УДК 517.547.2

ON THE LINDELÖF THEOREM

Yaroslav Ya. Prytula

Ya.Ya. Prytula. *On the Lindelöf theorem*, Matematychni Studii, **8**(1997) 31–42.

Necessary and sufficient conditions are established on the coefficients of an entire Dirichlet series that defines a function F for fulfilling the correlations $\ln M(\sigma,F) = \Phi\big(\sigma+o(1)\big)$ and $\ln M(\sigma,F)=\Phi\big((1+o(1))\sigma\big)$, whenever $\sigma\to\infty$, where $M(\sigma,F) = \sup\big\{|F(\sigma+it)| : \ t\in\mathbb{R}\big\}$, and Φ is a positive on $(-\infty,+\infty)$ function such that $0\le\Phi'(x)\uparrow+\infty$ $(-\infty<x\to+\infty)$.

1. Introduction

Let f be an entire function defined by the power series

$$f(z)=\sum_{n=0}^{+\infty} a_n z^n, \tag{1}$$

and $M_f(r) = \max\{|f(z)| : \ |z|=r\}$. The most important characteristics of the growth of the function f are its order ϱ and for $0<\varrho<+\infty$ its type τ that are defined by the following expressions

$$\varrho=\varlimsup_{r\to\infty}\frac{\ln\ln M_f(r)}{\ln r},\quad \tau=\varlimsup_{r\to\infty}\frac{\ln M_f(r)}{r^\varrho}$$

and are calculated by the following formulas

$$\varrho=\varlimsup_{n\to\infty}\frac{n\ln n}{-\ln|a_n|},\quad \tau=\frac{1}{\mathrm{e}\varrho}\varlimsup_{n\to\infty} n|a_n|^{\varrho/n}$$

E. Lindelöf [1] showed that $\ln M_f(r)=\big(1+o(1)\big)\tau r^\varrho$ $(r\to+\infty)$, $\tau\in(0,+\infty)$, iff for all $\varepsilon>0$:

1) there exists $n_0\in\mathbb{N}$ such that for all $n\ge n_0$

$$\ln|a_n|\le-\frac{n}{\varrho}\ln\frac{n}{\mathrm{e}\varrho\tau(1+\varepsilon)};$$

2) there exists an increasing subsequence (n_k) of natural numbers such that $n_{k+1}\sim n_k$ $(k\to\infty)$ and

$$\ln|a_{n_k}|\ge-\frac{n_k}{\varrho}\ln\frac{n_k}{\mathrm{e}\varrho\tau(1-\varepsilon)}.$$

1991 *Mathematics Subject Classification.* 30B10, 30B50, 30D15, 30D20.

For $\varrho = 1$ the Lindelöf theorem was redemonstrated in [2]. Counterparts of the Lindelöf theorem were established in [3] for entire functions of the finite logarithmic order and in [4] for the case of two-term asymptotics of entire functions of the finite order. Recently M.M. Sheremeta and M.V. Zabolotskii [5] obtained a generalization of the Lindelöf theorem for the case of correlation $\ln M_f(r) = \big(1 + o(1)\big)\Phi(\ln r)$, $r \to +\infty$, where Φ is a positive on $(-\infty, +\infty)$ function such that its derivative Φ' is a non-negative, continuous and increasing to $+\infty$ on $(-\infty, +\infty)$ function. Natural is the question about conditions on the coefficients of the series (1) for fulfilling the correlations $\ln M_f(r) = \Phi\big(\ln r(1 + o(1))\big)$ and $\ln M_f(r) = \Phi\big(\ln r + o(1)\big)$, $r \to +\infty$. We will obtain the answer to this question for the natural generalization of the power series — Dirichlet series with positive and increasing to $+\infty$ exponents.

Let $\Lambda = (\lambda_n)$, $0 = \lambda_0 < \lambda_n \uparrow +\infty$ $(n \to +\infty)$, and the Dirichlet series

$$F(s) = \sum_{n=0}^{+\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it \tag{2}$$

has the abscissa of absolute convergence $A \in (-\infty, +\infty]$. For $-\infty < \sigma < A$ let $M(\sigma, F) = \sup\big\{|F(\sigma+it)| : \ t \in \mathbb{R}\big\}$ and let $\mu(\sigma, F) = \max\big\{|a_n| \exp\{\sigma\lambda_n\} : \ n \geq 0\big\}$ be the maximal term of series (2).

By $\Omega(A)$ we denote the class of positive on $(-\infty, A)$ functions Φ such that their derivatives Φ' are non-negative, continuous and increasing to $+\infty$ on $(-\infty, A)$ functions.

For $\Phi \in \Omega(A)$ let φ be the inverse function to Φ', and $\psi(\sigma) = \sigma - \Phi(\sigma)/\Phi'(\sigma)$ be the function associative with Φ in the sence of Newton. Clearly, φ is defined on $(0, +\infty)$, $\varphi(x) \to -\infty$ if $x \to 0$ and $\varphi(x) \to A$ if $x \to +\infty$. In [6] it is showed that the function ψ increases on $(-\infty, A)$ and $\psi(x) \to A$ if $x \to A$.

According to [7] under certain conditions on the exponents λ_n the correlation

$$\gamma\big(\ln M(\sigma, F)\big) \sim \gamma\big(\ln \mu(\sigma, F)\big), \qquad \sigma \to +\infty$$

holds, therefore we need conditions on a_n for fulfilling the relations $\ln \mu(\sigma, F) = \Phi\big(\sigma(1 + o(1))\big)$ and $\ln \mu(\sigma, F) = \Phi\big(\sigma + o(1)\big)$ if $\sigma \to +\infty$. The following theorems, which are main in this article, give us such conditions.

