. We consider $\gamma_0^{\pm} u$, $u \in H^1(\Omega)$ as the restrictions of $\gamma_{0,\Sigma}^{\pm} u_{\pm} \in H^{1/2}(\Sigma)$ on S with the norm (4.2), i.e. $\gamma_0^{\pm} u \in H^{1/2}(S)$.

Analogously $\gamma_1^{\pm} u$, $u \in H^1(\Omega, L)$, are the restrictions of $\gamma_{1,\Sigma}^{\pm} u_{\pm} \in H^{-1/2}(S)$ on S, i.e. $\gamma_1^{\pm} u = \tilde{r}_S \gamma_{1,\Sigma}^{\pm} u_{\pm}$. By definition we have

$$\langle \gamma_{1,\Sigma}^{\pm} u_{\pm}, p_0 g\rangle = \langle \gamma_1^{\pm} u, g\rangle, \tag{4.4}$$

where $p_0 g \in H_{\bar{S}}^{1/2}(\Sigma)$, $g \in H_{00}^{1/2}(S)$.

Denote by $[\gamma_0]u = \gamma_0^+ u - \gamma_0^- u$, $[\gamma_1]u = \gamma_1^+ u - \gamma_1^- u$ the jump across S for $u \in H^1(\Omega)$ and $u \in H^1(\Omega, L)$ respectively. Then $[\gamma_i]_\Sigma u = \gamma_{i,\Sigma}^+ u_+ - \gamma_{i,\Sigma}^- u_-$, $i = 0, 1$, and $[\gamma_0]_\Sigma u \in H^{1/2}(\Sigma)$, $[\gamma_1]_\Sigma u \in H^{-1/2}(\Sigma)$. $[\gamma_0]u \in H^{1/2}(S)$ is the restriction of $[\gamma_0]_\Sigma u$ on S, $[\gamma_1]u \in H^{-1/2}(S)$ is the restriction of $[\gamma_1]_\Sigma u$ on S. The restrictions on S_0 are denoted by $\gamma_{0,S_0}^{\pm} u$, $\gamma_{1,S_0}^{\pm} u$, $[\gamma_0]_{S_0} u$, $[\gamma_1]_{S_0} u$ and are defined analogously.

Note that $\gamma_0^{\pm} u$, $\gamma_1^{\pm} u$, $[\gamma_0]u$, $[\gamma_1]u$ don't depend on the choice of S_0.

If the coefficients a_{ij}, a_0 of the operator L satisfy conditions (2.3) and (2.4) in Ω as a consequence of Lemma 2.1 and Lemma 3.1 we see that the norm $\|\cdot\|_{H^1(\Omega)}$ defined by (4.1) and the following norm

$$\|u\|_L^2 = \|u_+\|_L^2 + \|u_-\|_L^2$$

are equivalent. Thus we can consider $H^1(\Omega)$ for a given operator L as a Hilbert space with the norm $\|\cdot\|_L$ and the inner product

$$(u, v)_L = (u_+, v_+)_L + (u_-, v_-)_L.$$

Denote by $r_D u$ the restriction of $u \in H^1(\Omega)$ on a domain $D \subset \Omega$.

The following lemma gives us a qualitative characteristic of belonging to the spaces $H^1(\Omega)$, $H^1(\Omega, L)$ and probably is well-known.

Lemma 4.1. *The following assertions are true:*

(1) *if $u \in H^1(\Omega)$ then $[\gamma_0]_{S_0} u = 0$, i.e. $[\gamma_0]u \in H_{00}^{1/2}(S)$;*
(2) *if $u_{\pm} \in H^1(\Omega_{\pm})$ and $\gamma_{0,S_0}^+ u_+ = \gamma_{0,S_0}^- u_-$ then $u_{\pm} = r_{\Omega_{\pm}} u$, where $u \in H^1(\Omega)$ and $[\gamma_0]_{S_0} u = 0$;*
(3) *if $u \in H^1(\Omega, L)$ then $[\gamma_1]_{S_0} u = 0$, i.e. $[\gamma_1]u \in H_{00}^{-1/2}(S)$;*
(4) *if $u_{\pm} \in H^1(\Omega_{\pm}, L)$ and $\gamma_{0,S_0}^+ u_+ = \gamma_{0,S_0}^- u_-$, $\gamma_{1,S_0}^+ u_+ = \gamma_{1,S_0}^- u_-$ then $u_{\pm} = r_{\Omega_{\pm}} u$, where $u \in H^1(\Omega, L)$ and $[\gamma_0]_{S_0} u = 0$, $[\gamma_1]_{S_0} u = 0$.*

Proof. (1) Since the properties of $\gamma^{\pm}_{0,S_0}$ are local, we consider a bounded Lipschitz domain $D \subset \Omega$ such that $D = \Omega_+ \cup S_0 \cup \Omega'_-$ where $\Omega'_- \subset \Omega_-$. Then $w = r_D u \in H^1(D)$ and for all $v \in C^1_0(\overline{D}) = \{v \in C^1(\overline{D}) : v(x) = 0, x \in \partial D\}$:

$$\int\limits_D \frac{\partial w}{\partial x_i} v\,dx = \int\limits_{D_+} \frac{\partial w_+}{\partial x_i} v_+\,dx + \int\limits_{D_-} \frac{\partial w_-}{\partial x_i} v_-\,dx = -\int\limits_D w\frac{\partial v}{\partial x_i}\,dx + \int\limits_{S_0} [\gamma_0]_{S_0} wv\cos(\vec{n},\vec{x}_i)\,ds.$$

As a consequence we get $\int_{S_0}[\gamma_0]_{S_0} wv\cos(\vec{n},\vec{x}_i)ds = 0$. So far as $v \in C_0(\bar{S}_0) = \{v \in C(\bar{S}_0) : v(x) = 0, x \in \partial S_0\}$ and $C_0(\bar{S}_0)$ is dense in $L_2(S_0)$, we have $[\gamma_0]_{S_0} u = 0$. Then since $[\gamma_0]_\Sigma u \in H^{1/2}(\Sigma)$, we have $[\gamma_0]u \in H^{1/2}_{00}(S)$.

