УДК 517.95

THE WELL-POSEDNESS OF A FOURIER PROBLEM FOR QUASILINEAR PARABOLIC EQUATIONS OF ARBITRARY ORDER IN UNISOTROPIC SPACES

M.M. Bokalo, V.M. Sikorsky

M.M. Bokalo, V.M. Sikorsky. *The well-posedness of a Fourier problem for quasilinear parabolic equations of arbitrary order in unisotropic spaces*, Matematychni Studii, **8**(1997) 53–70.

The existence and uniqueness of a generalized solution of a Fourier problem has been proved for some class of quasilinear parabolic equations of higher order in unisotropic Sobolev spaces. There are no restrictions on the behaviour of solution and increasing of the entrance data at $t \to -\infty$. Moreover, the continuous dependence of the generalized solution of this problem on the right side of the equation has been obtained.

Introduction. The uniqueness of solutions of boundary value problems in unbounded domains for linear and a variety of nonlinear parabolic equations has been proved in the classes of functions of arbitrary behaviour on infinity, and existence have been proved if there are some boundary restrictions at infinity for entrance data or about conjunction entrance data with the geometry of a domain [1–6].

But there are such nonlinear equations which have unique solutions of the correspondent boundary value problems without any restrictions on its behaviour at infinity whereas entrance data have free arriving at infinity [7–10].

For the Cauchy Problem such results have been obtained, for example, for quasilinear parabolic equations of free degree with monotone space part. A generalized solution of this problem belongs to an isotropic space of Sobolev [7–11]. In our work these results are transmitted to the same equations with different degrees of nonlinearity of different derivatives. It led us to considering generalized solutions in unisotropic spaces of Sobolev. Besides this, the continuous dependence of generalized solution of this problem on the right side had been proved in the classes of functions with free behaviour as $t \to -\infty$. This result is also new in the case of isotropic spaces.

1. Statement of problem. Let $Q = \Omega \times (-\infty, T)$, where Ω is an unbounded domain in $\mathbb{R}^n_x$ with partly-smooth boundary $\partial\Omega, -\infty < T \leqslant +\infty$. Let $\Sigma = \partial\Omega \times (-\infty, T)$.

1991 *Mathematics Subject Classification.* 35D05, 35D99, 35K25, 35K35.
The first author is partly supported by the ISSEP grant APU 061007.

We consider the problem

$$u_t + \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha a_\alpha(x,t,\delta u) = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha f_\alpha(x,t) \quad \text{in} \quad Q, \tag{1}$$

$$\frac{\partial^j u}{\partial \nu^j} = 0 \quad \text{on} \quad S\ , \quad j = 0,1,...,m-1. \tag{2}$$

From now on the text α, β, γ denote multiindexes of length n; δu a vector consisting of all possible derivatives

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} ... \partial x_n^{\beta_n}},$$

with the degree $|\beta| = \beta_1 + ... + \beta_n$ less than or equal to m, and let N be the dimension of this vector; $\vec{\nu}$ is a unique vector of the outside normal to $\partial\Omega$. We assume that

(A_1) functions $a_\alpha(x,t,\xi)$, $|\alpha| \leqslant m$, are defined for almost all points $(x,t) \in Q$ and all vectors $\xi \in \mathbb{R}^N$ with coordinates ξ_β, $|\beta| \leqslant m$, and of Caratheodory type, that is they are measured on (x,t) for all ξ and continuous on ξ for almost all (x,t);

(A_2) there are some numbers $p_\alpha > 1$, $|\alpha| \leqslant m$, such that for all α, $|\alpha| \leqslant m$, $|a_\alpha(x,t,\xi)| \leqslant \sum_{|\beta \leq m} h_{\alpha\beta}(x,t)|\xi_\beta|^{p_\beta/p'_\alpha} + k_\alpha(x,t)$, where $h_{\alpha\beta}(x,t) \in L^\infty_{\rm loc}(\overline{Q})$, $k_\alpha(x,t) \in L^{p'_\alpha}_{\rm loc}(\overline{Q})$, $\frac{1}{p_\alpha} + \frac{1}{p'_\alpha} = 1$;

(A_3) for arbitrary vectors $\xi, \eta \in \mathbb{R}^N$ and $\sum_{|\alpha|\leq m}(a_\alpha(x,t,\xi) - a_\alpha(x,t,\eta))(\xi_\alpha - \eta_\alpha) \geqslant 0$ for almost all $(x,t) \in Q$;

(A_4) $f_\alpha(x,t) \in L^{p'_\alpha}_{\rm loc}(\overline{Q})$, $|\alpha| \leqslant m$.

Under the symbols $L^q_{\rm loc}(\overline{Q})$, with $q \geqslant 1$, we consider the space of functions measured on Q, and its restrictions on an arbitrary subset Q' of the set Q belongs to $L^q(Q')$.

Denote by $\vec{p} = (p_\alpha)$ an N-dimensioned vector whose coordinates are numbers p_α from condition (A_2). Under the symbol $\overset{o}{W}{}^m_{\vec{p}}(\Omega)$ we understand the space which is obtained by the completion of the space $C_0^\infty(\Omega)$ by the norm $\|v\| = \sum_{|\alpha|\leqslant m} \|D^\alpha v\|_{L^{p_\alpha}(\Omega)}$. Let $W^{-m}_{\vec{p}\,'}(\Omega)$ be the space which is adjoint to $\overset{o}{W}{}^m_{\vec{p}}(\Omega)$. It is known that the elements w of the space $W^{-m}_{\vec{p}\,'}(\Omega)$ are described as $w(x) = \sum_{|\alpha|\leqslant m} D^\alpha w_\alpha(x)$, where $w_\alpha(x) \in L^{p'_\alpha}(\Omega)$, $|\alpha| \leqslant m$.

Let $Q_{t_1,t_2} = \Omega \times (t_1,t_2)$, where t_1, t_2 are arbitrary numbers. Under $\overset{o}{W}{}^{m,0}_{\vec{p}}(Q_{t_1,t_2})$ we understand the space of functions which is obtained by the completion of the space $C_0^\infty(Q_{t_1,t_2})$ by the norm $||v|| = \sum_{|\alpha|\leqslant m} ||D^\alpha v||_{L^{p_\alpha}(Q_{t_1,t_2})}$. Denote by $\overset{o}{W}{}^{m,0}_{\vec{p},\rm loc}(\overline{Q})$ the space of measured on Q functions whose restrictions on Q_{t_1,t_2} for arbitrary numbers $t_1, t_2, -\infty < t_1 < t_2 \leqslant T$ belong to the space $\overset{o}{W}{}^{m,0}_{\vec{p}}(Q_{t_1,t_2})$.

Definition 1. Let the generalized solution of Problem (1),(2) be a function $u \in \overset{o}{W}{}^{m,0}_{\vec{p},\rm loc}(\overline{Q})$, for which the following integral identity

$$\iint\limits_Q \{-u\psi_t + \sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta u) D^\alpha \psi\}\, dxdt = \iint\limits_Q \sum_{|\alpha|\leqslant m} f_\alpha D^\alpha \psi \, dxdt \tag{3}$$

holds for arbitrary functions $\psi \in C_0^\infty(Q)$.

We'll study the existence of a generalized solution of Problem (1),(2), its uniqueness and continuous dependence on the right side of the equation.

2. Formulation of main results. Later in the text T is a finite positive number if otherwise is not indicated.

On the beginning we'll formulate the theorem of uniqueness of generalized solution.

Theorem 1. *Let us assume that a multiindex γ ($|\gamma| \leqslant m$) exists such that $p_\gamma > 2$ and for arbitrary $\xi, \eta \in \mathbb{R}^N$ and almost all $(x,t) \in Q$ the inequality holds*

$$\sum_{|\alpha|\leqslant m} (a_\alpha(x,t,\xi) - a_\alpha(x,t,\eta))(\xi_\alpha - \eta_\alpha) \geqslant \lambda(t)|\eta_\gamma - \xi_\gamma|^{p_\gamma}, \tag{4}$$

where $\lambda(t) \in L^\infty_{\rm loc}(-\infty,T)$, $\lambda(t) \geqslant 0$ and $\int_{-\infty}^0 \lambda(t)\,dt = +\infty$.

Then the generalized solution of Problem (1), (2) is unique.

Remark 1. The condition $p_\gamma > 2$ is necessary for the statement of Theorem 1. Indeed, let us consider the domain $Q = (0,\pi)\times(-\infty,T)$ and the following problem

$$u_t - u_{xx} + u = 0 \quad \text{in} \quad Q, \quad u|_{x=0} = 0, \quad u|_{x=\pi} = 0.$$

Here $p_0 = 2$, $p_1 = 2$. It is evident that all conditions of Theorem 1 hold for this problem (inequality (4) holds if $\lambda(t) = 1$), excepting the condition $p_\gamma > 2$ for any $\gamma \in \{0,1\}$ ($p_0 = p_1 = 2$). The following functions $u \equiv 0$ and $u = \sin x \cdot \exp\{-2t\}$ are its generalized solutions, i.e. we have no uniqueness of the solution in this case.

Let's pass to the theorem of existence of generalized solution.

Theorem 2. *Let us assume that a multiindex γ ($|\gamma| \leqslant m$) exists such that $p_\gamma > 2$ and for arbitrary $\xi \in \mathbb{R}^N$ and almost all $(x,t) \in Q$ the following inequality holds*

$$\sum_{|\alpha|\leqslant m} a_\alpha(x,t,\xi)\xi_\alpha \geqslant \sum_{|\alpha|\leqslant m} b_\alpha(t)|\xi_\alpha|^{p_\alpha} - b(x,t), \tag{5}$$

where $b_\alpha(t) \in L^\infty_{\rm loc}((-\infty,T])$, $\inf_{[a,c]} b_\alpha(t) > 0$ for every closed interval $[a,c] \subset (-\infty,T]$, $b(x,t) \in L^1_{\rm loc}(\overline{Q})$, $b(x,t) \geqslant 0$.