Theorem 1. *Suppose $\Phi \in \Omega(\infty)$. In order that $\ln \mu(\sigma, F) = \Phi\big(\sigma + o(1)\big)$ if $\sigma \to +\infty$, it is necessary and sufficient that for each $\varepsilon > 0$:*
1) there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\ln |a_n| \leq -\lambda_n \psi\big(\varphi(\lambda_n)\big) + \varepsilon\lambda_n; \tag{3}$$

2) there exists an increasing subsequence (n_k) of natural numbers such that

$$\ln |a_{n_k}| \geq -\lambda_{n_k} \psi\big(\varphi(\lambda_{n_k})\big) - \varepsilon\lambda_{n_k} \tag{4}$$

and

$$\lim_{k\to\infty} \Big\{ \frac{1}{\lambda_{n_{k+1}} - \lambda_{n_k}} \int_{\lambda_{n_k}}^{\lambda_{n_{k+1}}} \varphi(x)\, dx - \Phi^{-1}\Big(\frac{\lambda_{n_k}\lambda_{n_{k+1}}}{\lambda_{n_{k+1}} - \lambda_{n_k}} \int_{\lambda_{n_k}}^{\lambda_{n_{k+1}}} \frac{\Phi(\varphi(x))}{x^2}\, dx \Big) \Big\} = 0. \tag{5}$$

Theorem 2. *Suppose* $\Phi \in \Omega(A)$. *In order that* $\ln\mu(\sigma,F) = \Phi\big((1+o(1))\sigma\big)$ *if* $\sigma \to A$, *it is necessary and sufficient that for each* $\varepsilon > 0$*:*
1) there exists $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$

$$\ln|a_n| \leq -\frac{\lambda_n}{1+\varepsilon}\psi\Big(\varphi\Big(\frac{\lambda_n}{1+\varepsilon}\Big)\Big); \tag{6}$$

2) there exists an increasing subsequence (n_k) *of natural numbers such that*

$$\ln|a_{n_k}| \geq -\frac{\lambda_{n_k}}{1-\varepsilon}\psi\Big(\varphi\Big(\frac{\lambda_{n_k}}{1-\varepsilon}\Big)\Big) \tag{7}$$

and

$$\lim_{k\to\infty}\frac{\Phi^{-1}\Big(\frac{\lambda_{n_k}\lambda_{n_{k+1}}}{\lambda_{n_{k+1}}-\lambda_{n_k}}\int_{\lambda_{n_k}}^{\lambda_{n_{k+1}}}\frac{\Phi\big(\varphi(x)\big)}{x^2}\,dx\Big)}{\frac{1}{\lambda_{n_{k+1}}-\lambda_{n_k}}\int_{\lambda_{n_k}}^{\lambda_{n_{k+1}}}\varphi(x)\,dx} = 1. \tag{8}$$

2. Proof of Theorem 1

We will need the following lemma

Lemma ([6, c.6]). *Let* A *be the abscissa of absolute convergence of series (2),* $-\infty < A \leq +\infty$, *and* $\Phi \in \Omega(A)$. *In order that* $\ln\mu(\sigma,F) \leq \Phi(\sigma)$ *for all* $\sigma \in (\sigma_0, A)$, *it is necessary and sufficient that* $\ln|a_n| \leq -\lambda_n\psi\big(\varphi(\lambda_n)\big)$ *for all* $n \geq n_0$.

Suppose now that $\ln\mu(\sigma,F) = \Phi\big(\sigma + o(1)\big)$ if $\sigma \to \infty$. Then for each $\varepsilon > 0$ the inequality $\ln\mu(\sigma,F) \leq \Phi_1(\sigma) = \Phi(\sigma+\varepsilon)$ holds, if $\sigma \geq \sigma_0$. Then we have the corresponding functions $\psi_1(\sigma) = \psi(\sigma+\varepsilon)-\varepsilon$ and $\varphi_1(\sigma) = \varphi(\sigma)-\varepsilon$. Therefore, in accordance with Lemma, (3) holds for $n \geq n_0$.

For proving properties (4) and (5) define

$$G_1^\star(a,b,q) = \Phi^{-1}\Big(\frac{ab}{b-a}\int_a^b \frac{\Phi(\varphi(qx))}{x^2}\,dx\Big),$$
$$G_2^\star(a,b,q) = \frac{1}{b-a}\int_a^b \varphi(qx)\,dx,$$

where $0 < a < b < +\infty$ and $q > 0$, and let us show that

$$G_1^\star(a,b,q) < G_2^\star(a,b,q). \tag{9}$$

Consider the function

$$G^\star(x) = G_1^\star(a,x,q) - G_2^\star(a,x,q), \quad x > a.$$

Since

$$G_1^{\star\prime}(a,x,q)=\frac{1}{\Phi'\Big[\Phi^{-1}\Big(\frac{ax}{x-a}\int_a^x\frac{\Phi(\varphi(qt))}{t^2}\,dt\Big)\Big]}\frac{a}{(x-a)^2}\times$$

$$\times\Big\{\Phi\big(\varphi(xq)\big)-\frac{a}{x}\Phi\big(\varphi(xq)\big)+a\int_a^x\Phi(\varphi(tq))\,d\big(t^{-1}\big)\Big\}=$$

$$=\frac{1}{\Phi'\Big[\Phi^{-1}\Big(\frac{ax}{x-a}\int_a^x\frac{\Phi(\varphi(qt))}{t^2}\,dt\Big)\Big]}\frac{a}{(x-a)^2}\times$$

$$\times\Big\{\Phi\big(\varphi(xq)\big)-\Phi\big(\varphi(aq)\big)-aq\int_a^x d\varphi(tq)\Big\}=$$

$$=\frac{1}{\Phi'\Big[\Phi^{-1}\Big(\frac{ax}{x-a}\int_a^x\frac{\Phi(\varphi(qt))}{t^2}\,dt\Big)\Big]}\frac{aq}{(x-a)^2}\Big\{\int_a^x(t-a)\,d\varphi(tq)\Big\}=$$

$$=\frac{aq}{\Phi'\Big[\Phi^{-1}\Big(\frac{ax}{x-a}\int_a^x\frac{\Phi(\varphi(qt))}{t^2}\,dt\Big)\Big]}\frac{1}{(x-a)^2}\Big\{(x-a)\varphi(qx)-\int_a^x\varphi(tq)\,dt\Big\}$$

and

$$G_2^{\star\prime}(a,x,q)=\frac{1}{(x-a)^2}\Big\{(x-a)\varphi(qx)-\int_a^x\varphi(tq)\,dt\Big\}.$$

Since the function φ is increasing, we have

$$(x-a)\varphi(qx)-\int_a^x\varphi(tq)\,dt>0$$

and

$$\Phi'\Big[\Phi^{-1}\Big(\frac{ax}{x-a}\int_a^x\frac{\Phi(\varphi(qt))}{t^2}\,dt\Big)\Big]>aq,$$

then $G_1^{\star\prime}(a,x,q)<G_2^{\star\prime}(a,x,q)$. Hence, $G^{\star\prime}(x)<0$ for all $x>a$. Therefore, the function $G^\star(x)$ is non-decreasing on $(a,+\infty)$, and, since $G^\star(x)\to 0$ if $x\to a+$, we have $G^\star(x)<0$ for all $x>a$. Therefore, inequality (9) holds.

