УДК 512.5

ON ORBITS OF GENERAL POSITION

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A.S. Timoshin. *On orbits of general position*, Matematychni Studii, **8**(1997) 27–30.

We consider the actions of some linear groups on the dualizing spaces of their unipotent radicals and give some pre-normal form for the orbits of general position under such actions.

In the representation theory of "mixed" Lie groups (i.e. neither solvable nor reductive) the action on the dualizing space of the unipotent radical or some part of it often play an important role (cf. e.g. [3] or [5]). Usually, the classification of the orbits under such action is a very complicated ("wild") problem. In some cases (e.g. mentioned above) one can classify a "big" part of the orbit space ("orbits of general position"), though in more complicated cases even such partial results do not exist. In this paper we consider a special case of the so called *net subgroups* (cf. [1]) of the full linear groups. In this case we are able to give some pre-normal form for the orbits of general position. Though it does not solve completely the problem of classification of orbits, this form seems to be rather convenient and useful for further applications.

We use the technique of the "matrix problems", more precisely, the representations of boxes, just as in [3]. For the necessary definitions and elementary properties of such representations we refer to [2].

Let $S = (S_0, S_1)$ be a finite partially ordered set, where S is the point set and $S_1 = \{d_{xy} \mid x > y; x, y \in S_0\}$ is the relation set in S.

Call a subset $A \subset S_0$ *top stable*, if for any $a \in A$ and $a' \in S_0$ such that $a' > a$, we have $a' \in A$.

Similarly, a subset $B \subset S_0$ is *bottom stable*, if the condition $b' < b$, where $b \in B$, $b' \in S_0$, implies $b' \in B$.

Denote by $S^{]}$ and $S^{[}$ respectively the subsets of all maximal and minimal non-isolated points from S_0.

Call a *subdivision* of S a pair of subsets A, B of S_0 such that the following conditions hold:

1) $S_0 = A \cup B$;
2) $A \supset S^{]}$ and $B \supset S^{[}$;
3) A is top stable and B is bottom stable.

Consider the group G of automorphisms of a finitely generated projective (left) module P over the incidence algebra $A(S)$ over some field K. Thus $G = G(A(S), P)$ is a net subgroup (in the sense of [1]) of $GL(n, K)$, where $n = |S_0|$, i.e. that of block matrices, where the configuration of zero blocks is given by the order relation in S.

1991 *Mathematics Subject Classification.* 20G05.

Consider G as the semi-direct product $H \circ N$, where N is the unipotent part of G and H is its reductive part. Let $\mathcal{N}$ be the Lie algebra of the subgroup N. Then in order to describe irreducible unitary representations of the group G it is neccessary to determine the orbit space of the action of H on $\hat{N}$, where $\hat{N}$ is the space of irreducible unitary representations of N (cf. [4]). According to the Kirillov orbit method, $\hat{N} \simeq \mathcal{N}^*/N$, where $\mathcal{N}^*$ is the dual space of $\mathcal{N}$. Thus $\hat{N}/H \simeq (\mathcal{N}^*/N)/H = \mathcal{N}^*/(H \circ N) = \mathcal{N}^*/G$. The description of the orbit space $\mathcal{N}^*/G$ is equivalent to the classification of representations of some box Λ which is described as follows. The box Λ is the free box (C, V), where C is the free category with the object set the same as the point set S_0 and with the free generators family $\Sigma_0 = \{d_{xy} \mid x > y, y \notin S^{[}\}$. The kernel $\bar{V}$ of the box Λ is the free C-bimodule with the free generators family $\Sigma_1 = \{\xi_{yx} \mid x > y, x \notin S^{]}\}$. The differential δ of the box Λ is given as follows.

Let $d_{xy} \in \Sigma_0$, $\xi_{yx} \in \Sigma_1$. Then

$$\delta(d_{xy}) = \sum_{x>y>k} d_{xk}\xi_{ky} \qquad (k \notin S^{[}),$$
$$\delta(\xi_{yx}) = \sum_{x>k>y} \xi_{yk}\xi_{kx}.$$

Consider the mappings $m\colon S_0 \to Z^+$. Call such mappings the *dimensions* for the box Λ and denote by E the set of all such dimensions.

For each $x \in S$ define the following inequalities:

$$L_x : \sum_{x'>x} m(x') \le \sum_{x'<x} m(x'), \qquad \bar{L}_x : \sum_{x'>x} m(x') \ge \sum_{x'<x} m(x').$$

Thus $E = E(L_x) \cup E(\bar{L}_x)$, where $E(L_x)$ and $E(\bar{L}_x)$ denote the sets of all dimensions satisfying these inequalities.