Then Problem (1), (2) has at least one solution and each generalized solution $u(x,t)$ belongs to the space $C((-\infty;T];L^2(\Omega))$ and for arbitrary numbers t_1, t_2, δ such that $-\infty < t_1 < t_2 \leqslant T$, $\delta > 0$, satisfies the estimate

$$\sup_{[t_1,t_2]} \int_\Omega |u(x,t)|^2\,dx + \sum_{|\alpha|\leqslant m} \overline{b}_\alpha(t_1-\delta,t_2) \int_{t_1}^{t_2}\int_\Omega |D^\alpha u|^{p_\alpha}\,dxdt \leqslant C_1[\delta\cdot\overline{b}_\gamma(t_1-\delta,t_2)]^{-\frac{2}{p_\gamma-2}} +$$

$$+ 2\int_{t_1-\delta}^{t_2}\int_\Omega b(x,t)\,dxdt + C_2 \sum_{|\alpha|\leqslant m} [\overline{b}_\alpha(t_1-\delta,t_2)]^{-\frac{1}{p_\alpha-1}} \int_{t_1-\delta}^{t_2}\int_\Omega |f_\alpha(x,t)|^{p'_\alpha}\,dxdt, \tag{6}$$

where $\overline{b}_\alpha(t_1-\delta,t_2) = \operatorname{ess\,inf}_{[t_1-\delta,t_2]} b_\alpha(t)$, C_1, C_2 are positive constants depending on n, m, Ω, γ, p_α ($|\alpha| \leqslant m$).

Corollary 1. *Let the condition of Theorem 2 holds and* $\operatorname{ess\,inf}_{(-\infty,T)} b_\alpha(t) > 0$, $f_\alpha(x,t) \in L^{p_\alpha{}'}(Q)$, $|\alpha| \leqslant m$, $b(x,t) \in L^1(Q)$. *Then the generalized solution* $u(x,t)$ *of Problem (1),(2) belongs to the space* $\overset{o}{W}{}^{m,0}_{\vec{p},\mathrm{loc}}(\overline{Q}) \cap C((-\infty;T];L^2(\Omega))$ *and satisfies the estimate*

$$\sup_{(-\infty,T)} \int_\Omega u^2(x,t)\,dxdt + \iint_Q \sum_{|\alpha|\leqslant m} |D^\alpha u|^{p_\alpha}\,dxdt \leqslant C_3 \iint_Q [\sum_{|\alpha|\leqslant m} |f_\alpha|^{p_\alpha{}'} + b]\,dxdt, \tag{7}$$

where C_3 *is a positive constant depending on* n, m, Ω, γ, p_α, $\operatorname{ess\,inf}_{(-\infty,T)} b_\alpha(t)$ *(*$|\alpha| \leqslant m$*).*

Remark 2. Conditions of Theorem 2 doesn't provide the uniqueness of solution of Problem (1),(2). You can see the example in [7] as the evidence of this words.

Remark 3. Note that as a partial case of equation (1) for which all conditions of Theorems 1 and 2 hold, is the following equation

$$u_t + \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha(\hat{a}_\alpha(x,t)|D^\alpha u|^{p_\alpha-2}D^\alpha u) = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha f_\alpha(x,t),$$

where $\hat{a}_\alpha(x,t) \in L^\infty_{\mathrm{loc}}(\overline{Q})$, $\hat{a}_\alpha(x,t) \geqslant b_\alpha(t) > 0$, where $b_\alpha(t)$ are the same as in Theorem 2; $p_\alpha > 1$ for all α, $|\alpha| \leqslant m$, and there exists a multiindex γ ($|\gamma| \leqslant m$) such that $p_\gamma > 2$ and $\int_{-\infty}^0 b_\gamma(t)\,dt = +\infty$; $f_\alpha \in L^{p'_\alpha}_{\mathrm{loc}}(\overline{Q})$, $|\alpha| \leqslant m$. Inequality (4) for this equation has been obtained with the help of Lemma 1.2 [7].

Now we'll consider the question of well-posedness of Problem (1),(2), i.e. we'll study conditions under which the generalized solution

1) *exists for each functions* $f_\alpha(x,t) \in L^{p'_\alpha}_{\mathrm{loc}}(\overline{Q})$, $|\alpha| \leqslant m$*;*
2) *is unique;*
3) *continuously depends on the right side of equation (1).*

Continuous dependence of the generalized solution of Problem (1),(2) on the right side of (1) is understood as follows: for arbitrary sequences $\{f_{\alpha,k}\}_{k=1}^\infty \subset L^{p'_\alpha}_{\mathrm{loc}}(\overline{Q})$, $|\alpha| \leqslant m$, such that $f_{\alpha,k} \to f_\alpha$ in $L^{p'_\alpha}_{\mathrm{loc}}(\overline{Q})$, $|\alpha| \leqslant m$, the corresponding sequence of generalized solutions u_k of problems which are different from Problem (1),(2) only in the following way: the right sides of (1) contain $f_{\alpha,k}$ instead of f_α, $|\alpha| \leqslant m$, turns to the generalized solution u of Problem (1),(2) in $\overset{o}{W}{}^{m,0}_{\vec{p},\mathrm{loc}}(\overline{Q})$.

Recall that $g_k \to g$ in $L^{p_\alpha}_{\mathrm{loc}}(\overline{Q})(\overset{o}{W}{}^{m,0}_{\vec{p},\mathrm{loc}}(\overline{Q}))$, if for any numbers $t_1, t_2 (-\infty < t_1 < t_2 \leqslant T)$ $g_k \to g$ in $L^{p_\alpha}(Q_{t_1,t_2})(\overset{o}{W}{}^{m}_{\vec{p}}(Q_{t_1,t_2}))$.

Theorem 3. *Let us assume that a multiindex* γ *(*$|\gamma| \leqslant m$ *) exists such that* $p_\gamma > 2$ *and for arbitrary* $\xi, \eta \in \mathbb{R}^N$ *and for almost all* $(x,t) \in Q$ *the following inequality holds*

$$\sum_{|\alpha|\leqslant m} (a_\alpha(x,t,\xi) - a_\alpha(x,t,\eta))(\xi_\alpha - \eta_\alpha) \geqslant K_0 \sum_{|\alpha|\leqslant m} |\xi_\alpha - \eta_\alpha|^{p_\alpha}, \tag{8}$$

where $K_0 = \mathrm{const} > 0$.

Then Problem (1),(2) is well-posed and its generalized solution $u(x,t)$ belongs to the space $C((-\infty;T];L^2(\Omega))$ and for arbitrary numbers t_1,t_2,δ, such that $\infty < t_1 < t_2 \leqslant T$, $\delta > 0$, and satisfies the estimate

$$\sup_{[t_1,t_2]}\int_\Omega |u(x,t)|^2\,dx + \int_{t_1}^{t_2}\int_\Omega \sum_{|\alpha|\leqslant m}|D^\alpha u|^{p_\alpha}\,dxdt \leqslant$$

$$\leqslant C_4\delta^{-\frac{2}{p_\gamma-2}} + C_5\int_{t_1-\delta}^{t_2}\int_\Omega\sum_{|\alpha|\leqslant m}|f_\alpha(x,t)-a_\alpha(x,t,0)|^{p'_\alpha}\,dxdt, \qquad (9)$$

where C_4,C_5 are positive constants depending only on $n,m,\Omega,K_0,\gamma,p_\alpha$ $(|\alpha|\leqslant m)$.

Corollary 2. *Let us assume $f_\alpha(x,t)-a_\alpha(x,t,0)$ belongs to the space $L^{p_\alpha{}'}(Q)$ for every $\alpha, |\alpha|\leqslant m$, and the conditions of Theorem 3 hold. Then the generalized solution of Problem (1),(2), described in Theorem 3, belongs to the space $\overset{o}{W}{}^{m,0}_{\vec{p}}(Q)\cap C((-\infty,T];L^2(\Omega))$, moreover*

$$\sup_{(-\infty,T]}\int\limits_\Omega |u(x,t)|^2\,dx+\iint\limits_Q\sum_{|\alpha|\leqslant m}|D^\alpha u|^{p_\alpha}\,dx \leqslant \mathrm{C}_5\iint\limits_Q\sum_{|\alpha|\leqslant m}|f_\alpha-a_\alpha(x,t,0)|^{p'_\alpha}\,dxdt,$$

where C_5 is constant from (9).

Corollary 3. *Let us assume that $T=+\infty$ and the conditions of Problem 3 hold. Suppose that a number $\sigma>0$ exists such that for every multiindex α, $|\alpha|\leqslant m$,*
1) $a_\alpha(x,t+\sigma,\xi)=a_\alpha(x,t,\xi)$ for arbitrary $\xi\in\mathbb{R}^N$ and for nearly all $(x,t)\in Q$;
2) $f_\alpha(x,t+\sigma)=f_\alpha(x,t)$ for nearly all $(x,t)\in Q$.

Then the generalized solution of Problem (1),(2), described in Theorem 3, is periodic on t with the period σ (that is $u(x,t+\sigma)=u(x,t)$ for almost all $(x,t)\in Q$) and satisfies the estimate

$$\sup_{[0,\sigma]}\int\limits_\Omega |u(x,t)|^2\,dx + \int\limits_0^\sigma\int\limits_\Omega\sum_{|\alpha|\leqslant m}|D\alpha u|^{p_\alpha}\,dxdt \leqslant$$

$$\leq C_6 + C_7\int\limits_0^\sigma\int\limits_\Omega\sum_{|\alpha|\leqslant m}|f_\alpha(x,t)-a_\alpha(x,t,0)|^{p'_\alpha}\,dxdt, \qquad (10)$$

where C_6,C_7 are positive constants depending only on $n,m,\Omega,K_0,\gamma,p_\alpha$ $(|\alpha|\leqslant m)$.