Now, suppose on the contrary that there exist numbers $\tau\in(0,1)$, $\eta\in(0,1)$, and a sequence of intervals (n'_k,n''_k), $n'_k\uparrow\infty$, such that

$$G_2^\star(\lambda_{n'_k},\lambda_{n''_k},1)-G_1^\star(\lambda_{n'_k},\lambda_{n''_k},1)\ge\tau$$

and

$$\ln|a_n|\le-\lambda_n\psi(\varphi(\lambda_n))-\eta\lambda_n,\quad n'_k<n<n''_k.$$

Define

$$a_n^\star=\exp\big\{-\lambda_n\psi\big(\varphi(\lambda_n)\big)+\varepsilon\lambda_n\big\}$$

and consider the Dirichlet series

$$F^\star(s) = \sum_{n\le \eta'_k} a_n^\star \exp\{s\lambda_n\} + \sum_{n\ge n''_k} a_n^\star \exp\{s\lambda_n\}.$$

Suppose

$$\varkappa_k^\star = \frac{\ln a^\star_{n'_k} - \ln a^\star_{n''_k}}{\lambda_{n''_k} - \lambda_{n'_k}} = \frac{\Theta(\lambda_{n'_k}) - \Theta(\lambda_{n''_k})}{\lambda_{n''_k} - \lambda_{n'_k}} - \varepsilon,$$

where $\Theta(x) = x\psi\big(\varphi(x)\big)$. Then for all $n \le n'_k$ and $n \ge n''_k$ we have: $a_n^\star \exp\{\varkappa_k^\star \lambda_n\} \le a^\star_{n'_k} \exp\{\varkappa_k^\star \lambda_{n'_k}\} = a^\star_{n''_k} \exp\{\varkappa_k^\star \lambda_{n''_k}\} = \mu(\varkappa_k^\star, F^\star)$. Therefore,

$$\Phi^{-1}\big[\ln \mu(\varkappa_k^\star, F^\star)\big] - \varkappa_k^\star = \Phi^{-1}\big[\ln a^\star_{n'_k} + \varkappa_k^\star \lambda_{n'_k}\big] - \varkappa_k^\star =$$

$$= \Phi^{-1}\Big[\frac{\lambda_{n'_k}\lambda_{n''_k}}{\lambda_{n''_k} - \lambda_{n'_k}}\Big\{\psi\big(\varphi(\lambda_{n''_k})\big) - \psi\big(\varphi(\lambda_{n'_k})\big)\Big\}\Big] - \varkappa_k^\star =$$

$$= \Phi^{-1}\Big[\frac{\lambda_{n'_k}\lambda_{n''_k}}{\lambda_{n''_k} - \lambda_{n'_k}} \int_{\lambda_{n'_k}}^{\lambda_{n''_k}} \frac{\Phi\big(\varphi(t)\big)}{t^2}\, dt\Big] - \frac{1}{\lambda_{n''_k} - \lambda_{n'_k}} \int_{\lambda_{n'_k}}^{\lambda_{n''_k}} \varphi(t)\, dt + \varepsilon.$$

Since

$$\psi\big(\varphi(b)\big) - \psi\big(\varphi(a)\big) = \int_a^b \left(\frac{\Theta(x)}{x}\right)' dx =$$
$$= \int_a^b \frac{x\varphi(x) - x\psi\big(\varphi(x)\big)}{x^2}\, dx = \int_a^b \frac{\Phi\big(\varphi(x)\big)}{x^2}\, dx, \quad 0 < a < b < +\infty,$$

and

$$\Theta(b) - \Theta(a) = \int_a^b \varphi(x)\, dx, \quad 0 < a < b < +\infty,$$

we have

$$\Phi^{-1}\big[\ln \mu(\varkappa_k^\star, F^\star)\big] - \varkappa_k^\star = \Phi^{-1}\big[\ln a^\star_{n'_k} + \varkappa_k^\star \lambda_{n'_k}\big] - \varkappa_k^\star =$$

$$= G_1^\star(\lambda_{n'_k}, \lambda_{n''_k}, 1) - G_2^\star(\lambda_{n'_k}, \lambda_{n''_k}, 1) + \varepsilon \le -\tau + \varepsilon.$$

Hence, for $n \le n'_k$ and $n \ge n''_k$ we have $\ln|a_n| + \varkappa_k^\star \lambda_n \le \ln a_n^\star + \varkappa_k^\star \lambda_n \le \ln \mu(\varkappa_k^\star, F^\star) \le \Phi(\varkappa_k^\star - \tau + \varepsilon)$.

And for $n'_k < n < n''_k$, by Lemma,

$$\ln|a_n| + \varkappa_k^\star \lambda_n \le -\lambda_n \psi\big(\varphi(\lambda_n)\big) - \eta\lambda_n + \varkappa_k^\star \lambda_n \le$$

$$\le \max\big\{-\lambda_n \psi\big(\varphi(\lambda_n)\big) - \eta\lambda_n + \varkappa_k^\star \lambda_n,\ n \ge 0\big\} \le \Phi(\varkappa_k^\star - \eta).$$

Hence, $\ln \mu(\varkappa_k^\star, F^\star) \le \Phi\big(\varkappa_k^\star - \min\{\tau - \varepsilon, \eta\}\big)$, which is impossible because of arbitrarity of ε. The necessity is proved.

