УДК 512.544

# ONE GENERALIZATION OF FITTING'S THEOREM

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B. Petrenko. *One generalization of Fitting's theorem*, Matematychni Studii, **8**(1997) 21–26.

Fitting's theorem (in one of its group-theorethic versions) is generalized on modules with finite composition series over the ring $DG$, where $D$ is a Dedekind domain and $G$ is an XC-hypercentral group for some formation of finite groups X.

## Preliminaries

Let $R$ be an associative ring with identity and $A$ be an $R$-module. If $B$ is its $R$-submodule, this will be denoted by $B \le A$. It will always be clear over what ring the modules are considered, so there will be no need for any notation like $B \le_R A$. If $B \le C \le A$, then the module $C/B$ is called an $R$-quotient of $A$. If $C/B$ is a simple $R$-module (of course, $C/B \ne 0$), then it is called an $R$-chief quotient of $A$.

If $G$ is a group and $H$ is its subgroup, this will also be denoted by $H \le G$.

A class of groups X is a formation of finite groups if the following four conditions are satisfied:

1) every element of X is a finite group,
2) if $H \in \mathrm{X}$ and $H \cong G$, then $G \in \mathrm{X}$,
3) if $H$ is a normal subgroup of a group $G \in \mathrm{X}$, then $G/H \in \mathrm{X}$,
4) if $H_1$ and $H_2$ are normal subgroups of a group $G$ such that $G/H_1$, $G/H_2 \in \mathrm{X}$, then $G/(H_1 \cap H_2) \in \mathrm{X}$.

Let X be a formation of finite groups and let $G$ be a group. We define the XC-centre of the group $G$ as follows: $\mathrm{XC}(G) = \{g \in G \mid G/C_G(g^G) \in \mathrm{X}\}$. The group $G$ is called XC-hypercentral if it possesses a transitive ascending series of groups

$$I = X_0 \le X_1 \le \dots \le X_\alpha \le X_{\alpha+1} \le \dots X_\gamma = G, \qquad (*)$$

where $X_{\alpha+1} = \pi_\alpha^{-1}\big(\mathrm{XC}(G/X_\alpha)\big)$ ($\pi_\alpha \colon g \to gX_\alpha, g \in G$) and $X_\alpha = \bigcup_{\varrho<\alpha} X_\varrho$ if $\alpha$ is a limit ordinal. If $\gamma$ is finite, then the group $G$ is called XC-nilpotent.

If $G$ is an XC-hypercentral group and $H$ is its normal subgroup, then the quotient group $G/H$ is XC-hypercentral too.

1991 *Mathematics Subject Classification.* 20F22, 16D70.

Let $G$ be a group, $D$ a Dedekind domain (for the definition, see [1], Definition 18.4), X a group formation, and $A$ a $DG$-module. The XC-$DG$-centre of the module $A$ is by definition

$$\mathrm{XC}_{DG}(A) = \{a \in A \mid G/C_G(aDG) \in X\}.$$

The term $A_\delta$ in the series

$$0 = A_0 \leq A_1 \leq \cdots \leq A_\alpha \leq A_{\alpha+1} \leq \cdots \leq A_\delta,$$

where $A_{\alpha+1} = \pi_\alpha^{-1}\big(\mathrm{XC}_{DG}(G/A_\alpha)\big)$ ($\pi_\alpha\colon a \to a + A_\alpha$, $a \in A$) and $A_\alpha = \bigcup_{\varrho<\alpha} A_\varrho$ if $\alpha$ is a limit ordinal such that $\mathrm{XC}_{DG}(G/A_\gamma) = 0$, is called the upper XC-$DG$-hypercentre of $A$ and is denoted by $\mathrm{XC}^\infty_{DG}(A)$. The module $\mathrm{XC}^*_{DG}(A)$ is defined as follows:

$$\mathrm{XC}^*_{DG}(A) = \Sigma\{L \mid L \text{ is a } DG\text{-submodule of } A \text{ which has no } DG\text{-chief quotient } L_1/L_2 \text{ such that } G/C_G(L_1/L_2) \in X\}.$$

It is not difficult to prove the following (see [6]):
1) $\mathrm{XC}^\infty_{DG}(A) \cap \mathrm{XC}^*_{DG}(A) = 0$.
2) Let $\{X_\alpha\}_{\alpha\in I}$ be a system of $DG$-submodules of $A$ with the following property: every $X_\alpha$, $\alpha \in I$, has no $DG$-chief quotient $L_1/L_2$ such that $G/C_G(L_1/L_2) \in X$. Then $\sum_{\alpha\in I} L_\alpha$ itself has this property. (Therefore $\mathrm{XC}^\infty_{DG}(A)$ has no $DG$-chief quotient $L_1/L_2$ such that $G/C_G(L_1/L_2) \in X$; consequently $\mathrm{XC}^\infty_{DG}(A)$ is the maximal $DG$-submodule of $A$ with this property.)