Remark 4. The conditions of Theorem 3 include the conditions of Theorems 1 and 2. Let us note that as an example of equation satisfying the conditions of Theorem 3 may serve the equation from Remark 3 with additional requirement $p_\alpha\geqslant 2$, $\beta_\alpha(t)=\beta_0=\mathrm{const}>0$, $|\alpha|\leqslant m$.

3. Auxiliary statements. Let us prove some statements indispensable in the sequel.

Lemma 1. *Let us assume that $v\in\overset{o}{W}{}^{m}_{\vec{p}}(\Omega)$. Then $v\in L^{p_\beta}(\Omega)$ for any β, $|\beta|\leqslant m$, and*

$$\int_\Omega |v|^{p_\beta}\,dx\leqslant C_8\int_\Omega |D^\beta v|^{p_\beta}\,dx, \qquad (11)$$

where $C_8 > 0$ is a constant depending only on Ω, β, p_β. Moreover, if $p_\gamma > 2$ for some γ, $|\gamma| \leqslant m$, then

$$\Big(\int_\Omega |v|^2\,dx\Big)^{1/2} \leqslant C_9\Big(\int_\Omega |D^\gamma v|^{p_\gamma}\,dx\Big)^{1/p_\gamma}, \tag{12}$$

where $C_9 > 0$ is a constant depending only on Ω, γ, p_γ.

Proof. Because of the density of the set $C_0^\infty(\Omega)$ in $\overset{o}{W}{}^m_{\vec{p}}(\Omega)$, it suffices to prove the inequalities (11) and (12) for an arbitrary function $v \in C_0^\infty(\Omega)$.

Since Ω is a bounded domain in $\mathbb{R}^n_x$, there exists an n-dimensional parallelepiped $K = \{x : -\infty < a_j \leqslant x_j \leqslant b_j < \infty,\ j = \overline{1,n}\}$ such that $\Omega \subset K$. Let us extend v by zero outside Ω. Then $v \in C_0^\infty(K)$. Let us assume $|\beta| > 0, \beta_l \neq 0$. Set $w(x) = \frac{\partial^{|\beta|-1} v(x)}{\partial x_1^{\beta_1} \ldots \partial x_l^{\beta_l - 1} \ldots \partial x_n^{\beta_n}}$. Then $w(x) = \int_{a_l}^{x_l} \frac{\partial w(x)}{\partial x_l}\,dx_l$. Therefore,

$$|w(x)| \leqslant \int_{a_l}^{b_l} \Big|\frac{\partial w(x)}{\partial x_l}\Big|\,dx_l. \tag{13}$$

From (13), using the Hölder inequality, we obtain

$$|w|^{p_\beta} \leqslant \Big(\int_{a_l}^{b_l} |\frac{\partial w(x)}{\partial x_l}|\,dx_l\Big)^{p_\beta} \leqslant (b_l - a_l)^{p_\beta - 1}\int_{a_l}^{b_l} |\frac{\partial w(x)}{\partial x_l}|^{p_\beta}\,dx_l. \tag{14}$$

After integrating the inequality (14) on the Ω, we get

$$\int_\Omega |w|^{p_\beta}\,dx \leqslant C_{10}\int_\Omega \Big|\frac{\partial w}{\partial x_l}\Big|^{p_\beta}\,dx, \tag{15}$$

where $C_{10} > 0$ is a constant depending only on Ω and p_β. Reminding the definition of w, from (15) we get

$$\int_\Omega |D^{\beta'} v|^{p_\beta}\,dx \leqslant C_{10}\int_\Omega |D^\beta v|^{p_\beta}\,dx, \tag{16}$$

where $\beta' = (\beta_1, ..., \beta_{l-1}, \beta_l - 1, \beta_{l+1}, ..., \beta_n)$.

By integrating the inequality (16), we obtain the inequality (11), and afterwards, using the continuity of the inclusion $L^{p_\gamma}(\Omega) \subset L^2(\Omega)$, we obtain (12). Lemma 1 is proved.

Corollary 4. *Let us assume $v \in \overset{o}{W}{}^{m,0}_{\vec{p},\mathrm{loc}}(\overline{Q})$. Then $v \in L^{p_\beta}_{\mathrm{loc}}(\overline{Q})$ for any β, $|\beta| \leqslant m$. If $p_\gamma > 2$ for some γ, $|\gamma| \leqslant m$, then $v \in L^2_{\mathrm{loc}}(\overline{Q})$.*

Lemma 2. *Let $v \in \overset{o}{W}{}^{m,0}_{\vec{p},\mathrm{loc}}(\overline{Q_{T_0,T}}) \cap L^2_{\mathrm{loc}}(\overline{Q_{T_0,T}})$ and $g_\alpha \in L^{p'_\alpha}_{\mathrm{loc}}(\overline{Q_{T_0,T}})$, $|\alpha| \leqslant m$, where $-\infty \leqslant T_0 < T \leqslant +\infty$, satisfy the integral identity*

$$\iint_Q \{-v\psi_t + \sum_{|\alpha|\leqslant m} g_\alpha D^\alpha \psi\}\,dxdt = 0 \tag{17}$$

for an arbitrary function $\psi \in C_0^\infty(Q_{T_0,T})$.

Then v belongs to the space $C((-\infty,T];L^2(\Omega))$ and for arbitrary partly-smooth on $(-\infty,T]$ function θ and for any t_1,t_2 $(t_1<t_2\leqslant T)$ the following equality holds

$$\int\limits_{\Omega} v^2\theta|_{t=t_2}\,dx-\int\limits_{\Omega} v^2\theta|_{t=t_1}\,dx-\int\limits_{t_1}^{t_2}\int\limits_{\Omega} v^2\theta'\,dxdt+2\int\limits_{t_1}^{t_2}\int\limits_{\Omega}\theta\sum_{|\alpha|\leqslant m} g_\alpha D^\alpha v\,dxdt=0. \tag{18}$$

Let us note that $\overset{o}{W}{}^{m,0}_{\vec{p},\mathrm{loc}}(\overline{Q}_{T_0,T})=\overset{o}{W}{}^{m,0}_{\vec{p}}(Q_{T_0,T})$, $L^q_{\mathrm{loc}}(\overline{Q}_{T_0,T})=L^q(Q_{T_0,T})$, $q\geqslant 1$, if $T_0>-\infty$ and $T<+\infty$.

Proof. Firstly we consider the case: $T_0=-\infty$, $T=+\infty$.

Let us assume that $\rho_1(t)=C_{11}\exp\{-\frac{t^2}{t^2-1}\}$, if $t\in(-1,1)$, and $\rho_1(t)=0$ if $t\notin(-1,1)$, where $C_{11}>0$ is such that $\int_{-\infty}^{+\infty}\rho_1(t)\,dt=1$. For an arbitrary $k\in\mathbb{N}$ let $\rho_k(t)=k\rho_1(kt)$.

Substituting into the integral identity (16)$\psi(x,t)=\tilde\psi(x)\rho_k(\tau-t)$, where $\tilde\psi(x)$ is an arbitrary function belonging to the space $C_0^\infty(\Omega)$, and $\tau\in\mathbb{R}^1,k\in\mathbb{N}$ are arbitrary numbers, we obtain

$$\int\limits_{\Omega} v_k'(x,\tau)\cdot\tilde\psi(x)\,dx+\sum_{|\alpha|\leqslant m}\int\limits_{\Omega} g_{\alpha,k}(x,\tau)D^\alpha\tilde\psi(x)\,dx=0 \tag{19}$$

for any $\tilde\psi\in C_0^\infty(\Omega)$, where

$$v_k(x,\tau)=\int_{-\infty}^{+\infty} v(x,t)\rho_k(\tau-t)\,dt,\quad g_{\alpha,k}(x,\tau)=\int_{-\infty}^{+\infty} g_\alpha(x,t)\rho_k(\tau-t)\,dt$$

are averagings of functions v and g_α, $|\alpha|\leqslant m$ respectively. It is easy to check that $v_k(\cdot,\tau)\in C^\infty(\mathbb{R}^1;\overset{o}{W}{}^m_{\vec{p}}(\Omega)\bigcap L^2(\Omega))$.

Let k,l be arbitrary natural numbers. From (19) we have

$$\int\limits_{\Omega} w_{kl}'(x,\tau)\tilde\psi(x)\,dx+\sum_{|\alpha|\leqslant m}\int\limits_{\Omega} g_{\alpha,kl}(x,\tau)D^\alpha\tilde\psi(x)\,dx=0, \tag{20}$$

where $w_{kl}=v_k-v_l, g_{\alpha,kl}=g_{\alpha,k}-g_{\alpha,l}$.