Let us prove the sufficiency. By Lemma, from (3) it follows that $\ln\mu(\sigma,F)\le\Phi(\sigma+\varepsilon)$. Let $\{n_k\}$ be the sequence from conditions (4) and (5). Define

$$\widetilde{a}_k=\exp\bigl\{-\lambda_{n_k}\psi(\varphi(\lambda_{n_k}))-\varepsilon\lambda_{n_k}\bigr\}.$$

and consider the Dirichlet series

$$\widetilde{F}(s)=\sum_{k=1}^{\infty}\widetilde{a}_k\exp\{s\lambda_{n_k}\}.$$

From (4) it follows that $\ln\mu(\sigma,F)\ge\ln\mu(\sigma,\widetilde{F})$ and, by Lemma, $\ln\mu(\sigma,\widetilde{F})\le\Phi(\sigma-\varepsilon)$. But $\ln\widetilde{a}_k+\bigl(\varphi(\lambda_{n_k})+\varepsilon\bigr)\lambda_{n_k}=-\lambda_{n_k}\psi\bigl(\varphi(\lambda_{n_k})\bigr)+\lambda_{n_k}\varphi(\lambda_{n_k})=\Phi\bigl(\varphi(\lambda_{n_k})\bigr)$, that is

$$\ln\mu\bigl(\varphi(\lambda_{n_k})+\varepsilon,\widetilde{F}\bigr)=\Phi\bigl(\varphi(\lambda_{n_k})+\varepsilon-\varepsilon\bigr).$$

Suppose

$$\widetilde{\varkappa}_k=\frac{\ln\widetilde{a}_k-\ln\widetilde{a}_{k+1}}{\lambda_{n_{k+1}}-\lambda_{n_k}}=\frac{\Theta(\lambda_{n_{k+1}})-\Theta(\lambda_{n_k})}{\lambda_{n_{k+1}}-\lambda_{n_k}}+\varepsilon=\frac{1}{\lambda_{n_{k+1}}-\lambda_{n_k}}\int_{\lambda_{n_k}}^{\lambda_{n_{k+1}}}\bigl(\varphi(t)+\varepsilon\bigr)\,dt,$$

then $\varphi(\lambda_{n_k})+\varepsilon<\widetilde{\varkappa}_k<\varphi(\lambda_{n_{k+1}})+\varepsilon$.

Clearly, $\ln\widetilde{a}_k+\widetilde{\varkappa}_k\lambda_{n_k}=\ln\widetilde{a}_{k+1}+\widetilde{\varkappa}_k\lambda_{n_{k+1}}$. Moreover,
$\ln\widetilde{a}_k+\sigma\lambda_{n_k}\ge\ln\widetilde{a}_{k+1}+\sigma\lambda_{n_{k+1}}$, whenever $\varphi(\lambda_{n_k})+\varepsilon\le\sigma\le\widetilde{\varkappa}_k$ and
$\ln\widetilde{a}_k+\sigma\lambda_{n_k}\le\ln\widetilde{a}_{k+1}+\sigma\lambda_{n_{k+1}}$, whenever $\widetilde{\varkappa}_k\le\sigma\le\varphi(\lambda_{n_{k+1}})+\varepsilon$. Therefore,

$$\ln\mu(\sigma,\widetilde{F})=\begin{cases}\ln\widetilde{a}_k+\sigma\lambda_{n_k}, & \varphi(\lambda_{n_k})+\varepsilon\le\sigma\le\widetilde{\varkappa}_k;\\ \ln\widetilde{a}_{k+1}+\sigma\lambda_{n_{k+1}}, & \widetilde{\varkappa}_k\le\sigma\le\varphi(\lambda_{n_{k+1}})+\varepsilon.\end{cases}$$

Now, if $\varphi(\lambda_{n_k})+\varepsilon\le\sigma\le\widetilde{\varkappa}_k$, then

$$\bigl(\Phi^{-1}(\ln\widetilde{a}_k+\sigma\lambda_{n_k})-\sigma\bigr)'=$$

$$=\frac{1}{\Phi'\bigl[\Phi^{-1}(\ln\widetilde{a}_k+\sigma\lambda_{n_k})\bigr]}\Bigl\{\lambda_{n_k}-\Phi'\bigl[\Phi^{-1}\bigl(\ln\widetilde{a}_k+\sigma\lambda_{n_k}\bigr)\bigr]\Bigr\}\le$$

$$\le\frac{1}{\Phi'\bigl[\Phi^{-1}(\ln\widetilde{a}_k+\sigma\lambda_{n_k})\bigr]}\Bigl\{\lambda_{n_k}-\Phi'\bigl[\Phi^{-1}\bigl(\ln\widetilde{a}_k+(\varphi(\lambda_{n_k})+\varepsilon)\lambda_{n_k}\bigr)\bigr]\Bigr\}=0,$$

and, if $\widetilde{\varkappa}_k\le\sigma\le\varphi(\lambda_{n_{k+1}})+\varepsilon$, then, analogously,

$$\bigl(\Phi^{-1}(\ln\widetilde{a}_{k+1}+\sigma\lambda_{n_{k+1}})-\sigma\bigr)'\ge0,$$

i.e. $\widetilde{\varkappa}_k$ is the minimum of the function $\bigl\{\Phi^{-1}\bigl[\ln\mu(\sigma,\widetilde{F})\bigr]-\sigma\bigr\}$ on the given segment.

Hence, for $\varphi(\lambda_{n_k})+\varepsilon<\widetilde{\varkappa}_k<\varphi(\lambda_{n_{k+1}})+\varepsilon$ we have

$$\Phi^{-1}\bigl[\ln\mu(\sigma,F)\bigr]-\sigma\ge\Phi^{-1}\bigl[\ln\mu(\sigma,\widetilde{F})\bigr]-\sigma\ge$$

$$\ge\Phi^{-1}\bigl[\ln\mu(\widetilde{\varkappa}_k,\widetilde{F})\bigr]-\widetilde{\varkappa}_k=G_1^{\star}(\lambda_{n_k},\lambda_{n_{k+1}},1)-G_2^{\star}(\lambda_{n_k},\lambda_{n_{k+1}},1)-\varepsilon.$$

So, from (5) it follows that $\ln\mu(\sigma,F)\ge\Phi(\sigma-\delta)$ for all $\delta>0$ and $\sigma>\sigma_0$. Theorem 1 is proved.