If $A = \mathrm{XC}^\infty_{DG}(A) \oplus \mathrm{XC}^*_{DG}(A)$, then we say that the module $A$ has an XC-$DG$-decomposition.

Some XC-$DG$-decompositions were studied in [3], [4], [6], and [7]. If $X = E$, the class of all identity groups, and $D = \mathbb{Z}$, the rings of integers, then an XC-$DG$-decomposition is a $\mathbb{Z}$-decomposition; for infinite groups, it was for the first time introduced and studied in [3] and [7]. Namely, in [7] D.I. Zaitsev proved that any artinian $\mathbb{Z}G$-module $A$ over a hyperfinite locally soluble group $G$ has an FC-$\mathbb{Z}G$-decomposition. Depending on this, he proved that if $E$ is an extension of anabelian group $A$ by a hyperfinite locally soluble group $G$ and $A$ is an artinian $\mathbb{Z}g$-module, then $E$ splits conjugately over $A$ modulo $\mathrm{FC}^\infty_{\mathbb{Z}G}(A)$. This result was generalized by Z.Y. Duan in [3]. In [6], FC-$DG$-decompositions were studied for $D$ belonging to a certain class of Dedekind domains which contians any ring of integers of an algebraic number field. There, the results on FC-$\mathbb{Z}G$-decompositions of [3] and [7] were generalized.

The main result of our paper is Theorem 2. If $X = E$, $D = \mathbb{Z}$, the composition length of $A$ is finite, and $G$ is a finite group, then Theorem 2 becomes Fitting's theorem (or lemma) in one of its group-theoretic versions. Since we do not rely on this result, our arguments can serve as its independent proof.

## Proof of the main result

Let X be a formation of finite groups and $G$ an XC-nilpotent group. Since any quotient group $X_{i+1}/X_i$ in the series $(*)$ is finite, we have $G/C_G(X_{i+1}/X_i) \in X$

for all $i \in \{0, \dots, n-1\}$. Let $N = I_{i=0}^{n-1} C_G(X_{i+1}/X_i)$. The definition of group formation implies that $G/N \in X$. Considering the series of groups

$$1 = X_0 \cap N \leq X_1 \cap N \leq \cdots \leq X_{n-1} \cap N \leq X_n \cap N = N,$$

we conclude that $N$ is nilpotent (of class $\leq n$). Then $G/\operatorname{Fitt}(G) \in X$, where $\operatorname{Fitt}(G)$ denotes the Fitting subgroup of $G$.

**Lemma 1.** *Let* X *be a formation of finite groups, $G$ a finite* XC*-nilpotent group, $F$ a field, and $A$ an $FG$-module. Suppose that $A$ includes an $FG$-submodule $B$ such that:*
*1) $B$ and $A/B$ are simple $FG$-modules;*
*2) $G/C_G(B) \in \mathrm{X}$;*
*3) $G/C_G(A/B) \notin \mathrm{X}$.*
*Then there exists an $FG$-submodule $C$ of $A$ such that $A = B \oplus C$.*

*Proof.* Put $p = \operatorname{char} F$. If $p = 0$ or $p \notin \pi(G)$, then the existence of the above-mentioned module $C$ follows from Maschke's theorem (see [5], Th.20.2.2).

Let us consider the case of $p \in \pi(G)$. Without loss of generality we may assume that $C_G(A) = 1$. Put

$$H = \bigcap \{W \mid W \text{ is a normal subgroup of } G \text{ such that } G/W \in X\}.$$

Since $G$ is XC-nilpotent, we have $G/\operatorname{Fitt}(G) \in X$. It is obvious that $H \leq \operatorname{Fitt}(G)$, thus $H$ is nilpotent. By Burnside-Wielandt's theorem (see [5], Th. 17.1.4), $H = P \times Q$, where $P$ is the Sylow $p$-subgroup of‘$H$ and $Q$ is the Sylow $p'$-subgroup of $H$.

Suppose that $Q = I$. Then $H = P \leq O_p(G)$. Since $A/B$ is a simple $FG$-module and $\operatorname{char} F = p$, we have $O_p\big(G/C_G(A/B)\big) = I$ (see [2], Th. A.13.6). Thus we obtain $O_p(G) \leq C_G(A/B)$, in particular, $H \leq C_G(A/B)$. From $G/H \in X$ and the definition of group formation it follows that $G/C_G(A/B) \in X$ which contradicts (3) of our lemma.