Let us note that in (20), taking into account the density of $C_0^\infty(\Omega)$ in $\overset{o}{W}{}^m_{\vec{p}}(\Omega)$, one can take $\tilde\psi$ from the space $\overset{o}{W}{}^m_{\vec{p}}(\Omega)$. Let us substitute in (20) $\tilde\psi(x)=w_{kl}(x,\tau)$ for fixed τ:

$$\int\limits_{\Omega} w_{kl}'(x,\tau)w_{kl}(x,\tau)\,dx+\sum_{|\alpha|\leqslant m}\int\limits_{\Omega} g_{\alpha,kl}(x,\tau)D^\alpha w_{kl}(x,\tau)\,dx=0. \tag{21}$$

Let us multiply (21) by an arbitrary partly-smooth function $\theta(\tau)$ and integrate the obtained equality on τ from t_1 to t_2

$$\iint_{Q_{t_1,t_2}} w_{kl}'(x,\tau)w_{kl}(x,\tau)\theta\,dxd\tau+\sum_{|\alpha|\leqslant m}\iint_{Q_{t_1,t_2}}\theta g_{\alpha,kl}(x,\tau)D^\alpha w_{kl}(x,\tau)\,dxd\tau=0. \tag{22}$$

We change the first item of the left side (22)

$$\iint_{Q_{t_1,t_2}} w'_{kl}(x,\tau)w_{kl}(x,\tau)\theta\,dxd\tau = \frac{1}{2}\int_\Omega\int_{t_1}^{t_2}(w_{kl}^2)'\theta\,d\tau dx =$$
$$= \frac{1}{2}\int_\Omega w_{kl}^2(x,t_2)\theta(t_2)\,dx - \frac{1}{2}\int_\Omega w_{kl}^2(x,t_1)\theta(t_1)\,dx - \frac{1}{2}\iint_{Q_{t_1,t_2}} w_{kl}^2\theta'\,dxd\tau. \quad (23)$$

Then from (22) and (23) we get

$$\int_\Omega w_{kl}^2(x,t_2)\theta(t_2)\,dx - \int_\Omega w_{kl}^2(x,t_1)\theta(t_1)\,dx - \iint_{Q_{t_1,t_2}} w_{kl}^2\theta'\,dxd\tau +$$
$$+2\sum_{|\alpha|\leqslant m}\iint_{Q_{t_1,t_2}}\theta g_{\alpha,kl}(x,\tau)D^\alpha w_{kl}(x,\tau)\,dxd\tau = 0. \quad (24)$$

Let τ_0,τ_1,τ_2 be such that $-\infty<\tau_0<\tau_1<\tau_2<+\infty$.

Let's choose a function $\theta(\tau)\in C^1(\mathbb{R}^1)$ so that $0\leqslant\theta(\tau)\leqslant 1$, $\theta(\tau)=1$ on $[\tau_1,\tau_2]$, $\theta(\tau)=0$ на $(-\infty,\tau_0]$. Then from (24), taking $t_1=\tau_0, t_2\in[\tau_1,\tau_2]$ and using the Hölder inequality, we get

$$\max_{[\tau_1,\tau_2]}\int_\Omega w_{kl}^2(x,t)\,dx \le$$
$$\leqslant C_{12}\int_{\tau_0}^{\tau_2} w_{kl}^2\,dxd\tau + 2\sum_{|\alpha|\leqslant m}\Big(\int_{\tau_0}^{\tau_2}|g_{\alpha,kl}|^{p_\alpha{}'}dxdt\Big)^{\frac{1}{p_\alpha{}'}}\Big(\int_{\tau_0}^{\tau_2}|D^\alpha w_{kl}|^{p_\alpha}dxdt\Big)^{\frac{1}{p_\alpha}}, \quad (25)$$

where $C_{12}=\max_{[\tau_1,\tau_2]}|\theta'(\tau)|$.

Taking into account $w_{kl}\to 0$ in $L^2(Q_{\tau_0,\tau_2})$, $g_{\alpha,kl}\to 0$ in $L^{p'_\alpha}(Q_{\tau_0,\tau_2})$, $D^\alpha w_{kl}\to 0$ in $L^{p_\alpha}(Q_{\tau_0,\tau_2})$ at $k,l\to+\infty$, from (25) we obtain that the sequence $\{v_k(x,t)\}$ is Cauchy in the space $C([\tau_1,\tau_2];L^2(\Omega))$.

Because $v_k\to v$ as $k\to+\infty$ in the norm $L^2([\tau_1,\tau_2];L^2(\Omega))$, v_k tends to v in the space $C([\tau_1,\tau_2];L^2(\Omega))$ and, therefore, the function $v(x,t)$ belongs to the space $C([\tau_1,\tau_2];L^2(\Omega)$.

Starting from (19) and arguing as in the case of the equality (24), we obtain the equality

$$\int\limits_\Omega v_k^2(x,t_2)\theta(t_2)\,dx - \int\limits_\Omega v_k^2(x,t_1)\theta(t_1)\,dx - \iint\limits_{Q_{t_1,t_2}} v_k^2\theta'\,dxd\tau + 2\sum_{|\alpha|\leqslant m}\iint\limits_{Q_{t_1,t_2}}\theta g_{\alpha,k}D^\alpha v_k\,dxd\tau = 0,$$

where $\theta(t)$ is an arbitrary partially-smooth function.

Passing to limit in this equality, at $k\to+\infty$, we obtain (18).

2. Let us assume $T_0=-\infty, T<+\infty$. Making in (16) the substitution $t=s-\frac{1}{k}$, where $k\in\mathbb{N}$, we obtain the following integral identity

$$\int_{-\infty}^{T+\frac{1}{k}}\int_\Omega\{-v(x,s-\frac{1}{k})\psi'(x,s)+\sum_{|\alpha|\leqslant m}g_\alpha(x,s-\frac{1}{k})D^\alpha\psi(x,s)\}\,dxds = 0 \quad (26)$$

for any $\psi\in C_0^\infty(\Omega\times(-\infty,T+\frac{1}{k}))$

Let us sign $v^{\langle k\rangle}(x,s) = v(x, s-\frac{1}{k})$, $g_\alpha^{\langle k\rangle} = g_\alpha(x, s-\frac{1}{k})$, $|\alpha| \leqslant m$, and consider the function $\chi^{\langle k\rangle}(s) \in C^\infty(\mathbb{R}^1)$ such that $0 \leqslant \chi^{\langle k\rangle}(s) \leqslant 1$, $\chi^{\langle k\rangle}(s) = 1$ if $s \leqslant T + \frac{1}{2k}$, $\chi^{\langle k\rangle}(s) = 0$ if $s \geqslant T + \frac{1}{k}$.

Let us substitute $\tilde\psi\chi^{\langle k\rangle}$ instead of ψ into the integral identity (26), where $\tilde\psi \in C_0^\infty(\Omega\times\mathbb{R}^1)$. As a result of a simple conversion we'll obtain the equality

$$\int_{-\infty}^{+\infty}\int_\Omega \{-(v^{\langle k\rangle}\chi^{\langle k\rangle})\tilde\psi' - [v^{\langle k\rangle}(\chi^{\langle k\rangle})']\tilde\psi + \sum_{|\alpha|\leqslant m}(g_\alpha^{\langle k\rangle}\chi^{\langle k\rangle})D^\alpha\tilde\psi\}\,dxds = 0 \qquad (27)$$

for an arbitrary $\tilde\psi \in C_0^\infty(\Omega\times\mathbb{R}^1)$.

Let k and l be arbitrary natural numbers. Substituting $w^{\langle k,l\rangle} = v^{\langle k\rangle}\chi^{\langle k\rangle} - v^{\langle l\rangle}\chi^{\langle l\rangle}$, $g_\alpha^{\langle k,l\rangle} = g_\alpha^{\langle k\rangle}\chi^{\langle k\rangle} - g_\alpha^{\langle l\rangle}\chi^{\langle l\rangle}$, $0 < |\alpha| \leqslant m$, $g_0^{\langle k,l\rangle} = -[v^{\langle k\rangle}(\chi^{\langle k\rangle})' - v^{\langle l\rangle}(\chi^{\langle l\rangle})'] + g_0^{\langle k\rangle}\chi^{\langle k\rangle} - g_0^{\langle l\rangle}\chi^{\langle l\rangle}$, from (27) we'll have

$$\int_{-\infty}^{+\infty}\int_\Omega\{-w^{\langle k,l\rangle}\tilde\psi' + \sum_{|\alpha|\leqslant m} g_\alpha^{\langle k,l\rangle}D^\alpha\tilde\psi\}\,dxds = 0$$

for any $\tilde\psi \in C_0^\infty(\Omega\times\mathbb{R}^1)$.

It is evident that functions $v = w^{\langle k,l\rangle}$, $g_\alpha = g^{\langle k,l\rangle}$, $|\alpha| \leqslant m$, satisfy the conditions of lemma in the case $Q = \Omega\times(-\infty,+\infty)$. Therefore we have $w^{\langle k,l\rangle} \in C(\mathbb{R}^1; L^2(\Omega))$ and

$$\int_\Omega [w^{\langle k,l\rangle}(x,t_2)]^2\theta(t_2)\,dx - \int_\Omega [w^{\langle k,l\rangle}(x,t_1)]^2\theta(t_1)\,dx -$$
$$- \int_{t_1}^{t_2}\int_\Omega [w^{\langle k,l\rangle}]^2\theta'\,dxdt + 2\sum_{|\alpha|\leqslant m}\iint_{Q_{t_1,t_2}} \theta g_\alpha^{\langle k,l\rangle}D^\alpha v^{\langle k,l\rangle}\,dxds = 0 \qquad (28)$$

for an arbitrary function $\theta(t) \in C^1(\mathbb{R}^1)$ and any $t_1, t_2, -\infty < t_1 < t_2 < +\infty$.

Let τ_0, τ_1 be arbitrary numbers with $-\infty < \tau_0 < \tau_1 < T$. Let us choose a function θ such that $\theta(\tau_0) = 0$ and $\theta(t) = 1$ for $t \in [\tau_1, T]$. Then (28) and $t_1 = \tau_0$, $t_2 \in [\tau_1, T]$ imply

$$\max_{[\tau_1,T]}\int_\Omega |v^{\langle k\rangle}(x,t) - v^{\langle l\rangle}(x,t)|^2\,dx \leqslant$$
$$\leqslant C_{13}\iint_{Q_{\tau_0,T}} |v^{\langle k\rangle} - v^{\langle l\rangle}|^2\,dxdt + 2\iint_{Q_{\tau_0,T}} (g_\alpha^{\langle k\rangle} - g_\alpha^{\langle l\rangle})D^\alpha(v^{\langle k\rangle} - v^{\langle l\rangle})\,dxds,$$

where $C_{13} = \text{const} > 0$ does not depend on k and l.