3. Proof of Theorem 2

Suppose that $\ln\mu(\sigma,F)=\Phi\big((1+o(1))\sigma\big)$ if $\sigma\to A$. Then for each $\varepsilon>0$ and $\sigma\in(\sigma_0,A)$ we have $\ln\mu(\sigma,F)\le\Phi_1(\sigma)=\Phi\big((1+\varepsilon)\sigma\big)$. As we may easily check, the corresponding functions are $\psi_1(\sigma)=\frac{1}{1+\varepsilon}\psi\big((1+\varepsilon)\sigma\big)$ and $\varphi_1(\sigma)=\frac{1}{1+\varepsilon}\varphi\big(\sigma/1+\varepsilon\big)$. Therefore, in accordance with Lemma, we have (6) for $n\ge n_0$.

Remark that, as in the proof of Theorem 1,

$$G_1^\star(a,b,q)<G_2^\star(a,b,q).$$

For proving properties (7) and (8) suppose on the contrary that there are exist such numbers $\tau\in(0,1)$, $\eta\in(0,1)$ and a sequence of intervals (n_k',n_k''), $n_k'\uparrow\infty$, such that

$$G_1^\star(\lambda_{n_k'},\lambda_{n_k''},1)\le(1-\tau)G_2^\star(\lambda_{n_k'},\lambda_{n_k''},1)$$

and

$$\ln|a_n|\le-\frac{\lambda_n}{1-\eta}\psi\Big(\varphi\Big(\frac{\lambda_n}{1-\eta}\Big)\Big),\quad n_k'<n<n_k''.$$

Define

$$a_n^\star=\exp\Big\{-\frac{\lambda_n}{1+\varepsilon}\psi\Big(\varphi\Big(\frac{\lambda_n}{1+\varepsilon}\Big)\Big)\Big\}$$

and consider the Dirichlet series

$$F^\star(s)=\sum_{n\le\eta_k'}a_n^\star\exp\{s\lambda_n\}+\sum_{n\ge n_k''}a_n^\star\exp\{s\lambda_n\}.$$

Since $\big(x\psi(\varphi(x))\big)'=\varphi(x)$, the function $\Theta(x)=x\psi\big(\varphi(x)\big)$ is convex. And, if we define

$$\varkappa_k^\star=\frac{\ln a_{n_k'}^\star-\ln a_{n_k''}^\star}{\lambda_{n_k''}-\lambda_{n_k'}}=\frac{\Theta\Big(\frac{\lambda_{n_k'}}{1+\varepsilon}\Big)-\Theta\Big(\frac{\lambda_{n_k''}}{1+\varepsilon}\Big)}{\lambda_{n_k''}-\lambda_{n_k'}}\nearrow A\quad(k\to\infty),$$

then for all $n\le n_k'$, $n\ge n_k''$ we have

$$a_n^\star\exp\{\varkappa_k^\star\lambda_n\}\le a_{n_k'}^\star\exp\{\varkappa_k^\star\lambda_{n_k'}\}=a_{n_k''}^\star\exp\{\varkappa_k^\star\lambda_{n_k''}\}=\mu(\varkappa_k^\star,F^\star).$$

Therefore,

$$\frac{1}{\varkappa_k^\star}\Phi^{-1}\big[\ln\mu(\varkappa_k^\star,F^\star)\big]=\frac{1}{\varkappa_k^\star}\Phi^{-1}\big[\ln a_{n_k'}^\star+\varkappa_k^\star\lambda_{n_k'}\big]=\frac{1}{\varkappa_k^\star}\Phi^{-1}\Big[\frac{-\lambda_{n_k'}\ln a_{n_k''}^\star+\lambda_{n_k''}\ln a_{n_k'}^\star}{\lambda_{n_k''}-\lambda_{n_k'}}\Big]=$$

$$=\frac{1}{\varkappa_k^\star}\Phi^{-1}\Big[\frac{\lambda_{n_k'}\lambda_{n_k''}}{\lambda_{n_k''}-\lambda_{n_k'}}\Big\{\psi\Big(\varphi\Big(\frac{\lambda_{n_k''}}{1+\varepsilon}\Big)\Big)-\psi\Big(\varphi\Big(\frac{\lambda_{n_k'}}{1+\varepsilon}\Big)\Big)\Big\}\Big]=\frac{G_1^\star\big(\lambda_{n_k'},\lambda_{n_k''},\frac{1}{1+\varepsilon}\big)}{G_2^\star\big(\lambda_{n_k'},\lambda_{n_k''},\frac{1}{1+\varepsilon}\big)}.$$

Since $G_1^\star(a,b,q)\to G_1^\star(a,b,1)$ and $G_2^\star(a,b,q)\to G_2^\star(a,b,1)$ if $q\to1$, we can take small $\varepsilon>0$ such that

$$\frac{G_1^\star\big(\lambda_{n_k'},\lambda_{n_k''},\frac{1}{1+\varepsilon}\big)}{G_2^\star\big(\lambda_{n_k'},\lambda_{n_k''},\frac{1}{1+\varepsilon}\big)}\le\frac{G_1^\star\big(\lambda_{n_k'},\lambda_{n_k''},1\big)}{G_2^\star\big(\lambda_{n_k'},\lambda_{n_k''},1\big)}+\frac{\tau}{2}\le1-\frac{\tau}{2}.$$

Hence, for $n \le n'_k$, $n \ge n''_k$ we have

$$\ln|a_n| + \varkappa_k^\star \lambda_n \le \ln a_n^\star + \varkappa_k^\star \lambda_n \le \ln \mu(\varkappa_k^\star, F^\star) \le \Phi\big((1-\tau/2)\varkappa_k^\star\big). \tag{10}$$

And if $n'_k < n < n''_k$, then in accordance with Lemma

$$\ln|a_n| + \varkappa_k^\star \lambda_n \le -\frac{\lambda_n}{1-\eta}\psi\Big(\varphi\Big(\frac{\lambda_n}{1-\eta}\Big)\Big) + \varkappa_k^\star \lambda_n \le$$

$$\le \max\Big\{-\frac{\lambda_n}{1-\eta}\psi\Big(\varphi\Big(\frac{\lambda_n}{1-\eta}\Big)\Big) + \varkappa_k^\star \lambda_n,\ n \ge 0\Big\} \le \Phi\big((1-\eta)\varkappa_k^\star\big). \tag{11}$$

From (10) and (11) it follows that $\ln \mu(\varkappa_k^\star, F^\star) \le \Phi\big(\max\{(1-\eta),(1-\tau/2)\}\varkappa_k^\star\big)$, that is impossible because $\ln \mu(\sigma, F) = \Phi\big((1+o(1))\sigma\big)$, $\sigma \to A$. The necessity of conditions 1) and 2) of Theorem 2 is proved.