(2) Consider a function $u(x)$, $x \in \Omega$, such that $r_{\Omega_\pm} u = u_\pm$. We have to show that $u \in H^1(\Omega)$, i.e. $\frac{\partial u}{\partial x_i} \in L_2(\Omega)$, where

$$\int\limits_\Omega \frac{\partial u}{\partial x_i} v dx = -\int\limits_\Omega u\frac{\partial v}{\partial x_i} dx$$

for all $v \in C^\infty_0(\Omega)$. Since $u_\pm \in H^1(\Omega_\pm)$, we get

$$\langle \frac{\partial u}{\partial x_i}, v\rangle = -\int\limits_\Omega u\frac{\partial v}{\partial x_i} dx + \int\limits_{\Omega_+} \frac{\partial u_+}{\partial x_i} v_+ dx + \int\limits_{\Omega_-} \frac{\partial u_-}{\partial x_i} v_- dx - \int\limits_{S_0} [\gamma_0]_{S_0} uv\cos(\vec{n},\vec{x}_i)ds.$$

Since $[\gamma_0]_{S_0} u = 0$, we have $\int\limits_{S_0} [\gamma_0]_{S_0} uv\cos(\vec{n},\vec{x}_i)ds = 0$ or

$$\langle \frac{\partial u}{\partial x_i}, v\rangle = \int\limits_{\Omega_+} \frac{\partial u_+}{\partial x_i} v_+ dx + \int\limits_{\Omega_-} \frac{\partial u_-}{\partial x_i} v_- dx.$$

It means that $r_{\Omega_\pm}\frac{\partial u}{\partial x_i} = \frac{\partial u_\pm}{\partial x_i} \in L_2(\Omega_\pm)$ and hence $\frac{\partial u}{\partial x_i} \in L_2(\Omega)$.

(3) Let $u \in H^1(\Omega, L)$ and $v \in H^1_{\bar{D}}(\mathbb{R}^3) = \{v \in H^1(R^3) : \text{supp } v \subset \bar{D}\}$. So far as $Lu \in L_2(\Omega)$, using the first Green formula in D, we have for $u \in H^1(\Omega, L)$, $v \in H^1_{\bar{D}}(\mathbb{R}^3)$:

$$(u, v)_L = (Lu, v)_{L_2(D)} = (Lu, v)_{L_2(\Omega)}. \tag{4.5}$$

In Ω_+ and Ω_- we can get

$$(u_\pm, v_\pm)_L = (Lu_\pm, v_\pm)_{L_2(\Omega_\pm)} \pm \langle \gamma^\pm_{1,\Sigma} u_\pm, \gamma^\pm_{0,\Sigma} v_\pm\rangle. \tag{4.6}$$

By the first proposition of the lemma $\gamma^+_{0,S_0} v_+ = \gamma^-_{0,S_0} v_- = g$ and by definition $\gamma^+_{0,S} v_+ = \gamma^-_{0,S} v_- = 0$, i.e. $g \in H^{1/2}_{00}(S_0)$ or $\gamma^\pm_{0,\Sigma} v_\pm \in H^{1/2}_{\bar{S}_0}(\Sigma)$. Then we have $\langle \gamma^+_{1,\Sigma} u_+, \gamma^+_{0,\Sigma} v_+\rangle - \langle \gamma^-_{1,\Sigma} u_-, \gamma^-_{0,\Sigma} v_-\rangle = \langle [\gamma_1]_{S_0} u, g\rangle$.

Since $(u_+, v_+)_L + (u_-, v_-)_L = (u, v)_L$, $(Lu_+, v_+)_{L_2(\Omega_+)} + (Lu_-, v_-)_{L_2(\Omega_-)} = (Lu, v)_{L_2(\Omega)}$, from (4.5) and (4.6) we obtain $\langle [\gamma_1]_{S_0} u, g\rangle = 0$.

Let us show that for all $g \in H^{1/2}_{00}(S_0)$ there exists a function $v \in H^1_{\bar{D}}(\mathbb{R}^3)$ with $\gamma^\pm_{0,S_0} v_\pm = g$. Surjectivity of $\gamma^+_{0,\Sigma}$ implies existence of $v_+ \in H^1(\Omega_+)$ such that

$\gamma^+_{0,\Sigma}v_+ \in H^{1/2}_{\bar S_0}(\Sigma)$ and $\gamma^+_{0,S_0}v_+ = g$. Analogously there exists $v'_- \in H^1(\Omega'_-)$ with $\gamma^-_{0,\partial\Omega'_-}v'_- \in H^{1/2}_{\bar S_0}(\partial\Omega'_-)$ and $\gamma^-_{0,S_0}v'_- = g$. Then by proposition (2) of the lemma we have a function $v \in H^1_{\bar D}(\mathbb{R}^3)$ such that $r_{\Omega_+}v = v_+$, $r_{\Omega'_-}v = v'_-$. Thus we have shown that $\langle[\gamma_1]_{S_0}u, g\rangle = 0$ for all $g \in H^{1/2}_{00}(S_0)$, i.e. $[\gamma_1]_{S_0}u = 0$. It means that $[\gamma_1]_\Sigma u \in H^{-1/2}_{\bar S}(\Sigma)$ or $[\gamma_1]u \in H^{-1/2}_{00}(S)$ and

$$\langle[\gamma_1]u, g\rangle = \langle[\gamma_1]_\Sigma u, pg\rangle, \tag{4.7}$$

where $u \in H^1(\Omega, L)$ and $pg \in H^{1/2}(\Sigma)$ is an arbitrary extension of $g \in H^{1/2}(S)$.