From the continuity of the Lebesgue integral it follows that the left side of the inequality tends to zero whenever $k, l \to +\infty$. This implies that the sequence $\{v^{\langle k\rangle}\}$ is Cauchy in the space $C([\tau_1, T]; L^2(\Omega))$. Since $v^{\langle k\rangle}(x,t) \to v(x,t)$ if $k \to \infty$ for almost all $(x,t) \in Q$, $v(x,t) \in C((-\infty, T]; L^2(\Omega))$.

3. Now we consider the case $T_0 > -\infty$, $T < +\infty$. Let the function $\chi(t) \in C^\infty(\mathbb{R}^1)$ be such that $\chi(t) = 1$ in a neighbourhood of T and $\chi(t) = 0$ in a neighbourhood of T_0. We'll extend v by zero outside $Q_{T_0,T}$ and put $v_1 = \chi v$, $v_2 = (1-\chi)v$. Then

$v = v_1 + v_2$. From integral identity (17), substituting $\chi\psi$ instead of ψ, after simple transformation we obtain the identity

$$\int_{-\infty}^{T} \int_{\Omega} \{-v_1\psi_t + (\chi g_0 - v\chi')\psi + \sum_{0<|\alpha|\leqslant m} \chi g_\alpha D^\alpha \psi\} \, dxdt = 0$$

for an arbitrary $\psi \in C_0^\infty(\Omega \times (-\infty, T))$.

From here and from the item 2 it follows that $v_1 \in C((-\infty, T]; L^2(\Omega))$.

Similarly, substituting in the integral identity (17) $(1-\chi)\psi$ instead of ψ, we conclude $v_2 \in C([T_0, +\infty); L^2(\Omega))$. In this way, $v = v_1 + v_2 \in C([T_0, T]; L^2(\Omega))$. Lemma 2 is proved.

Corollary 5. *If the conditions of lemma 2 hold, then the function $y(t) = \int_\Omega v^2(x,t)\,dx$ is absolutely continuous (after changing on a set of zero measure, if necessary) and*

$$\frac{dy(t)}{dt} + 2\int_\Omega \sum_{|\alpha|\leqslant m} g_\alpha D^\alpha v \, dx = 0 \tag{29}$$

for almost all $t \in (-\infty, T)$.

Proof. Assume that $\theta(t) = 1$, $t_1 = t$, $t_2 = t + \Delta t$ in (18). We obtain

$$\int_\Omega v^2(x, t+\Delta t)\,dx - \int_\Omega v^2(x,t)\,dx + 2\int_t^{t+\Delta t} \int_\Omega \{\sum_{|\alpha|\leqslant m} g_\alpha D^\alpha v\} \, dxdt = 0.$$

Divide the obtained equality by Δt and take the limit at $\Delta t \to 0$. Using Theorem of Lebesgue on the differentiation of integral we obtain (29) for almost all $t \in (-\infty, T]$. Corollary 5 is proved.

4. Proof of basic results.

Proof of Theorem 1. Let u_1, u_2 be two generalized solutions of problem (1),(2). Then, by subtracting from integral identity (3) for u_1 the same identity for u_2, we obtain

$$\int_Q \{-w\psi_t \, dxdt + \sum_{|\alpha|\leqslant m} [a_\alpha(x,t,\delta u_1) - a_\alpha(x,t,\delta u_2)] D^\alpha \psi\} \, dxdt = 0, \tag{30}$$

where $w = u_1 - u_2$, and ψ is an arbitrary function from $C_0^\infty(Q)$.

From the integral identity (30) and from Corollaries 4 and 5 we obtain

$$\frac{dy(t)}{dt} + 2\int_\Omega \sum_{|\alpha|\leqslant m} [a_\alpha(x,t,\delta u_1) - a_\alpha(x,t,\delta u_2)] D^\alpha w \, dx = 0,$$

where $y(t) = \int_\Omega w^2(x,t)\,dx$.

This and inequality (4) imply the inequality

$$\frac{dy(t)}{dt} + 2\lambda(t) \int_\Omega |D^\gamma w|^{p_\gamma} \, dx \leqslant 0. \tag{31}$$

Using lemma 1 we obtain

$$\int_\Omega |D^\gamma w|^{p_\gamma}\,dx \geqslant C_{14}\Big(\int_\Omega w^2\,dx\Big)^{p_\gamma/2}, \tag{32}$$

where $C_{14} = C_7^{-p_\gamma}$.

Therefore, from (31) and (32) we obtain $\frac{dy(t)}{dt} + C_{15}\lambda(t)y^{p_\gamma/2} \leqslant 0$, where $C_{15} = 2C_{14}$. This and Lemma 1.1 of [1] imply that $y(t) = 0$ on $(-\infty, T]$, i.e. $u_1 = u_2$ almost everywhere on Q. Theorem 1 is proved.

Proof of Theorem 2. First we obtain the apriory estimate (6) of a generalized solution of Problem (1),(2). For this the Young inequality is needed: $ab \leqslant \varepsilon a^p + M(\varepsilon,p)b^{p'}$, where $a \geqslant 0, b \geqslant 0, \varepsilon > 0, p > 1, \frac{1}{p} + \frac{1}{p'} = 1, M(\varepsilon,p) = \varepsilon^{-1/(p-1)}p^{-p'}(p-1)$.

Take a function $\theta_1(t)$ from the space $C^\infty(\mathbb{R}^1)$ with the following properties: $0 \leqslant \theta_1(t) \leqslant 1$, $\theta_1'(t) \geqslant 0$ on $\mathbb{R}^1$, $\theta_1(t) = 0$ if $t \in (-\infty,-1]$, $\theta_1(t) = \exp\{-1/(t+1)\}$ if $t \in (-1,-1/2]$, $\theta_1(t) \geqslant \exp\{-2\}$ if $t \in (-1/2, 0)$, $\theta_1(t) = 1$ if $t \in [0,+\infty)$.

It is clear that

$$\sup \theta_1'(t)\theta_1^{-\varkappa}(t) \leqslant C_{16}, \tag{33}$$

where $0 < \varkappa < 1$, $C_{16} > 0$ is a constant depending only on $\varkappa$.

Let $u(x,t)$ be a generalized solution of Problem (1),(2) and t_1, t_2, δ, be arbitrary numbers such that $\infty < t_1 < t_2 \leqslant T$, $\delta > 0$. From integral identity (3), Corollary 4 and Lemma 2, substituting $\theta(t) = \theta_1(\frac{t-t_1}{\delta})$ instead of t_1 $t_1 = t_1 - \delta$, $t_2 = s$, where s is arbitrary number from the interval $[t_1, t_2]$, we obtain

$$\int_\Omega u^2(x,s)\,dx - \int_{t_1-\delta}^{s}\int_\Omega u^2\theta'\,dxdt + \\ + 2\int_{t_1-\delta}^{s}\int_\Omega \theta \sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta u)D^\alpha u\,dxdt = 2\int_{t_1-\delta}^{s}\int_\Omega \theta \sum_{|\alpha|\leqslant m} f_\alpha D^\alpha u\,dxdt. \tag{34}$$

Here $\theta = \theta_1(\frac{t-t_1}{\delta})$.

Let us estimate the second term of the left side of the equality (34), taking into account (12),(33) and using the Young inequality, in the following way:

$$\int_{t_1-\delta}^{s}\int_\Omega u^2\theta'\,dxdt = \int_{t_1-\delta}^{t_1} \|u\|^2_{L^2(\Omega)}\theta'\,dt \leqslant C_7^2 \int_{t_1-\delta}^{t_1} \|D^\gamma u\|^2_{L^{p_\gamma}(\Omega)}\theta'\,dt = \\ = C_7^2 \int_{t_1-\delta}^{t_1} \|D^\gamma u\|^2_{L^{p_\gamma}(\Omega)}\frac{\theta^{2/p_\gamma}}{\theta^{2/p_\gamma}}\theta'\,dt \leqslant \varepsilon_\gamma \int_{t_1-\delta}^{t_1} \|D^\gamma u\|^{p_\gamma}_{L^{p_\gamma}(\Omega)}\theta\,dt + \\ + C_{17}\varepsilon_\gamma^{-\frac{2}{p_\gamma-2}} \int_{t_1-\delta}^{t_1} (\theta'\theta^{-2/p_\gamma})^{\frac{p_\gamma}{p_\gamma-2}}\,dt \leqslant \varepsilon_\gamma \int_{t_1-\delta}^{s}\int_\Omega |D^\gamma u|^{p_\gamma}\theta\,dxdt + C_{18}\,[\delta\,\varepsilon_\gamma]^{-\frac{2}{p_\gamma-2}}, \tag{35}$$

where $\varepsilon_\gamma > 0$ is an arbitrary number, $C_{17}, C_{18} > 0$ are constants depending only on Ω, γ, p_γ.