For the subset $C \subset S_0$ we denote by L_C the system of all inequalities $\{L_c\}_{c \in C}$. Denote by $E(L_C)$ the cone in E defined by all these inequalities: $E(L_C) = \bigcap_{c \in C} E(L_c)$. Obviously, for arbitrary subsets C, $C' \subset S_0$, we have: $E(L_{C \cup C'}) = E(L_C) \cap E(L_{C'})$.

Proposition 1. *Let $S = A|B$ be some subdivision. Then $E(L_A) = E(L_{A^{[}})$ and $E(\bar{L}_B) = E(\bar{L}_{B^{]}})$.*

Proof. It is easy to see that $E(L_A) = \bigcap_{a \in A^{[}} E(L_a) \cap \bigcap_{a \in A \setminus A^{[}} E(L_a) \subset \bigcap_{a \in A^{[}} E(L_a) = E(L_{A^{[}})$. Thus $E(L_A) \subset E(L_{A^{[}})$. Similarly, $E(\bar{L}_B) \subset E(\bar{L}_{B^{]}})$.

Let $x, y \in S_0$, so that $x > y$. Then $E(L_x) \supset E(L_y)$ (and $E(\bar{L}_x) \subset E(\bar{L}_y)$). Indeed,

$$\sum_{x'>x} m(x') < \sum_{x'>y} m(x'); \qquad \sum_{x'<x} m(x') > \sum_{x'<y} m(x').$$

Suppose that for some $m \in E$ the inequality L_y holds. Then

$$\sum_{x'>x} m(x') < \sum_{x'>y} m(x') \le \sum_{x'<y} m(x') < \sum_{x'<x} m(x')$$

and, therefore, m satisfies also the inequality L_x, whence the inclusion $E(L_x) \supset E(L_y)$ follows.

By analogy, we obtain that $E(\bar{L}_x) \subset E(\bar{L}_y)$.

Finally show that $E(L_A) \supset E(L_{A^[})$ and $E(\bar{L}_B) \supset E(\bar{L}_{B^]})$. Indeed, we see that $E(L_A) = \bigcap_{a\in A} E(L_a) = \bigcap_{a\in A^[} E(L_a) \cap \bigcap_{a\in A\setminus A^[} E(L_a) \supset \bigcap_{a\in A^[} E(L_a) \cap \bigcap_{a\in A^[} E(L_a) = E(L_{A^[})$. This completes the proof of the proposition.

For a subdivision $S = A|B$ denote $E(A|B) = E(L_A) \cap E(\bar{L}_B)$. One can easily check that the subset $E(A|B)$ is not empty.

Lemma 1. *For any subdivision $S = A|B$ and any top stable subset C such that $C \subset A$, $C \bigcap S^[= \varnothing$, we have $\bigcup_{C\subseteq A} E(A|B) = E(L_C)$.*

We prove this lemma by induction on $|S_0 \setminus C|$.

If $C = S_0 \setminus S^[$, then $S = C \bigcup S^[$ is a subdivision. Moreover, if $A'|B'$ is another subdivision for which $C \subset A'$, then $A' = C$, $B' = S^[$. Therefore, $\bigcup_{C\subseteq A} E(A|B) = E(C|S^[) = E(L_C) \cap E(\bar{L}_{S^[}) = E(L_C)$. Now suppose that $C \neq S_0 \setminus S^[$. Denote by M the set of maximal elements from $S_0 \setminus (C \bigcup S^[)$. Then $C \bigcup \{b\} \subseteq A$ for some $b \in M$. Hence

$$\bigcup_{C\subseteq A} E(A|B) = E(C|S_0 \setminus C) \cup \Big(\bigcup_{b\in B^]} \bigcup_{C\cup\{b\}} E(A|B)\Big).$$

According to the induction hypothesis, for any $b \in M$ the equality $\bigcup_{C\cup\{b\}} E(A|B) = E(L_{C\cup\{b\}}) E(L_C) \cap E(L_b)$ holds. Moreover, $E_{C\cup(S_0\setminus C)} = E(L_C) \cap E(L_{S_0\setminus C})$. By Proposition 1 we obtain $E(L_{S_0\setminus C}) = \bigcap_{x\in S_0\setminus C} E(L_x) = \bigcap_{b\in M} E(L_b) = E(L_M)$, whence $E_{C\cup\bar{C}} = E(L_C) \cap E(\bar{L}_M)$. Thus we have

$$\bigcup_{C\subset A} E(A|B) = (E(L_C) \cap E(\bar{L}_M) \cup \Big(\bigcup_{b\in M} E(L_C) \cap E(L_M)\Big) = E(L_C)$$

and Lemma 1 is proved.