(4) Let $\varphi \in C_0^\infty(\Omega)$, $\varphi_\pm = r_{\Omega_\pm}\varphi$. Using the second Green formula in $\Omega_\pm$ we have

$$(Lu_\pm, \varphi_\pm)_{L_2(\Omega_\pm)} = (u_\pm, L\varphi_\pm)_{L_2(\Omega_\pm)} \pm \langle\gamma^\pm_{1,\Sigma}\varphi_\pm, \gamma^\pm_{0,\Sigma}u_\pm\rangle \mp \langle\gamma^\pm_{1,\Sigma}u_\pm, \gamma^\pm_{0,\Sigma}\varphi_\pm\rangle.$$

Since $\gamma^+_{0,S_0}\varphi_+ = \gamma^-_{0,S_0}\varphi_- \in H^{1/2}_{\bar S_0}(\Sigma)$, $\gamma^+_{1,S_0}\varphi_+ = \gamma^-_{1,S_0}\varphi_- \in C(\bar S_0)$, we obtain

$$\begin{aligned}(Lu_+, \varphi_+)_{L_2(\Omega_+)} + (Lu_-, \varphi_-)_{L_2(\Omega_-)} &= (u_+, L\varphi_+)_{L_2(\Omega_+} + (u_-, L\varphi_-)_{L_2(\Omega_-}\\ &+ \langle\gamma^\pm_{1,S_0}\varphi_\pm, \gamma^+_{0,S_0}u_+ - \gamma^-_{0,S_0}u_-\rangle - \langle\gamma^+_{1,S_0}u_+ - \gamma^-_{1,S_0}u_-, \gamma^\pm_{0,S_0}\varphi_\pm\rangle\\ &= (u_+, L\varphi_+)_{L_2(\Omega_+)} + (u_-, L\varphi_-)_{L_2(\Omega_-)}.\end{aligned}$$

By proposition (2) of the lemma we have a function $u \in H^1(\Omega)$ such that $r_{\Omega_\pm}u = u_\pm$. Then we get

$$(Lu_+, \varphi_+)_{L_2(\Omega_+)} + (Lu_-, \varphi_-)_{L_2(\Omega_-)} = (u, L\varphi)_{L_2(\Omega)}. \tag{4.8}$$

Let $Lu_\pm = f_\pm$. Then $f \in L_2(\Omega)$ if $f_\pm = r_{\Omega_\pm}f$.

To prove this assertion of the lemma we have to show that $Lu = f$, where derivatives in Lu are in the sence of distributions, i.e. $\langle Lu, \varphi\rangle = \langle u, L\varphi\rangle$ for all $\varphi \in C_0^\infty(\Omega)$. Using (4.8) we have

$$\begin{aligned}\langle Lu, \varphi\rangle = \langle u, L\varphi\rangle = (u, L\varphi)_{L_2(\Omega)} &= (f_+, \varphi_+)_{L_2(\Omega_+)}\\ &+ (f_-, \varphi_-)_{L_2(\Omega_-)} = (f, \varphi)_{L_2(\Omega)} = \langle f, \varphi\rangle.\end{aligned}$$

Then $\langle Lu, \varphi\rangle = \langle f, \varphi\rangle$ for all $\varphi \in C_0^\infty(\Omega)$ that means $Lu = f$ or $Lu \in L_2(\Omega)$, i.e. $u \in H^1(\Omega, L)$.$\square$

If we denote by $H^1(\Omega_+ \cup \Omega_-)$ (respectively $H^1(\Omega_+ \cup \Omega_-, L)$) a space of functions $u(x)$, $x \in \mathbb{R}^3$, with the norm (4.1) (respectively with the norm $\|\cdot\|_{H^1(\Omega,L)}$) then Lemma 4.1 yields us that $H^1(\Omega)$ is a subspace of $H^1(\Omega_+ \cup \Omega_-)$ and coincides with $\ker[\gamma_0]_{S_0}$. On its turn, $H^1(\Omega, L)$ is a subspace of $H^1(\Omega_+ \cup \Omega_-, L)$ and coincides with $\ker[\gamma_{0,1}]_{S_0} = \{u \in H^1(\Omega_+ \cup \Omega_-, L) : [\gamma_0]_{S_0}u = 0, [\gamma_1]_{S_0}u = 0\}$.

By using the Green formulae (2.1), (2.2), (3.2), (3.3) for Ω_+ and Ω_-, (4.4), (4.7) and Lemma 4.1 we can get for $u \in H^1(\Omega, L)$, $v \in H^1(\Omega)$ the first Green formula in Ω

$$(u, v)_L = (Lu, v)_{L_2(\Omega)} + \langle\gamma_1^+u, [\gamma_0]v\rangle + \langle[\gamma_1]u, \gamma_0^-v\rangle, \tag{4.9}$$

and for $u, v \in H^1(\Omega, L)$ the second Green formula

$$\begin{aligned}(Lu, v)_{L_2(\Omega)} - (u, Lv)_{L_2(\Omega)} &= \langle\gamma_1^+v, [\gamma_0]u\rangle + \langle[\gamma_1]v, \gamma_0^-u\rangle\\ &- \langle\gamma_1^+u, [\gamma_0]v\rangle - \langle[\gamma_1]u, \gamma_0^-v\rangle. \end{aligned}\tag{4.10}$$