Now we estimate the right side of (34) using the Young inequality

$$2\int_{t_1-\delta}^{s}\int_{\Omega}\theta\sum_{|\alpha|\leqslant m} f_\alpha D^\alpha u\,dxdt \le$$
$$\leqslant \sum_{|\alpha|\leqslant m}\varepsilon_\alpha\int_{t_1-\delta}^{s}\int_{\Omega}|D^\alpha u|^{p_\alpha}\theta\,dxdt+\sum_{|\alpha|\leqslant m}C_\alpha\varepsilon_\alpha^{-1/(p_\alpha-1)}\int_{t_1-\delta}^{s}\int_{\Omega}|f_\alpha|^{p'_\alpha}\theta\,dxdt, \tag{36}$$

where $\varepsilon_\alpha > 0$ are arbitrary numbers, $C_\alpha > 0$ are constants depending only on $p_\alpha, |\alpha| \leqslant m$. From (5), (32)–(36), substituting $\varepsilon_\alpha = \overline{b_\alpha}(t_1-\delta, s)$, where $\overline{b_\alpha}(t_1-\delta, s) = \operatorname{ess\,inf}_{[t_1-\delta,s]} b_\alpha(t)$, at $\alpha \neq \gamma$, and $\varepsilon_\gamma = \frac{1}{2}\overline{b}_\gamma(t_1-\delta, s)$, we obtain

$$\int\limits_{\Omega} u^2(x,s)\,dx + 2\int\limits_{t_1-\delta}^{s}\int\limits_{\Omega}[\sum_{|\alpha|\leqslant m} b_\alpha(t)|D^\alpha u|^{p_\alpha} - b(x,t)]\theta\,dxdt \leqslant C_{19}[\delta\overline{b}_\gamma(t_1-\delta_1,s)]^{-\frac{2}{p_\gamma-2}}+$$
$$\sum_{|\alpha|\leqslant m}\overline{b}_\alpha(t_1-\delta,s)\int\limits_{t_1-\delta}^{s}\int\limits_{\Omega}|D^\alpha u|^{p_\alpha}\theta\,dxdt + C_{20}\sum_{|\alpha|\leqslant m}[\overline{b}_\alpha(t_1-\delta_1,s)]^{-\frac{1}{p_\alpha-1}}\int\limits_{t_1-\delta}^{s}\int\limits_{\Omega}|f_\alpha|^{p'_\alpha}\theta\,dxdt,$$

where $C_{20} = \max\limits_{\alpha} C_\alpha$. From this we obtain

$$\int\limits_{\Omega} u^2(x,s)\,dx + \sum_{|\alpha|\leqslant m}\overline{b}_\alpha(t_1-\delta_1,s)\int\limits_{t_1-\delta}^{s}\int\limits_{\Omega}|D^\alpha u|^{p_\alpha}\theta\,dxdt \leqslant C_{19}[\delta\overline{b}_\gamma(t_1-\delta_1,s)]^{-\frac{2}{p_\gamma-2}}+$$
$$2\int\limits_{t_1-\delta}^{s}\int\limits_{\Omega} b(x,t)\theta\,dxdt + C_{20}\sum_{|\alpha|\leqslant m}[\overline{b}_\alpha(t_1-\delta_1,s)]^{-\frac{1}{p_\alpha-1}}\int\limits_{t_1-\delta}^{s}\int\limits_{\Omega}|f_\alpha|^{p_\alpha'}\theta\,dxdt. \tag{37}$$

Taking into account that $s \in [t_1, t_2]$, from (37) we obtain (6) with $C_1 = 2C_{19}$, $C_2 = 2C_{20}$.

Now we construct a sequence of functions approximating the generalized solution of problem (1),(2) in some sense. Let $Q_\mu = \Omega \times (T-\mu, T)$, $\Sigma_\mu = \partial\Omega \times (T-\mu, T)$, where $\mu \in \mathbb{N}$. Let us consider the family of mixed problems ($\mu \in \mathbb{N}$):

$$\hat{u}_{\mu t} + \sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^\alpha a_\alpha(x,t,\delta\hat{u}_\mu) = \sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^\alpha f_\alpha(x,t) \text{ in } Q_\mu, \tag{1_μ}$$
$$\frac{\partial^j\hat{u}_\mu}{\partial\nu^j} = 0 \quad \text{on} \quad \Sigma_\mu, \quad j = 0,1,...,m-1, \tag{2_μ}$$
$$\hat{u}_\mu(x, T-\mu) = 0. \tag{3_μ}$$

A function $\hat{u}_\mu(x,t) \in \overset{o}{W}{}^{m,0}_{\vec{p}}(Q_\mu)$ *is called a generalized solution of the problem* (1_μ)–(3_μ) *if it satisfies the integral identity*

$$\int_{Q_\mu}\{-\hat{u}_\mu\psi_t + \sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta\hat{u}_\mu)D^\alpha\psi\}\,dxdt = \int_{Q_\mu}\sum_{|\alpha|\leqslant m} f_\alpha D^\alpha\psi\,dxdt \tag{4_μ}$$

for arbitrary $\psi \in C^\infty(\overline{Q_\mu})$ *such that* $\psi = 0$ *in the neighbourhood of plane* $\{t = T\}$ *and lateral surface* $\partial\Omega \times (T - \mu, T)$.

The existence and the uniqueness of a generalized solution $\hat{u}_\mu$ of the problem $(1_\mu) - (3_\mu)$ follows from the results of [12]. Let us extend the function $\hat{u}_\mu$ by zero on $\overline{Q} \setminus \overline{Q}_\mu$ and denote this extension by u_μ. Put $f_{\alpha,\mu}(x,t) = f_\alpha(x,t)$ for $(x,t) \in Q_\mu$ and $f_{\alpha,\mu}(x,t) = a_\alpha(x,t,0)$ for $(x,t) \in Q \setminus Q_\mu$, $|\alpha| \leqslant m$. It is evident that u_μ are generalized solutions of Fourier problem

$$u_\mu + \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha a_\alpha(x,t,\delta u_\mu) = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha f_{\alpha,\mu}(x,t) \text{ in } Q, \qquad (5_\mu)$$

$$\frac{\partial^j u_\mu}{\partial \nu^j} = 0 \quad \text{on} \quad \Sigma, \quad j = 0,1,...,m-1, \qquad (6_\mu)$$

i.e. u_μ belongs to the space $\overset{o}{W}{}^{m,0}_{\vec{p}}(Q) \subset \overset{o}{W}{}^{m,0}_{\vec{p},\mathrm{loc}}(Q)$ and satisfies the identity

$$\iint_Q \{-u_\mu \psi_t + \sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta u_\mu) D^\alpha \psi - \sum_{|\alpha|\leqslant m} f_{\alpha,\mu}(x,t) D^\alpha \psi\}\, dxdt = 0 \qquad (7_\mu)$$

for any $\psi \in C_0^\infty(Q)$. Since problem $(5_\mu), (6_\mu)$ for any $\mu \in \mathbb{N}$ is different from the problem (1), (2) only by $f_{\alpha,\mu}$ instead of f_α, $|\alpha| \leqslant m$, in the right side of (1), the estimates (6) and (7) hold with the substitution $f_{\alpha,\mu}$ by f_α. Besides this, from the condition (A_2) we obtain

$$\int_{t_1}^{t_2} \int_\Omega |a_\alpha(x,t,\delta u_\mu(x,t))|^{p'_\alpha}\, dxdt \leqslant C(t_1,t_2), \quad |\alpha| \leqslant m, \qquad (38)$$

where t_1, t_2 are arbitrary numbers from the interval $(-\infty, T]$, $t_1 < t_2$, $C(t_1,t_2)$ is a constant depending only on t_1 and t_2, but not depending on μ, whenever μ is sufficiently large. These estimates imply the existence of subsequence of the sequence $\{u_\mu\}$ (we denote it also by $\{u_\mu\}$) and functions $u \in \overset{o}{W}{}^{m,0}_{\vec{p},\mathrm{loc}}(\overline{Q})$, $\chi_\alpha \in L^{p'_\alpha}_{\mathrm{loc}}(\overline{Q})$, $|\alpha| \leqslant m$, such that

$$u_\mu(\cdot,t) \to u(\cdot,t) \quad * -\text{weakly in} \quad L^\infty_{\mathrm{loc}}((-\infty;T];L^2(\Omega)), \qquad (39)$$

$$D^\alpha u_\mu \to D^\alpha u \quad \text{weakly in} \quad L^{p_\alpha}_{\mathrm{loc}}(\overline{Q}), \quad |\alpha| \leqslant m, \qquad (40)$$

$$a_\alpha(x,t,\delta u_\mu(x,t)) \to \chi_\alpha(x,t) \quad \text{weakly in} \quad L^{p'_\alpha}_{\mathrm{loc}}(\overline{Q}), \quad |\alpha| \leqslant m. \qquad (41)$$

Take the limit at $\mu \to \infty$ in (7_μ), taking into account (39)–(41). We obtain

$$\iint_Q \{-u\psi_t + \sum_{|\alpha|\leqslant m} \chi_\alpha(x,t) D^\alpha \psi - \sum_{|\alpha|\leqslant m} f_\alpha D^\alpha \psi\}\, dxdt = 0 \qquad (42)$$

for arbitrary $\psi \in C_0^\infty(Q)$.

To finish the proof we have to show that

$$\iint_Q \sum_{|\alpha|\leqslant m} \chi_\alpha(x,t) D^\alpha \psi\, dxdt = \int_Q \sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta u) D^\alpha \psi\, dxdt \qquad (43)$$

for any $\psi \in \overset{o}{C}{}_0^\infty(Q)$. Let us do this with the help of the monotonicity method [12].