Let us prove the sufficiency. From (6), in accordance with Lemma, it follows that $\ln \mu(\sigma, F) \le \Phi\big((1+\varepsilon)\sigma\big)$, $\sigma \in (\sigma_0(\varepsilon), A)$. Let $\{n_k\}$ be the sequence from conditions (7) and (8). Define

$$\widetilde{a}_k = \exp\Big\{-\frac{\lambda_{n_k}}{1-\varepsilon}\psi\Big(\varphi\Big(\frac{\lambda_{n_k}}{1-\varepsilon}\Big)\Big)\Big\}$$

and consider the Dirichlet series

$$\widetilde{F}(s) = \sum_{k=1}^{\infty} \widetilde{a}_k \exp\{s\lambda_{n_k}\}.$$

From condition (4) it follows that $\ln \mu(\sigma, F) \ge \ln \mu(\sigma, \widetilde{F})$ for all $\sigma \in (-\infty, A)$, and, in accordance with Lemma, $\ln \mu(\sigma, \widetilde{F}) \le \Phi\big((1-\varepsilon)\sigma\big)$, $\sigma \in [\sigma_0(\varepsilon), A)$. On the other hand,

$$\ln \widetilde{a}_k + \Big(\frac{\varphi\left(\lambda_{n_k}/(1-\varepsilon)\right)}{1-\varepsilon}\Big)\lambda_{n_k} =$$

$$= -\frac{\lambda_{n_k}}{1-\varepsilon}\psi\Big(\varphi\Big(\frac{\lambda_{n_k}}{1-\varepsilon}\Big)\Big) + \frac{\lambda_{n_k}}{1-\varepsilon}\varphi\Big(\frac{\lambda_{n_k}}{1-\varepsilon}\Big) = \Phi\Big(\varphi\Big(\frac{\lambda_{n_k}}{1-\varepsilon}\Big)\Big), \tag{12}$$

that is

$$\ln \mu\Big(\frac{\varphi\left(\lambda_{n_k}/(1-\varepsilon)\right)}{1-\varepsilon}, \widetilde{F}\Big) = \Phi\Big[(1-\varepsilon)\frac{\varphi\left(\lambda_{n_k}/(1-\varepsilon)\right)}{1-\varepsilon}\Big]. \tag{13}$$

Suppose

$$\widetilde{\varkappa}_k = \frac{\ln \widetilde{a}_k - \ln \widetilde{a}_{k+1}}{\lambda_{n_{k+1}} - \lambda_{n_k}} = \frac{\Theta\big(\frac{\lambda_{n_{k+1}}}{1-\varepsilon}\big) - \Theta\big(\frac{\lambda_{n_k}}{1-\varepsilon}\big)}{\lambda_{n_{k+1}} - \lambda_{n_k}} = \frac{1}{\lambda_{n_{k+1}} - \lambda_{n_k}} \int_{\lambda_{n_k}}^{\lambda_{n_{k+1}}} \frac{\varphi\big(t/(1-\varepsilon)\big)}{1-\varepsilon}\,dt.$$

Since the function φ is increasing, we have

$$\frac{\varphi\big(\lambda_{n_k}/(1-\varepsilon)\big)}{1-\varepsilon} < \widetilde{\varkappa}_k < \frac{\varphi\big(\lambda_{n_{k+1}}/(1-\varepsilon)\big)}{1-\varepsilon}.$$

As we can easily check, $\ln\widetilde{a}_k+\widetilde{\varkappa}_k\lambda_{n_k}=\ln\widetilde{a}_{k+1}+\widetilde{\varkappa}_k\lambda_{n_{k+1}}$,

$$\begin{aligned}&\ln\widetilde{a}_k+\sigma\lambda_{n_k}\ge\ln\widetilde{a}_{k+1}+\sigma\lambda_{n_{k+1}}\ \text{whenever}\ \frac{\varphi\big(\lambda_{n_k}/(1-\varepsilon)\big)}{1-\varepsilon}\le\sigma\le\widetilde{\varkappa}_k\ \text{and}\\&\ln\widetilde{a}_k+\sigma\lambda_{n_k}\le\ln\widetilde{a}_{k+1}+\sigma\lambda_{n_{k+1}}\ \text{whenever}\ \widetilde{\varkappa}_k\le\sigma\le\frac{\varphi\big(\lambda_{n_{k+1}}/(1-\varepsilon)\big)}{1-\varepsilon}.\end{aligned}\tag{14}$$

Therefore, by (13) and (14),

$$\ln\mu(\sigma,\widetilde{F})=\begin{cases}\ln\widetilde{a}_k+\sigma\lambda_{n_k}, & \frac{\varphi(\lambda_{n_k}/(1-\varepsilon))}{1-\varepsilon}\le\sigma\le\widetilde{\varkappa}_k;\\ \ln\widetilde{a}_{k+1}+\sigma\lambda_{n_{k+1}}, & \widetilde{\varkappa}_k\le\sigma\le\frac{\varphi(\lambda_{n_{k+1}}/(1-\varepsilon))}{1-\varepsilon}.\end{cases}$$

Now if $\varphi\big(\lambda_{n_k}/(1-\varepsilon)\big)/(1-\varepsilon)\le\sigma\le\widetilde{\varkappa}_k$, then

$$\Big(\frac{\Phi^{-1}[\ln\widetilde{a}_k+\sigma\lambda_{n_k}]}{\sigma}\Big)'=\frac{1}{\sigma^2}\Big\{\frac{\sigma\lambda_{n_k}}{\Phi'\big[\Phi^{-1}(\ln\widetilde{a}_k+\sigma\lambda_{n_k})\big]}-\Phi^{-1}(\ln\widetilde{a}_k+\sigma\lambda_{n_k})\Big\}=$$

$$=\frac{1}{\sigma^2}\frac{1}{\Phi'\big[\Phi^{-1}(\ln\widetilde{a}_k+\sigma\lambda_{n_k})\big]}\Big\{\ln\frac{1}{\widetilde{a}_k}-\psi\big(\Phi^{-1}(\ln\widetilde{a}_k+\sigma\lambda_{n_k})\big)\Phi'\big[\Phi^{-1}(\ln\widetilde{a}_k+\sigma\lambda_{n_k})\big]\Big\}.$$