Put $\gamma_0 = (\gamma_0^-, [\gamma_0])$, $\gamma_1 = ([\gamma_1], \gamma_1^+)$ and introduce the functional spaces

$$Z(S) = \{f = (f_1, f_2) : f_1 \in H^{1/2}(S),\ f_2 \in H_{00}^{1/2}(S)\},$$
$$Y(S) = \{g = (g_1, g_2) : g_1 \in H_{00}^{-1/2}(S),\ g_2 \in H^{-1/2}(S)\},$$

with the norm

$$\|f\|^2_{Z(S)} = \|f_1\|^2_{H^{1/2}(S)} + \|f_2\|^2_{H_{00}^{1/2}(S)},$$
$$\|g\|^2_{Y(S)} = \|g_1\|^2_{H_{00}^{-1/2}(S)} + \|g_2\|^2_{H^{-1/2}(S)}.$$

Obviously, $(Z(S))' = Y(S)$ and if $f \in Z(S)$, $g \in Y(S)$ then $\langle g, f\rangle = \langle g_1, f_1\rangle + \langle g_2, f_2\rangle$. Then we can rewrite the formula (4.9) as

$$(u, v)_L = (Lu, v)_{L_2(\Omega)} + \langle \gamma_1 u, \gamma_0 v\rangle, \tag{4.11}$$

where $\gamma_1 u = ([\gamma_1]u, \gamma_1^+ u) \in Y(S)$, $\gamma_0 v = (\gamma_0^- v, [\gamma_0]v) \in Z(S)$.

Lemma 4.2. *The trace map $\gamma_0 \colon H^1(\Omega) \to Z(S)$ is continuous and surjective.*

Proof. $\gamma_0^\pm u \in H^{1/2}(S)$ for every $u \in H^1(\Omega)$. Since $[\gamma_0]_{S_0} u = 0$, we see that $[\gamma_0]u \in H_{00}^{1/2}(S)$. It means that $\gamma_0 u \in Z(S)$. Let us show the continuity of γ_0. We have

$$\|\gamma_0^\pm u\|_{H^{1/2}(S)} \le \|\gamma_{0,\Sigma}^\pm u_\pm\|_{H^{1/2}(\Sigma)} \le c_\pm \|u_\pm\|_{H^1(\Omega_\pm)}.$$

So far as $\|g\|_{H_{00}^{1/2}(S)} = \|p_0 g\|_{H^{1/2}(\Sigma)}$, where $p_0 g$ is an extension by zero on Σ, we have

$$\|\gamma_0 u\|^2_{Z(S)} = \|\gamma_0^- u\|^2_{H^{1/2}(S)} + \|[\gamma_0]u\|^2_{H_{00}^{1/2}(S)} \le \|\gamma_0^- u\|^2_{H^{1/2}(\Sigma)} + \|p_0[\gamma_0]u\|^2_{H^{1/2}(\Sigma)} \le c\|u\|^2_{H^1(\Omega)}.$$

Here $c_\pm, c$ are some positive constants.

Suppose $f \in H^{1/2}(S)$ and $g \in H_{00}^{1/2}(S)$. Then $pf \in H^{1/2}(\Sigma)$ and $p_0 g \in H^{1/2}(\Sigma)$ are extensions on Σ. For given function pf there exists $u_- \in H^1(\Omega_-)$ such that $\gamma_{0,\Sigma}^- u_- = pf$. For $pf + p_0 g$ we have $u_+ \in H^1(\Omega_+)$ with the boundary condition $\gamma_{0,\Sigma}^+ u_+ = pf + p_0 g$. Lemma 4.1 yields that the function $u(x) = u_\pm(x)$, $x \in \Omega_\pm$, belongs to $H^1(\Omega)$, because $[\gamma_0]_{S_0} u = 0$. Hence for every $(f, g) \in Z(S)$ there exists $u \in H^1(\Omega)$ with $\gamma_0 u = (f, g)$.□

Corollary 4.1. *The trace map $[\gamma_0] \colon H^1(\Omega) \to H_{00}^{1/2}(S)$ is continuous and surjective.*

Lemma 4.3. *The trace map $\gamma_1 \colon H^1(\Omega, L) \to Y(S)$ is continuous and surjective.*

Proof. Let $u \in H^1(\Omega, L)$. Since $u_\pm \in H^1(\Omega_\pm, L)$, we have $\gamma_{1,\Sigma}^\pm u_\pm \in H^{-1/2}(\Sigma)$ or $\gamma_1^\pm u \in H^{-1/2}(S)$. From Lemma 4.1 we have $[\gamma_1]_{S_0} u = 0$, i.e. $[\gamma_1]u \in H_{00}^{-1/2}(S)$. Thus $\gamma_1 u \in Y(S)$.

Since $\gamma_1^\pm u$ are restrictions of $\gamma_{1,\Sigma}^\pm u_\pm$ (see also Lemma 2.6 and Lemma 3.6), we have

$$\|\gamma_1^\pm u\|_{H^{-1/2}(S)} \le \|\gamma_{1,\Sigma}^\pm u_\pm\|_{H^{-1/2}(\Sigma)} \le c_\pm \|u_\pm\|_{H^1(\Omega_\pm, L)}.$$

Whereas $[\gamma_1]_\Sigma u \in H_{\bar{S}}^{-1/2}(\Sigma)$, i.e. $\|[\gamma_1]u\|_{H_{00}^{-1/2}(S)} = \|[\gamma_1]_\Sigma u\|_{H^{-1/2}(\Sigma)}$, we have

$$\|[\gamma_1]u\|_{H_{00}^{-1/2}(S)} = \|\gamma_{1,\Sigma}^{+} u_+ - \gamma_{1,\Sigma}^{-} u_-\|_{H^{-1/2}(\Sigma)} \leq (c_+ + c_-)\|u\|_{H^1(\Omega,L)}.$$