Let $v(x,t) \in \overset{o}{W}{}^{m,0}_{\vec{p},\mathrm{loc}}(\overline{Q})$, $\theta(t) \in C_0^\infty(-\infty,T)$ be arbitrary functions. Consider the expression

$$M_\mu = \iint\limits_Q \theta \sum_{|\alpha|\leqslant m} (a_\alpha(x,t,\delta v) - a_\alpha(x,t,\delta u_\mu))(D^\alpha v - D^\alpha u_\mu)\,dxdt.$$

Using the condition (A$_3$) we see that $M_\mu \geqslant 0$. Express M_μ as

$$M_\mu = \iint\limits_Q \theta \sum_{|\alpha|\leqslant m} \{a_\alpha(x,t,\delta v)(D^\alpha v - D^\alpha u_\mu) - a_\alpha(x,t,\delta u_\mu)D^\alpha v + \\ + a_\alpha(x,t,\delta u_\mu)D^\alpha u_\mu\}\,dxdt \geqslant 0. \quad (44)$$

Integral identity (7_μ) and Lemma 2 imply

$$\iint\limits_Q \theta \sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta u_\mu)D^\alpha u_\mu dxdt = \iint\limits_Q \theta \sum_{|\alpha|\leqslant m} f_{\alpha,\mu}D^\alpha u_\mu dxdt + \frac{1}{2}\iint\limits_Q |u_\mu|^2\theta' dxdt. \quad (45)$$

Substituting (45) into (44) we obtain

$$M_\mu = \iint\limits_Q \theta \sum_{|\alpha|\leqslant m} \{a_\alpha(x,t,\delta v)(D^\alpha v - D^\alpha u_\mu) - a_\alpha(x,t,\delta u_\mu)D^\alpha v\}\,dxdt + \\ + \iint\limits_Q \theta \sum_{|\alpha|\leqslant m} f_{\alpha,\mu}D^\alpha u_\mu\,dxdt + \frac{1}{2}\iint\limits_Q |u_\mu|^2\theta'\,dxdt \geqslant 0. \quad (46)$$

Let t_1, t_2 be numbers such that $\operatorname{supp}\theta' \subset [t_1,t_2] \subset (-\infty,T]$. From the definition of u_μ, $\mu \in \mathbb{N}$, it follows that $u_\mu \in L^q(t_1,t_2;\overset{o}{W}{}^m_{\vec{p}}(\Omega))$ for every $\mu \in \mathbb{N}$, where $q = \min_\alpha p_\alpha$, and there exists the derivative u'_μ of u_μ (by t), as an element of the space of generalized functions $D'(t_1,t_2;\overset{o}{W}{}^{-m}_{\vec{p}'}(\Omega))$, and, as it follows from (A_2) and (A_4), u'_μ, $\mu \in \mathbb{N}$, can be considered as elements of the space $L^{q^*}(t_1,t_2;\overset{o}{W}{}^{-m}_{\vec{p}'}(\Omega))$, where $q^* = \min_\alpha p_\alpha{}'$. Using the estimate (46) and condition (A_4) we conclude that the sequence $\{u_\mu{}'\}$ is bounded in the norm of the space $L^{q^*}(t_1,t_2;\overset{o}{W}{}^{-m}_{\vec{p}'}(\Omega))$. From here and from results [12, theorem 5.1] and [11] it follows that the sequence $\{u_\mu\}$ is compact in $L^q(t_1,t_2;L^2(\Omega))$. Passing to a subsequence, if necessary, we may assume that $||u_\mu(\cdot,t)||^2_{L^2(\Omega)} \to ||u(\cdot,t)||^2_{L^2(\Omega)}$ almost everywhere on (t_1,t_2). Taking this into account, the boundedness of $||u_\mu(\cdot,t)||^2_{L^2(\Omega)}$ in $L^\infty(t_1,t_2)$ and a lemma in [12] imply

$$\iint_Q |u_\mu|^2\theta'\,dxdt \to \iint_Q |u|^2\theta'\,dxdt \quad \text{if} \quad \mu \to +\infty. \quad (47)$$

Take the limit at $\mu \to +\infty$ in (45), taking into account (40),(41),(47). We obtain

$$\iint\limits_Q \theta \sum_{|\alpha|\leqslant m} \{a_\alpha(x,t,\delta v)(D^\alpha v - D^\alpha u) - \chi_\alpha D^\alpha v\}\, dxdt + \\ + \iint\limits_Q \theta \sum_{|\alpha|\leqslant m} f_\alpha D^\alpha u\, dxdt + \frac{1}{2}\iint\limits_Q |u|^2\theta'\, dxdt \geqslant 0. \quad (48)$$

From identity (48) and from Lemma 2 we have

$$\frac{1}{2}\iint\limits_Q u^2\theta'\, dxdt + \iint\limits_Q \theta \sum_{|\alpha|\leqslant m} f_\alpha D^\alpha u\, dxdt = \iint\limits_Q \theta \sum_{|\alpha|\leqslant m} \chi_\alpha(x,t) D^\alpha u\, dxdt. \quad (49)$$

Substitute (49) into (48). After simple conversions we obtain

$$\iint\limits_Q \theta \sum_{|\alpha|\leqslant m} \left(a_\alpha(x,t,\delta v) - \chi_\alpha(x,t)\right)\left(D^\alpha v - D^\alpha u\right) dxdt \geq 0. \quad (50)$$

Let us take $v = u + \lambda\psi$, where $\psi \in C_0^\infty(Q)$, and $\lambda > 0$ is arbitrary. Then

$$\lambda \iint\limits_Q \theta \sum_{|\alpha|\leqslant m} (a_\alpha(x,t,\delta u + \lambda\delta\psi) - \chi_\alpha(x,t)) D^\alpha\psi\, dxdt \geqslant 0. \quad (51)$$

Divide this inequality by λ and take the limit at $\lambda \to 0$, taking into account conditions (A_1), (A_2) and the theorem of Lebesgue on taking limits under the sign of integral. Then

$$\iint\limits_Q \theta \sum_{|\alpha|\leqslant m} (a_\alpha(x,t,\delta u) - \chi_\alpha(x,t)) D^\alpha\psi\, dxdt \geqslant 0 \quad (52)$$

for any $\psi \in C_0^\infty(Q)$.

Equality (43) follows from (52). Taking into account (42) and (43), we have (3). Therefore, the existence of generalized solution of Problem (1),(2) is proved. From Lemmas 1 and 2 it follows that $u \in C((-\infty,T]; L^2(\Omega))$. Thus, Theorem 2 is proved.

Proof of Corollary 1. Let us take the limit in (6) at $\delta \to +\infty$. As a result, taking into account arbitrarity t_1 and t_2 and the condition $\operatorname{ess\,inf}_{(-\infty,T]} b_\alpha(t) > 0$, $|\alpha| \leqslant m$, we obtain estimate (7). This implies the required statement.

Lemma 3. *Let $u(x,t)$ be a generalized solution of Problem (1),(2) and $\tilde u(x,t)$ a generalized solution of a problem which differs from Problem (1),(2) only by $\tilde f_\alpha$, $|\alpha| \leq m$ instead of f_α the right part of equation (1). Suppose that the conditions of Theorem 3 hold.*

Then for arbitrary numbers t_1, t_2, δ such that $-\infty < t_1 < t_2 \leqslant T, \delta > 0$, the following inequality holds

$$\max_{[t_1,t_2]} \int_\Omega |u(x,t) - \tilde u(x,t)|^2\, dx + \int_{t_1}^{t_2}\int_\Omega \sum_{|\alpha|\leqslant m} |D^\alpha - D^\alpha \tilde u|^{p_\alpha}\, dxdt \leqslant \\ \leqslant C_{21}\delta^{-\frac{2}{p_\gamma - 2}} + C_{22}\int_{t_1-\delta}^{t_2}\int_\Omega \sum_{|\alpha|\leqslant m} |f_\alpha - \tilde f_\alpha|^{p'_\alpha}\, dxdt, \quad (53)$$

where C_{21}, C_{22} are constants depending only on $n, m, \Omega, \gamma, K_0, p_\alpha(|\alpha| \leqslant m)$.

Proof. Let us subtract from integral identity (3) for u the same integral identity for $\tilde{u}$. As a result, taking $w = u - \tilde{u}$, we obtain

$$\iint\limits_Q \{-w\psi_t + \sum_{|\alpha|\leqslant m} (a_\alpha(x,t,\delta u) - a_\alpha(x,t,\delta\tilde{u}))D^\alpha w\}\, dxdt = \\ = \iint\limits_Q \sum_{|\alpha|\leqslant m} (f_\alpha - \tilde{f}_\alpha) D^\alpha w\, dxdt \quad (54)$$

for arbitrary $\psi \in C_0^\infty(Q)$.

Arguing like in the proof of Theorem 2 during obtaining estimate (6), from (54) we obtain (53). Lemma 3 is proved.

Proof of Theorem 3. Uniqueness. Let u_1, u_2 be two generalized solutions of the problem (1),(2). From Lemma 3, substituting $u = u_1, \tilde{u} = u_2, f_\alpha = \tilde{f}_\alpha, |\alpha| \leqslant m$, we obtain

$$\max_{[t_1,t_2]} \int_\Omega |u_1(x,t) - u_2(x,t)|^2\, dx \leqslant C_{21}\delta^{-\frac{2}{p_\gamma - 2}}, \quad (55)$$

where δ is an arbitrary number and t_1, t_2 are any numbers such that $\infty < t_1 < t_2 \leqslant T$. Let us take the limit in (55) at $\delta \to +\infty$. As a result, taking into account the independence C_{21} from δ and arbitrarity t_1, t_2 and δ, we have $u_1 = u_2$ almost everywhere on Q. The uniqueness of the generalized solution of Problem (1),(2) is proved. Note that this result also follows from Theorem 1.

Existence. It is easy to check that the conditions of Theorem 2 hold in this case. This implies the existence of a generalized solution of this problem. It turns out that conditions of Theorem 3 give the possibility to simplify the proof and to obtain more strong results on the convergence of sequence of approximations of a generalized solution of this problem.

Indeed, from Lemma 3, taking $\tilde{f}_\alpha(x,t) = a_\alpha(x,t,0), |\alpha| \leqslant m, \tilde{u} = 0$, we obtain the apriory estimate (9) of a generalized solution u of the problem (1),(2).

Then we create a sequence $\{u_\mu\}$ of generalized solutions of a Fourier problems (5_μ), (6_μ), $\mu \in \mathbb{N}$, by the same way as we had done it in the correspondent part of the proof of Theorem 2. Let us show, that the sequence of restrictions of u_μ on $Q_\varkappa$, where $\varkappa$ — is an arbitrary number, is fundamental in the space $\overset{o}{W}{}^{m,0}_{\vec{p}}(Q_\varkappa) \cap C([T-\varkappa, T]; L^2(\Omega))$. Indeed, let k, l be arbitrary natural numbers such that $k > 2\varkappa, l > 2\varkappa$. Using Lemma 3 in the case, for $u = u_k, \tilde{u} = u_l, f_\alpha = f_{\alpha,k}, \tilde{f}_\alpha = f_{\alpha,l}(|\alpha| \leqslant m) t_1 = T - \varkappa, t_2 = T$, $\delta = \min\{k - \varkappa, l - \varkappa\}$, we obtain

$$\max_{[T-\varkappa,T]} \int\limits_\Omega |u_k(x,t) - u_l(x,t)|^2\, dx + \int\limits_{T-\varkappa}^{T} \int\limits_\Omega \sum_{|\alpha|\leqslant m} |D^\alpha u_k - D^\alpha u_l|^{p_\alpha}\, dxdt \leqslant C_{21}\delta^{-\frac{2}{p_\gamma - 2}}.$$

Since $\delta^{-\frac{2}{p_\gamma-2}} \to 0$ as $\delta \to +\infty$, and C_{21} does not depend on δ, from (56) it follows that for arbitrary small $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that the left side of (56) is less then ε for $k > N$ and $l > N$.