Since the functions ψ and Φ' are increasing and, by (12),

$$\Big(\frac{\Phi^{-1}[\ln\widetilde{a}_k+\sigma\lambda_{n_k}]}{\sigma}\Big)'\le\frac{1}{\sigma^2}\frac{1}{\Phi'\big[\Phi^{-1}(\ln\widetilde{a}_k+\sigma\lambda_{n_k})\big]}\Big\{-\ln\widetilde{a}_k-$$

$$-\psi\Big(\Phi^{-1}\Big(\ln\widetilde{a}_k+\frac{\varphi\big(\lambda_{n_k}/(1-\varepsilon)\big)}{1-\varepsilon}\lambda_{n_k}\Big)\Big)\Phi'\Big[\Phi^{-1}\Big(\ln\widetilde{a}_k+\frac{\varphi\big(\lambda_{n_k}/(1-\varepsilon)\big)}{1-\varepsilon}\lambda_{n_k}\Big)\Big]\Big\}=$$

$$=\frac{1}{\sigma^2}\frac{1}{\Phi'\big[\Phi^{-1}(\ln\widetilde{a}_k+\sigma\lambda_{n_k})\big]}\Big\{-\ln\widetilde{a}_k-\psi\Big(\varphi\Big(\frac{\lambda_{n_k}}{(1-\varepsilon)}\Big)\Big)\frac{\lambda_{n_k}}{1-\varepsilon}\Big\}=0,$$

and if $\widetilde{\varkappa}_k\le\sigma\le\varphi\big(\lambda_{n_{k+1}}/(1-\varepsilon)\big)/(1-\varepsilon)$, then

$$\Big(\frac{\Phi^{-1}[\ln\widetilde{a}_{k+1}+\sigma\lambda_{n_{k+1}}]}{\sigma}\Big)'\ge\frac{1}{\sigma^2}\frac{1}{\Phi'\big[\Phi^{-1}(\ln\widetilde{a}_{k+1}+\sigma\lambda_{n_{k+1}})\big]}\Big\{-\ln\widetilde{a}_{k+1}-$$

$$-\psi\Big(\Phi^{-1}\Big(\ln\widetilde{a}_{k+1}+\frac{\varphi\big(\lambda_{n_{k+1}}/(1-\varepsilon)\big)}{1-\varepsilon}\lambda_{n_{k+1}}\Big)\Big)\times$$

$$\times\Phi'\Big[\Phi^{-1}\Big(\ln\widetilde{a}_{k+1}+\frac{\varphi\big(\lambda_{n_{k+1}}/(1-\varepsilon)\big)}{1-\varepsilon}\lambda_{n_{k+1}}\Big)\Big]\Big\}=0.$$

Hence, for

$$\frac{\varphi(\lambda_{n_k}/(1-\varepsilon))}{1-\varepsilon}<\widetilde{\varkappa}_k<\frac{\varphi(\lambda_{n_{k+1}}/(1-\varepsilon))}{1-\varepsilon}$$

we have

$$\frac{1}{\sigma}\Phi^{-1}\big[\ln\mu(\sigma,F)\big] \geq \frac{1}{\sigma}\Phi^{-1}\big[\ln\mu(\sigma,\widetilde{F})\big] \geq \frac{1}{\widetilde{\varkappa}_k}\Phi^{-1}\big[\ln\mu(\widetilde{\varkappa}_k,\widetilde{F})\big] =$$

$$= \frac{1}{\widetilde{\varkappa}_k}\Phi^{-1}\big[\ln\widetilde{a}_k + \widetilde{\varkappa}_k\lambda_{n_k}\big] = \frac{G_1^\star\big(\lambda_{n_k},\lambda_{n_{k+1}},1/(1-\varepsilon)\big)}{G_2^\star\big(\lambda_{n_k},\lambda_{n_{k+1}},1/(1-\varepsilon)\big)}. \quad (15)$$

Since

$$\frac{G_1^\star\big(\lambda_{n_k},\lambda_{n_{k+1}},1/(1-\varepsilon)\big)}{G_2^\star\big(\lambda_{n_k},\lambda_{n_{k+1}},1/(1-\varepsilon)\big)} \to \frac{G_1^\star(\lambda_{n_k},\lambda_{n_{k+1}},1)}{G_2^\star(\lambda_{n_k},\lambda_{n_{k+1}},1)}, \qquad \varepsilon\to 0,$$

from (8) and (15) it follows that $\ln\mu(\sigma,F) \geq \Phi\big((1-\delta)\sigma\big)$ for all $\delta>0$ and $\sigma\in[\sigma_0(\delta),A)$. Theorem 2 is proved.

4. Corollaries and remarks

Remark 1. The classic Lindelöf theorem follows from Theorem 1, if $\Phi(\sigma)=\tau e^{\rho\sigma}$ and $\lambda_n=n$, and from the correlation $\ln M(\sigma,F)\sim\ln\mu(\sigma,F)$, $\sigma\to+\infty$, that holds in this case.

Since the expressions in relations (5) and (8) are very intricate, it would be intresting to show conditions on the function Φ that reduce these relations.

For that suppose

$$\varkappa_k = \frac{1}{\lambda_{n_{k+1}}-\lambda_{n_k}}\int_{\lambda_{n_k}}^{\lambda_{n_{k+1}}}\varphi(t)\,dt.$$

Corollary 1. *Let $\Phi\in\Omega(A)$ such that*

$$\varlimsup_{x\to A}\frac{\Phi^{-1}\big[\Phi'\big((1-\tau)x\big)\tau x + \Phi\big((1-\tau)x\big)\big]}{x} < 1, \quad 0<\tau<1. \quad (16)$$

Then $\ln\mu(\sigma,F)=\Phi\big((1+o(1))\sigma\big)$, $\sigma\to A$, iff for each $\varepsilon>0$ conditions 1) and 2) of Theorem 2 hold with a subsequence $\{n_k\}$ such that

$$\varphi(\lambda_{n_k})\sim\varkappa_k, \quad k\to\infty. \quad (17)$$

Proof. Remark that for all k we have $\varphi(\lambda_{n_k})\leq\varkappa_k$ and since

$$1 \geq \frac{G_1^\star(\lambda_{n_k},\lambda_{n_{k+1}},1)}{G_2^\star(\lambda_{n_k},\lambda_{n_{k+1}},1)} \geq \frac{\varphi(\lambda_{n_k})}{\varkappa_k},$$

the sufficiency is proved.