And finally

$$\|\gamma_1 u\|_{Y(S)}^2 = \|\gamma_1^+ u\|_{H^{-1/2}(S)}^2 + \|[\gamma_1]u\|_{H_{00}^{-1/2}(S)}^2 \leq c\|u\|_{H^1(\Omega,L)}^2.$$

Since $\ker L \in H^1(\Omega, L)$, surjectivity of γ_1 follows from Lemma 4.7.□

Corollary 4.2. *The trace map* $[\gamma_1]\colon H^1(\Omega, L) \to H_{00}^{-1/2}(S)$ *is continuous and surjective.*

Consider the following kernels: $\ker[\gamma_0] = \{u \in H^1(\Omega) : [\gamma_0]u = 0\}$, $\ker\gamma_1 = \{u \in H^1(\Omega, L) : \gamma_1^{\pm} u = 0\}$, $\ker[\gamma_1] = \{u \in H^1(\Omega, L) : [\gamma_1]u = 0\}$, $\ker L = \{u \in H^1(\Omega) : Lu = 0\}$.

Obviously, they are closed subspaces of $H^1(\Omega)$ or $H^1(\Omega, L)$ respectively.

By definition for every $u \in H^1(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ we have

$$\langle Lu, \varphi\rangle = \langle u, L\varphi\rangle = (u, \varphi)_L.$$

Since $(u, \varphi)_L$ is a continuous bilinear form on $H^1(\Omega) \times H^1(\Omega)$ and $C_0^\infty(\Omega)$ is dence in $H_0^1(\Omega)$, we can extend by continuity the equality $\langle Lu, \varphi\rangle = (u, \varphi)_L$ on $H_0^1(\Omega)$ and obtain

$$\langle Lu, v\rangle = (u, v)_L, \tag{4.12}$$

where $u \in H^1(\Omega)$, $v \in H_0^1(\Omega)$ and L is a continuous operator from $H^1(\Omega)$ into $H^{-1}(\Omega)$.

Lemma 4.4. $H^1(\Omega) = \ker\gamma_0 \oplus \ker L$.

Proof. Since $\ker\gamma_0$ is a closed subspace of $H^1(\Omega)$, we have $H^1(\Omega) = \ker\gamma_0 \oplus \ker{\gamma_0}^{\perp}$. Let $u \in \ker L$, $v \in \ker\gamma_0$. From (4.12) we can get $(u, v)_L = 0$, i.e. $\ker L \subset \ker\gamma_0^{\perp}$. Let now $u \in \ker\gamma_0^{\perp}$, $v \in \ker\gamma_0$. Then $\langle Lu, v\rangle = 0$. So far as v is an arbitrary function from $\ker\gamma_0$ and $Lu \in H^{-1}(\Omega) = (H_0^1(\Omega))'$, we have $Lu = 0$ or $\ker\gamma_0^{\perp} \subset \ker L$. Thus $\ker L = \ker\gamma_0^{\perp}$.□

Lemma 4.5. *The operator* $L\colon \ker\gamma_0 \to H^{-1}(\Omega)$ *is an isomorphism.*

Proof. Let $u, v \in \ker\gamma_0$. Then from (4.12) we can get $\langle Lu, u\rangle = \|u\|_L^2$ and we see that L is positively defined on $\ker\gamma_0$, i.e. L is an isomorphism. Moreover, L is a canonical isometry from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$.□

Analogously as in sections 2 and 3 we define the following boundary value problems in Ω.

Dirichlet problem: find a function $u \in H^1(\Omega)$ such that $Lu = f$, $\gamma_0^{\pm} u = g_{\pm}$, where $f \in H^{-1}(\Omega)$, $(g_-, g_+ - g_-) \in Z(S)$. This problem can be reduced to the following independent ones: $Lv = 0$, $\gamma_0^{\pm} v = g_{\pm}$ and $Lw = f$, $\gamma_0^{\pm} w = 0$. Then $u = v + w$.

Neumann problem: find a function $u \in H^1(\Omega, L)$ such that $Lu = f$, $\gamma_1^{\pm} u = g_{\pm}$, where $f \in L_2(\Omega)$, $(g_+ - g_-, g_+) \in Y(S)$. This problem can be represented as: $Lv = 0$, $\gamma_1^{\pm} v = g_{\pm}$ and $Lw = f$, $\gamma_1^{\pm} w = 0$, where $u = v + w$.

Corollary 4.3. *The boundary value problem* $Lu = f$, $\gamma_0^{\pm} u = 0$, $u \in H^1(\Omega)$, *has a unique solution for every* $f \in H^{-1}(\Omega)$.

Lemma 4.6. *The trace map* $\gamma_0 \colon \ker L \to Z(S)$ *is an isomorphism.*

Proof. Injectivity of γ_0 on $\ker L$ follows From (4.11). Thus the lemma is a consequence of Lemma 4.2 and the representation of $H^1(\Omega)$ (Lemma 4.4).□

Corollary 4.4. *The Dirichlet problem* $Lu = 0$, $\gamma_0^{\pm} u = g_{\pm}$, $u \in H^1(\Omega)$, *has a unique solution for every* $g = (g_-, g_+ - g_-) \in Z(S)$.