Corollary. $\bigcup_{A|B} E(A|B) = E$, *where $A|B$ runs through all subdivisions of S.*

Denote by $R(\Lambda)$ the category of representations of the box Λ and $R_d(\Lambda)$, where $d \in E$, the set of the representations of the dimension d. Remind that the *dimension* of the representation U is the function $\dim U \colon S_0 \to Z^+$ which maps x to $\dim U(x)$ (cf. [2]).

Let $U \in R(\Lambda)$ and $X, Y \subset S_0$ be two subsets such that $x > y$ for each $x \in X$, $y \in Y$. In this case we write $X > Y$. Then the mapping $U_{XY} \colon \bigoplus_{x\in X} U(x) \to \bigoplus_{y\in Y} U(y)$ is well-defined. Now we say that a representation U is *of general position* if for each pair $X, Y \subset S_0$ such that $X > Y$ the mapping U_{XY} is either monomorphism or epimorphism.

Theorem 1. *Let $S = A|B$ be any subdivision, $U \in R(\Lambda)$ be a representation of general position and* $\dim U \in E(A|B)$. *Then there exists a representation $U_0 \in R(\Lambda)$ and an isomorphism $\Phi \colon U \to U_0$ with the following properties:*
1) if for some $x, y \in S_0, U_0(d_{xy}) \neq 0$, then $x \in A, y \in B$;
2) if for some $x, y \in S_0, \Phi(\xi_{yx}) \neq 0$, then $x \in A, y \in B$.

This theorem is also proved by induction on $|S_1|$. We may suppose, of course, that S has no isolated points.

If $A = S^]$ and $B = S^[$, then, of course, $E(A|B) = E$ and the conditions 1), 2) trivially hold for each representation $M \in R(\Lambda)$.

Suppose that $S = A|B$ is such that at least one of the conditions $A \neq S^{]}$ or $B \neq S^{[}$ hold. Then the set $S_0 \setminus (S^{[} \cup S^{]})$ is not empty. Let a be a maximal point and b be a minimal point from $S_0 \setminus (S^{[} \cup S^{]})$, where $b \leq a$. Consider the set $S^{]}_a = \{x \in S^{]} \mid x > a\}$, $S^{[}_b = \{y \in S^{[} \mid y < b\}$. Suppose that

$$\sum_{x \in S^{]}_a} m(x) \leq \sum_{y \in S^{[}_b} m(y).$$

In this case, obviously, $a \in A$. Moreover, in this case the mapping $U_{S^{]}_a, S^{[}_b}$ is a monomorphism. Then one can easily construct a representation $U' \in R(\Lambda)$ and a morphism $\Phi' \colon U \to U'$ such that $U'(d_{xa}) = 0$ for all $x \in S^{]}_a$ and $U'(d_{xy}) = U(d_{xy})$ for all $y \neq a$. Moreover, if $\Phi'(\xi_{yx}) \neq 0$, then $x = a, y \in S^{[}_b$.

Let $S' = (S'_0, S'_1)$ be such that $S'_0 = S_0$, $S'_1 = S_1 \setminus \{\bigcup_{x \in S^{]}_a} d_{xa}\}$. As $c > a$ implies also $c > b$, S' also has no isolated points. Hence, $(S')^{]} = S^{]} \bigcup \{a\}, (S')^{[} = S^{[}$. Consider the subdivision $S' = A'|B'$ such that $(A')^{[} = A^{[}, (B')^{]} = B^{]}$. It is obvious that $E'(A'|B') = E(A|B)$.

Let the box $\Lambda' = (C', V')$ corresponds to the set S' in the above mentioned way. Then obviously C' is a subcategory in C and $V' \subset V$. There exists the obvious functor $F \colon R(\Lambda') \to R(\Lambda)$ identical on the representation dimensions and such that $\operatorname{Im} F$ includes all representations $W \in R(\Lambda)$ with $W(d_{xa}) = 0$ for all $x \in S^{]}_a$. In particular, $U' = F(U_1)$. Moreover, it is obvious that U_1 is also of general position.

According to induction hypothesis, there exists such representation $U_2 \in R(\Lambda')$ that $\dim U_2 = \dim U_1 = \dim U$ and a morphism $\Phi'' \colon U_1 \to U_2$, for which Properties 1), 2) of the theorem hold. Put $U_0 = F(U_2)$. Then the representation U_0 and the composition of the morphisms $\Phi = F(\Phi'') \circ \Phi' \colon U \to U_0$ also satisfy Properties 1), 2), which proves the theorem.

If

$$\sum_{x \in S^{]}_a} m(x) \geq \sum_{y \in S^{[}_b} m(y)$$

then $b \in B$ and the proof is quite similar.

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Received 5.10.96