To prove the necessity, suppose on the contrary that there exist $\tau \in (0,1)$ and a subsequence $\lambda_{n_{k(s)}}$ of the sequence λ_{n_k} such that $\varphi(\lambda_{n_{k(s)}}) \leq (1-\tau)\varkappa_{k(s)}$. Then

$$G_1^\star(\lambda_{n_{k(s)}},\lambda_{n_{k(s+1)}},1) = \Phi^{-1}\left[\frac{\lambda_{n_{k(s)}}\lambda_{n_{k(s+1)}}}{\lambda_{n_{k(s+1)}}-\lambda_{n_{k(s)}}}\{\psi(\varphi(\lambda_{n_{k(s+1)}}))-\psi(\varphi(\lambda_{n_{k(s)}}))\}\right] =$$

$$= \Phi^{-1}\Big[\frac{\lambda_{n_{k(s)}}\big(\lambda_{n_{k(s+1)}}\psi(\varphi(\lambda_{n_{k(s+1)}}))-\lambda_{n_{k(s)}}\psi(\varphi(\lambda_{n_{k(s)}}))\big)}{\lambda_{n_{k(s+1)}}-\lambda_{n_{k(s)}}} - \lambda_{n_{k(s)}}\psi\big(\varphi(\lambda_{n_{k(s)}})\big)\Big] =$$

$$= \Phi^{-1}\Big[\Phi'\big(\varphi(\lambda_{n_{k(s)}})\big)\varkappa_{k(s)} - \Phi'\big(\varphi(\lambda_{n_{k(s)}})\big)\psi\big(\varphi(\lambda_{n_{k(s)}})\big)\Big] \leq$$

$$\leq \Phi^{-1}\Big[\Phi'\big((1-\tau)\varkappa_{k(s)}\big)\big(\varkappa_{k(s)} - \psi\big((1-\tau)\varkappa_{k(s)}\big)\big)\Big] =$$

$$= \Phi^{-1}\Big[\tau\varkappa_{k(s)}\Phi'\big((1-\tau)\varkappa_{k(s)}\big) + \Phi\big((1-\tau)\varkappa_{k(s)}\big)\Big],$$

because the function $x\big(\varkappa_{k(s)} - \psi(\varphi(x))\big)$ increases on the interval $(0, \Phi'(\varkappa_{k(s)}))$. Hence,

$$\varliminf_{k\to\infty}\frac{G_1^\star(\lambda_{n_k},\lambda_{n_{k+1}},1)}{G_2^\star(\lambda_{n_k},\lambda_{n_{k+1}},1)} \leq \varlimsup_{m\to\infty}\frac{\Phi^{-1}\big[\tau\varkappa_{k(s)}\Phi'((1-\tau)\varkappa_{k(s)}) + \Phi((1-\tau)\varkappa_{k(s)})\big],}{\varkappa_{k(s)}} < 1,$$

that contradicts (8). Corollary 1 is proved.

Remark 2. In the proof of the sufficiency of Corollary 1 the condition (16) is not used. Therefore, if conditions 1) and 2) of Theorem 2 hold with the sequence λ_{n_k} such that (17) holds, then $\ln\mu(\sigma,F) = \Phi\big((1+o(1))\sigma\big)$, $\sigma\to A$, for every function $\Phi\in\Omega(A)$.

Remark 3. The example 1 from [5] shows that the conditions (8) and (17) do not coincide in the class $\Omega(\infty)$.

Corollary 3. *Let* $\Phi\in\Omega(\infty)$,

$$\varlimsup_{n\to\infty}\frac{\ln n}{\lambda_n\psi(\varphi(\lambda_n))} < 1$$

and for every slowly growing function γ

$$\varlimsup_{n\to\infty}\frac{\Phi^{-1}\{\ln n + \psi(\varphi(\lambda_n))\gamma[\psi(\varphi(\lambda_n))]\}}{\Phi^{-1}\{\psi(\varphi(\lambda_n))\gamma[\psi(\varphi(\lambda_n))]\}} = 1$$

holds. Then

$$\ln M(\sigma,F) = \Phi\big((1+o(1))\sigma\big), \quad \sigma\to+\infty,$$

iff conditions (6), (7) and (8) hold.

Corollary 3 follows from Theorem 2 and from the correlation $\Phi^{-1}\big(\ln M(\sigma,F)\big) \sim \Phi^{-1}\big(\ln\mu(\sigma,F)\big)$, $\sigma\to+\infty$, that is proved under the same conditions in [7].

REFERENCES

1. Lindelöf E. *Sur la determination de la croissance des fonctions entiere definies par un developpement de Taylor* // Bull. Soc. Math. 1903. 27. №1. P.1–62.
2. Говоров Н.В., Черных Н.М. *О признаках полной регулярности роста некоторых классов целых функций экспоненциального типа, представленных интегралами Бореля, рядами Ньютона, Дирихле и степенными рядами* // ДАН СССР. 1979. Т.249, №6. С.1295–1299.
3. Srivastava G.S., Junea O.P. *On entire functions of slow growth* // Ann. Soc. Math. Polon. Ser. 1. 1984. 24. P.133–141.
4. Шеремета М.Н. *Двучленная асимптотика целых рядов Дирихле* // Теория функций, функ. анализ и их прилож. 1990. 54. С.16–25.
5. Заболоцький М.В., Шеремета М.М. *Узагальнення теореми Ліндельофа* // Укр. мат. ж. (в друці).
6. Шеремета М.М., Притула Я.Я., Фединяк С.І. *Зростання рядів Діріхле* // Препринт Центру мат. моделювання ІППММ ім. Я.С. Підстригача НАН України. 1995. С.3–7.
7. Шеремета М.Н. *О соотношениях между максимальным членом и максимумом модуля целого ряда Дирихле* // Мат. заметки. 1992. Т.51, №5. С.141–148.

Lviv State University, Department of Mechanics and Mathematics

Received 3.12.96