Lemma 4.7. *The trace map* $\gamma_1 \colon \ker L \to Y(S)$ *is an isomorphism.*

Proof. Using (4.11) we can define γ_1 on $\ker L$ as $\langle \gamma_1 u, g\rangle = (u, \gamma_0^{-1} g)_L$, where $g \in Z(S)$ and $\gamma_0^{-1} \colon Z(S) \to \ker L$ is an inverse operator. So far as γ_0^{-1} is an isomorphism (see Lemma 4.6), the operator $(\gamma_0^{-1})' \colon (\ker L)' \to Y(S)$ is also an isomorphism. Let $J \colon \ker L \to (\ker L)'$ be a canonical isometry. Then we have $(u, \gamma_0^{-1} g)_L = \langle Ju, \gamma_0^{-1} g\rangle = \langle (\gamma_0^{-1})' Ju, g\rangle = \langle \gamma_1 u, g\rangle$. Thus we obtained that $\gamma_1 = (\gamma_0^{-1})' J$ and it is an isomorphism from $\ker L$ onto $Y(S)$ as a composition of two isomorphisms.□

Corollary 4.5. *The Neumann problem* $Lu = 0$, $\gamma_1^{\pm} u = g_{\pm}$, $u \in H^1(\Omega, L)$, *has a unique solution for every* $g = (g_+ - g_-, g_+) \in Y(S)$.

Lemma 4.8. *The operator* $L \colon \ker \gamma_1 \to L_2(\Omega)$ *is an isomorphism.*

Proof. If $u \in \ker \gamma_1$, $v \in H^1(\Omega)$ from (4.9) we have $\langle Lu, v\rangle = (u, v)_L$ and it is obvious that $\ker L \cap \ker \gamma_1 = \{0\}$. So far as $L_2(\Omega) \subset H^{-1}(\Omega)$ for an arbitrary $f \in L_2(\Omega)$, according to Lemma 4.5 we have a unique $u_0 \in H^1(\Omega, L)$, $u_0 \in \ker \gamma_0$ and $Lu_0 = f$. Then $\gamma_1 u_0 = g \in Y(S)$ and it means that there exists a unique function $u_1 \in \ker L$ and $\gamma_1 u_1 = g$. Consequently we have a unique $u = u_0 - u_1$ such that $Lu = Lu_0 = f$ and $\gamma_1 u = \gamma_1 u_0 - \gamma_1 u_1 = 0$.

We have shown that L is a surjective map onto $L_2(\Omega)$. This proves the lemma together with injectivity and continuity of L on $\ker \gamma_1$.□

Corollary 4.6. *The Neumann problem* $Lu = f$, $\gamma_1^{\pm} u = 0$, $u \in H^1(\Omega, L)$, *has a unique solution for every* $f \in L_2(\Omega)$.

Let us consider a question connected with the structure of $\ker L$. Put $\ker[\gamma_0]_L = \ker[\gamma_0] \cap \ker L$ and $\ker[\gamma_1]_L = \ker[\gamma_1] \cap \ker L$.

Lemma 4.9. $\ker L = \ker[\gamma_0]_L \oplus \ker[\gamma_1]_L$.

Proof. The first Green formula (4.9) yeilds us for $u, v \in \ker L :$ $(u, v)_L = \langle \gamma_1^{+} u, [\gamma_0] v\rangle + \langle [\gamma_1] u, \gamma_0^{-} v\rangle$. Since $\ker L$ is a Hilbert space with the norm $\|\cdot\|_{H^1(\Omega)}$, we have the representation $\ker L = \ker[\gamma_0]_L \oplus \ker[\gamma_0]_L^{\perp}$. Let $v \in \ker[\gamma_0]_L$, i.e. $v \in \ker L$ and $\gamma_0^{+} v = \gamma_0^{-} v = g$ on S . Then $(u, v)_L = \langle [\gamma_1] u, g\rangle$. If $u \in \ker[\gamma_0]_L^{\perp}$ we have $\langle [\gamma_1] u, g\rangle = 0$ for arbitrary $g \in H^{1/2}(S)$. Thus $u \in \ker[\gamma_1]_L$ or $\ker[\gamma_0]_L^{\perp} \subset \ker[\gamma_1]_L$. Let now $u \in \ker[\gamma_1]_L$. Then $(u, v)_L = 0$ for $v \in \ker[\gamma_0]_L$ or $u \in \ker[\gamma_0]_L^{\perp}$. This means that $\ker[\gamma_0]_L^{\perp} = \ker[\gamma_1]_L$ and proves the lemma.□

Corollary 4.7. *The trace map* $[\gamma_1] \colon \ker[\gamma_0]_L \to H_{00}^{-1/2}(S)$ *is an isomorphism.*

Corollary 4.8. *The trace map* $[\gamma_0] \colon \ker[\gamma_1]_L \to H_{00}^{1/2}(S)$ *is an isomorphism.*

5. Boundary integral equation method. Let $Q(x, y)$ be the fundamental solution of L, i.e. $L_y Q(x, y) = \delta(x - y)$, $x, y \in \mathbb{R}^3$. Applying the second Green formula (4.10) we can get for $u \in H^1(\Omega, L)$ (see, for example, [2] for a closed Lipschitz surface):

$$u(x) = \langle Lu, Q(x,\cdot)\rangle + \langle \gamma_1 Q(x,\cdot), [\gamma_0]u\rangle - \langle [\gamma_1]u, Q(x,\cdot)\rangle, \quad x \in \Omega. \tag{5.1}$$

The representation formula (5.1) implies that every function $u \in H^1(\Omega, L)$, i.e. $Lu = f \in L_2(\Omega)$, is the sum of the following potentials

$$u(x) = Df(x) + W[\gamma_0]u(x) - V[\gamma_1]u(x), \tag{5.2}$$

where

$$Df(x) = \int\limits_{\Omega} Q(x,y)f(y)dy, \quad W[\gamma_0]u(x) = \int\limits_{S} \frac{\partial Q(x,y)}{\partial n_y}[\gamma_0]u(y)ds_y,$$

and $V[\gamma_1]u(x) = \int\limits_S Q(x,y)[\gamma_1]u(y)ds_y$ if $[\gamma_1]u \in L_2(S)$.