It means that the sequence of restrictions u_μ on $Q_\varkappa, \mu \in \mathbb{N}$, is fundamental in the space $\overset{o}{W}{}^{m,0}_{\vec{p}}(Q_\varkappa) \cap C([T-\varkappa, T]; L^2(\Omega))$, where $\varkappa$ is an arbitrary natural number.

Therefore, there exists a function $u \in \overset{o}{W}{}^{m,0}_{\vec{p},\mathrm{loc}}(\overline{Q}) \cap C((-\infty,T];L^2(\Omega))$ such that

$$u_\mu \to u \quad \text{in} \quad C_{\mathrm{loc}}((-\infty,T];L^2(\Omega)), \tag{57}$$

$$D^\alpha u_\mu \to D^\alpha u \quad \text{strong in} \quad L^{p_\alpha}_{\mathrm{loc}}(\overline{Q}), |\alpha| \leqslant m. \tag{58}$$

Note that μ runs through all natural values.

From (58) and condition (A_2) it follows that there exists a subsequence $\{u_{\mu_i}\} \subset \{u_\mu\}$ and functions $\tilde{\chi}_\alpha \in L^{p'_\alpha}_{\mathrm{loc}}(\overline{Q}), |\alpha| \leqslant m$, such that

$$a_\alpha(x,t,\delta u_{\mu_i}(x,t)) \to \tilde{\chi}(x,t) \quad \text{weak in} \quad L^{p'_\alpha}_{\mathrm{loc}}(\overline{Q}),\ |\alpha| \leqslant m, \tag{59}$$

$$D^\alpha u_{\mu_i} \to D^\alpha u \quad \text{nearly everywhere on} \quad Q, |\alpha| \leqslant m. \tag{60}$$

From (59),(60), condition (A_1) and a lemma from [12], we have

$$a_\alpha(x,t,\delta u_{\mu_i}) \to a_\alpha(x,t,\delta u) \quad \text{weakly in} \quad L^{p'_\alpha}_{\mathrm{loc}}(\overline{Q}),\ |\alpha| \leqslant m. \tag{61}$$

Let us show that u is a generalized solution of the problem (1),(2). Indeed, let $\psi \in C_0^\infty(Q)$ be arbitrary (fixed) function. From (7_μ) we have

$$\iint_Q \{-u_{\mu_i}\psi_t + \sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta u_{\mu_i}) D^\alpha \psi\}\, dxdt = \iint_Q \sum_{|\alpha|\leqslant m} f_{\alpha,\mu_i} D^\alpha \psi \, dxdt. \tag{62}$$

Let us take the limit in (62) at $i \to +\infty$. As a result, taking into account (57),(58),(61) and the definition of f_{α,μ_i}, we obtain equality (3). This implies that u is a generalized solution of Problem (1),(2).

Thus we have shown the existence of a generalized solution of Problem (1),(2) and the strong convergence of the sequence of its approximations $\{u_\mu\}$ in the space $\overset{o}{W}{}^{m,0}_{\vec{p},\mathrm{loc}}(\overline{Q}) \cap C_{\mathrm{loc}}((-\infty,T];L^2(\Omega))$ (see (57),(58)).

Continuous dependence on the right side of the equation. Let $\{f_{\alpha,k}\}_{k=1}^\infty \subset L^{p'_\alpha}_{\mathrm{loc}}(\overline{Q})$, $|\alpha| \leqslant m$, be sequences of functions such that $f_{\alpha,k} \to f_\alpha$ in $L^{p'_\alpha}_{\mathrm{loc}}(\overline{Q})$, $|\alpha| \leqslant m$, and $\{u_k\}$ be a sequence of generalized solutions of corresponding problems. Let us show that $u_k \to u$ in $\overset{o}{W}{}^{m,0}_{\vec{p},\mathrm{loc}}(\overline{Q}) \cap C_{\mathrm{loc}}((-\infty,T];L^2(\Omega))$.

Indeed, let $\varepsilon > 0$ be an arbitrary number. Consider numbers t_1, t_2, $-\infty < t_1 < t_2 \leqslant T$. From Lemma 3 follows that

$$\max_{[t_1,t_2]} \int_\Omega |u_k(x,t) - u(x,t)|^2\, dx + \int_{t_1}^{t_2} \int_\Omega \sum_{|\alpha|\leqslant m} |D^\alpha u_k - D^\alpha u|^{p_\alpha}\, dxdt \leqslant$$

$$\leqslant C_{23}\delta^{-\frac{2}{p_\gamma - 2}} + C_{24} \int_{t_1-\delta}^{t_2} \int_\Omega \sum_{|\alpha|\leqslant m} |f_{\alpha,k} - f_\alpha|^{p'_\alpha}\, dxdt, \tag{63}$$

where $\delta > 0$ is an arbitrary number, $C_{23}, C_{24} > 0$ are constants which does not depend on δ.

Choose and fix $\delta > 0$ so that

$$C_{23}\delta^{-\frac{2}{p_\gamma - 2}} < \frac{\varepsilon}{2}. \tag{64}$$

From $f_{\alpha,k} \to f_\alpha$ in $L^{p'_\alpha}(Q_{t_1-\delta,t_2})$, $|\alpha| \leqslant m$, we obtain the existence of a number $k_0 \in \mathbb{N}$, such that for any $k \geqslant k_0$

$$C_{24} \int_{t_1-\delta}^{t_2} \int_{\Omega} \sum_{|\alpha|\leqslant m} |f_{\alpha,k} - f_\alpha|^{p'_\alpha}\, dxdt \leqslant \frac{\varepsilon}{2}. \qquad (65)$$

Taking into account (63)–(65), we conclude that the right side of (63) is less than ε for any $k \geqslant k_0$. Theorem 3 is proved.

Proof of Theorem 4. Theorem 3 implies the existence of a unique generalized solution of Problem (1),(2) and its continuous dependence on the right side of the equation (1). Let $\tilde{\psi}(x,t)$ be an arbitrary function from $C_0^\infty(Q)$. Put $\psi(x,t) = \tilde{\psi}(x,t-\sigma)$ at (3) and make the substitution t instead of $t+\sigma$ in the integrals. Taking into account the equality $a_\alpha(x,t+\sigma,\delta u(x,t+\sigma)) = a_\alpha(x,t,\delta u(x,t+\sigma))$ almost everywhere on Q (which follows from the theorem), we conclude, that $u(x,t+\sigma)$ is a generalized solution of the problem (1), (2). Now the uniqueness of the solution implies $u(x,t+\sigma) = u(x,t)$ for almost all $(x,t) \in Q$. Theorem 4 is proved.

REFERENCES

1. Тихонов А.Н.*Теоремы единственности для уравнения теплопроводности* // Мат. сб.– 1935.– Ґ2.– С.199-216.

2. Олейник О.А., Иосифьян Г.А. *Аналог принципа Сен-Венана и единственность решений краевых задач в неограниченных областях для параболических уравнений* // Успехи мат. наук.– 1976.– Т.31, Ґ6.– С.142-166.

3. Калашников А.С.*О задаче Коши в классах растущих функций для некоторых квазилинейных вырождающихся параболических уравнений второго порядка* // Дифференц. уравнения.– 1973.– Т.9, Ґ4.– С.682-691.

4. Шишков А.Е.*Классы единственности обобщённых решений краевых задач для параболических уравнений в неограниченных нецилиндрических областях* // Диф/-фе/-ренц. уравнения.–1990.–Т.26, Ґ9.– С.1627-1633.

5. Бокало Н.М. *Энергетические оценки решений и однозначная разрешимость задачи Фурье для линейных и квазилинейных параболических уравнений* // Дифференц. уравнения. 1994. Т.30, №8. С.1325-1334.

6. Сікорський В.М. *Задача Фур'є зі змішаною граничною умовою для систем квазілінійних параболічних рівнянь* // Вісник Львівського університету. Вип.45,– 1996,– С.45-56.

7. Бокало Н.М. *О задаче без начальных условий для некоторых классов нелинейных параболических уравнений* // Тр. семинара им. И.Г.Петровского. – М.: Изд-во Моск. ун-та,– 1989.– Вып. 14.– С.3-44.

8. Бокало Н.М.*Краевые задачи для полулинейных параболических уравнений в неограниченных областях без условий на бесконечности* // Сиб. мат. журнал.– 1996. –Т.37. Ґ5. С.977-985.

9. Бокало М.М. *Апріорна оцінка розв'язку та теорема типу Фрагмена-Ліндельофа для деяких квазілінійних параболічних систем у необмежених областях* // Вісник Львівського університету. Вип.45,– 1996,– С.26-35.

10. Bernis F. *Elliptic and parabolic semilinear problems without conditions at infinity* // Arch. Ration Mech. and Anal.– 1989.– V.106, Ґ3.– P.217-241.

11. Kufner A.,John O., Fučik S. Function Spaces.- Praha; Groningen: Academia; Noordhoff,1977.

12. Лионс Ж.-Л. Некоторые методы решения нелинейных краевых задач. – М.: Мир.– 1972.

Lviv University, Department of Mechanics and Mathematics

Received 1.03.1997