If $u \in \ker L$ from (5.2) we have

$$u(x) = W[\gamma_0]u(x) - V[\gamma_1]u(x). \tag{5.3}$$

Lemma 5.1. *The following assertions are true:*

(1) *for all* $u \in \ker[\gamma_0]_L$ *we have* $u = -V[\gamma_1]u$;
(2) *for all* $u \in \ker[\gamma_1]_L$ *we have* $u = W[\gamma_0]u$.

Proof. If $u \in \ker[\gamma_0]_L$ from (5.3) we have $u = -V[\gamma_1]u$. Analogously for $\ker[\gamma_1]_L$.□

Lemma 5.2 (jump relations). *Let* $\tau \in H_{00}^{-1/2}(S)$ *and* $\mu \in H_{00}^{1/2}(S)$. *Then:*

(1) $V\tau \in \ker[\gamma_0]_L$ *and* $[\gamma_1]V\tau = -\tau$,
(2) $W\mu \in \ker[\gamma_1]_L$ *and* $[\gamma_0]W\mu = \mu$.

Proof. Let $\tau \in H_{00}^{-1/2}(S)$. Then by Corollary 4.7 there exists a unique $u \in \ker[\gamma_0]_L$ such that $[\gamma_1]u = \tau$. By Lemma 5.1 we have $V\tau = V[\gamma_1]u = -u \in \ker[\gamma_0]_L$.

The second assertion of the lemma can be proved in the same way.□

Let $N\tau = \frac{1}{2}(\gamma_1^+ V\tau + \gamma_1^- V\tau)$, $M\mu = \frac{1}{2}(\gamma_0^+ W\mu + \gamma_0^- W\mu)$.

Corollary 5.1. *Let* $\tau \in H_{00}^{-1/2}(S)$, $\mu \in H_{00}^{1/2}(S)$. *Then* $\gamma_1^{\pm} V\tau = \mp\frac{1}{2}\tau + N\tau$, $\gamma_0^{\pm} W\mu = \pm\frac{1}{2}\mu + M\mu$.

Proof. $\pm\frac{1}{2}(\gamma_1^+ V\tau - \gamma_1^- V\tau) = \mp\frac{1}{2}\tau$, $N\tau = \frac{1}{2}(\gamma_1^+ V\tau + \gamma_1^- V\tau)$. Summing these two equalities we obtain the first expression of the corollary. Analogously for $\gamma_0^{\pm} W\mu$.□

Lemma 5.3. *The operators* $V\colon H_{00}^{-1/2}(S) \to \ker[\gamma_0]_L$ *and* $W\colon H_{00}^{1/2}(S) \to \ker[\gamma_1]_L$ *are isomorphisms.*

Proof. This lemma is a consequence of Corollary 4.7 and Corollary 4.8, because $V = -[\gamma_1]^{-1}$, $W = [\gamma_0]^{-1}$.□

Now we can consider questions connected with reduction of boundary value problems for the operator L in the domain Ω to equivalent boundary equations over the surface S. A number of works (see, for instance, [4,6,10-14]) were devoted to applying the boundary integral equation method for different types of elliptic boundary value problems for a smooth open boundary.

In the case of sufficiently smooth boundary value data the Dirichlet problem is reduced to the equivalent Fredholm integral equation of the first kind

$$K\tau \equiv \int\limits_S Q(x,y)\tau(y)ds_y = h(x), \quad x \in S, \tag{5.4}$$

and, in turn, the Neumann problem is equivalent to the following singular integro-differential equation of the first kind

$$H\mu \equiv \Big(\int\limits_S \frac{\partial^2 Q(x,y)}{\partial n_x \partial n_y}\mu(y)ds_y\Big)^{\pm} = g(x), \quad x \in S, \tag{5.5}$$

where $\tau(y) = u^+(y) - u^-(y)$, $\mu(y) = (\frac{\partial u}{\partial n_y})^+ - (\frac{\partial u}{\partial n_y})^-$ and $f^{\pm}(y)$ is a limiting value of function $f(x), x \in \Omega$, when x tends to $y \in S$ in nontangential way.

The equation (5.4) has a weak singularity and the equation (5.5) can be represented in the form

$$H\mu \equiv \int\limits_S ([\vec{n}_x \times \nabla_x Q_1(x,y)] \cdot [\vec{n}_y \times \nabla_y \mu(y)])ds_y + \int\limits_S Q_2(x,y)\mu(y)ds_y = g(x), \tag{5.6}$$

where $x \in S$, $[\vec{a} \times \vec{b}]$ and $(\vec{a} \cdot \vec{b})$ are the vector and scalar product respectively, ∇_x is the surface gradient in the point $x \in S$. The first integral in (5.6) is a two-dimensional singular integral and the second one has a weak singularity.

Let us consider the Dirichlet problem

$$Lu = 0, \quad \gamma_0^{\pm}u = g_{\pm}, \quad u \in H^1(\Omega), \quad g = (g_-, g_+ - g_-) \in Z(S), \tag{5.7}$$

and the following equation of the first kind

$$K\tau \equiv -\gamma_0^{\pm} V\tau = \tilde{g}, \tag{5.8}$$

where $\tilde{g} = \frac{1}{2}(g_- + g_+) - M(g_+ - g_-)$, $\tilde{g} \in H^{1/2}(S)$.

Theorem 5.1. *The Dirichlet problem (5.7) is equivalent to the equation (5.8), i.e. the solution u of the problem (5.7) has the form*

$$u = -V\tau + W(g_+ - g_-), \tag{5.9}$$

where $\tau = [\gamma_1]u \in H_{00}^{-1/2}(S)$ is a solution of the equation (5.8), and vice versa if τ is a solution of the equation (5.8) then the function u given by (5.9) is a solution of the problem (5.7).

Proof. Representation (5.9) follows from the boundary conditions and from (5.3). Using Lemma 5.2 and Corollary 5.1 we can get $\gamma_0^{\pm}u = -\gamma_0^{\pm}V\tau \pm \frac{1}{2}(g_+ - g_-) + M(g_+ - g_-)$.

Then if we add the left and the right sides of the obtained equality we have

$$K\tau = \frac{1}{2}(g_- + g_+) - M(g_+ - g_-) = \tilde{g}.$$

Let now u have form (5.9), where $g \in Z(S)$ and τ is a solution of (5.8). Then by Lemma 5.3 $u \in H^1(\Omega)$, $Lu = 0$ and by Lemma 5.2 and Corollary 5.1 $\gamma_0^{\pm}u = g_{\pm}$. Hence we have proved that equation (5.8) and the boundary value problem (5.7) are equivalent via representation (5.9) which is necessary and unique for u.□

Theorem 5.2. *The operator* $K\colon H_{00}^{-1/2}(S) \to H^{1/2}(S)$ *is an isomorphism.*

Proof. From Corollary 4.4 and from Theorem 5.1 in the case $g_+ = g_- \in H^{1/2}(S)$ we have that K is surjective and $\ker K = 0$. Since K is a composition of two continuous operators, it is an isomorphism. Moreover, from (4.10) it follows that K is selfadjoint from $H_{00}^{-1/2}(S) = (H^{1/2}(S))'$ onto $H^{1/2}(S)$. We can show that K is positively defined. Let $\tau \in H_{00}^{-1/2}(S)$. Then by Lemma 5.3 $V\tau \in \ker[\gamma_0]_L$ and from (4.9) we have $\|V\tau\|_L^2 = \langle \tau, K\tau\rangle$. Since V is an isomorphism from $H_{00}^{-1/2}(S)$ onto $\ker[\gamma_0]_L$ (Lemma 5.3), there exists a constant $c > 0$ such that for all $\tau \in H_{00}^{-1/2}(S)$, $\|V\tau\|_L^2 \geq c\|\tau\|^2_{H_{00}^{-1/2}(S)}$, and finally we have

$$\langle \tau, K\tau\rangle \geq c\|\tau\|^2_{H_{00}^{-1/2}(S)}.\Box$$

Let us consider the Neumann problem

$$Lu = 0, \quad \gamma_1^{\pm} u = g_{\pm}, \quad u \in H^1(\Omega, L), \quad g = (g_+ - g_-, g_+) \in Y(S), \tag{5.10}$$

and the following equation of the first kind

$$H\mu \equiv \gamma_1^{\pm} W\mu(x) = \tilde{g}, \tag{5.11}$$

where $\tilde{g}(x) = \frac{1}{2}(g_- + g_+) + N(g_+ - g_-)$, $\tilde{g} \in H^{-1/2}(S)$.

Theorem 5.3. *The Neumann problem (5.10) is equivalent to equation (5.11), i.e. the solution u of problem (5.10) has the form*

$$u = W\mu - V(g_+ - g_-), \tag{5.12}$$

where $\mu = [\gamma_0]u \in H_{00}^{1/2}(S)$ is a solution of equation (5.11), and vice versa if μ is a solution of equation (5.11) then the function u given by (5.12) is a solution of problem (5.10).

Proof. The boundary conditions $\gamma_1^{\pm} u = g_{\pm}$ and formula (5.3) imply the representation formula (5.12) which is necessary and unique for u. Equation (5.11) is a consequence of (5.12), Lemma 5.2, Corollary 5.1 and boundary conditions. Thus we have reduced boundary value problem (5.10) via representation (5.12) to the boundary equation (5.11).

Let u be given by formula (5.12), where $\mu \in H_{00}^{1/2}(S)$ is a solution of the equation $H\mu = \tilde{g}$. Then u satisfies the equation $Lu = 0$ and boundary conditions $\gamma_1^{\pm} u = g_{\pm}$. From Lemma 5.2 we have $u \in H^1(\Omega, L)$, because $\mu \in H_{00}^{1/2}(S)$ and $g_+ - g_- \in H_{00}^{-1/2}(S)$. Thus we have proved that problem (5.10) and equation (5.11) are equivalent.$\Box$

Theorem 5.4. *The operator* $H\colon H_{00}^{1/2}(S) \to H^{-1/2}(S)$ *is an isomorphism.*

Proof. From Corollary 4.5 and from Theorem 5.3 with $g_+ = g_- \in H^{-1/2}(S)$ we see that H is bijective. Since H is a composition of two continuous operators, it is an isomorphism.

Let $\mu, \nu \in H_{00}^{1/2}(S)$. Then from (4.10) for $u = W\mu$, $v = W\nu$ we have $\langle H\mu, \nu\rangle = \langle H\nu, \mu\rangle$, i.e. H is a selfadjoint operator from $H_{00}^{1/2}(S)$ into $H^{-1/2}(S) = (H_{00}^{1/2}(S))'$.

Let us show that, moreover, H is positively defined. Let $\mu \in H_{00}^{1/2}(S)$. By Lemma 5.3 $W\mu \in \ker[\gamma_1]_L$ and from (4.9) we have $\|W\mu\|_L^2 = \langle H\mu, \mu\rangle$. Since W

is an isomorphism from $H^{1/2}_{00}(S)$ onto $\ker[\gamma_1]_L$, there exists a constant $c > 0$ such that for all $\mu \in H^{1/2}_{00}(S)$ $\|W\mu\|^2_L \geq c\|\mu\|^2_{H^{1/2}_{00}(S)}$ and we obtain

$$\langle H\mu, \mu\rangle \geq c\|\mu\|^2_{H^{1/2}_{00}(S)}.\square$$

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