This contradiction shows that $Q \neq 1$. By Fitting's theorem (see [2], Th. A.12.5), we obtain the direct decomposition $A = C_A(Q) \oplus A\omega(FQ)$, where $\omega(FQ)$ is the augmentation ideal of the group ring $FQ$. Modules $C_A(Q)$ and $A\omega(FQ)$ are $FG$-submodules of $A$ because $Q$ is a normal subgroup of $G$. Since $Q \leq H$ and $G/(C_G(B) \in X$, we have $Q \leq C_G(B)$ and consequently $B \leq C_A(Q)$. From $C_G(A) = 1$ it follows that $A \neq C_A(Q)$. Thus $B = C_A(Q)$. Put $C = A\omega(FQ)$. This completes the proof of the lemma.

**Lemma 2.** *Let* X *be a formation of finite groups, $G$ a finite* XC*-nilpotent group, $F$ a field, and $A$ an $FG$-modules. Suppose that $A$ includes an $FG$-submodule $B$ such that:*
*1) $B$ and $A/B$ are simple $FG$-modules.*
*2) $G/C_G(B) \notin \mathrm{X}$.*
*3) $G/C_G(A/B) \in \mathrm{X}$.*
*Then there exists an $FG$-submodule $C$ of $A$ such that $A = B \oplus C$.*

*Proof.* Again we may assume that $\operatorname{char} F = p \in (G)$ and $C_G(A) = 1$. Put

$$H = \bigcap \{W \mid W \text{ is a normal subgroup of } G \text{ such that } G/W \in \mathrm{X}\}.$$

As in Lemma 1, it can be shown that $Q \neq 1$. From $Q \leq H$ we have $Q \leq C_G(A/B)$. By Maschke's theorem (see [5], Th. 20.2.2), $A = B \oplus E$ for some $FQ$-submodule $E$ of $A$. Since $Q \leq C_G(A/B)$, we have $A(x-1) \leq B$, in particular, $E(x-1) \leq B$ for all $x \in Q$. In addition, $E(x-1) \leq E$, thus $E(x-1) \leq E \cap B = 0$. Hence, $E \leq C_A(Q)$, in particular, $C_A(Q) \neq 0$. From $E \leq C_A(Q)$ we obtain $C_A(Q) \not\leq B$. The module $B$ is $FG$-simple, thus either $B \leq C_A(Q)$ or $B \cap C_A(Q) = 0$. If $B \leq C_A(Q)$, then from $FG$-simplicity of $A/B$ and from $C_A(Q) \not\leq B$ it follows that $C_A(Q) = A$. This, however, contradicts the equality $C_G(A) = 1$. As a result, $B \cap C_A(Q) = 0$. Since $A/B$ is a simple $FG$-module, it follows that $A = B \oplus C_A(Q)$. This completes the proof.

**Corollary 1.** *Let* X *be a formation of finite groups, $G$ a finite* XC*-nilpotent group, $D$ a Dedekind domain, and $A$ a $DG$-module. Suppose that $A$ includes a $DG$-submodule $B$ such that:*
*1) $B$ and $A/B$ are simple $DG$-modules.*
*2) $G/C_G(B) \in \mathrm{X}$.*
*3) $G/C_G(A/B) \notin \mathrm{X}$.*
*Then there exists a $DG$-submodule $C$ of $A$ such that $A = B \oplus C$.*

*Proof.* Since $G/C_G(B)$ is a finite group and $D$ is a noetherian ring, $B$ is a finitely generated noetherian $D$-module. Put $T = \{b \in B \mid b\lambda = 0 \text{ for some } 0 \neq \lambda \in D\}$, then $\operatorname{Ann}_D(T) = I \neq 0$. If $T \neq 0$, then $\operatorname{Ann}_D(B) \neq 0$ and $A$ can be viewed as a $D/\operatorname{Ann}_D(B)$-module, $D/\operatorname{Ann}_D(B)$ being a principal ideal ring (see [1], Corollary 18.22); consequently $BI \cap T = 0$. It is obvious that $BI$ and $T$ are $DG$-submodules of $B$. Since $B$ is a simple $DG$-module, it follows that either $T = 0$ or $BI = 0$.

If $T = 0$, then $B = \bigoplus_{i=1}^{n} E_i$, where $n$ is some natural number and $E_i$ is isomorphic with some fractional ideal of $D$ (see [1], Th. 22.5). It follows that $0 \neq BP \neq B$ for some maximal ideal $P$ of $D$, a contradiction.

Consequently, $BI = 0$ which implies $B = T$. Since $BJ$ is a $DG$-submodule of $B$ for every ideal $J$ of $D$, we see that $I$ is a maximal ideal of $D$. By analogy, $\operatorname{Ann}_D(A/B)$ is also a maximal ideal of $D$.

If $I \neq M$, then taking into account that

$$B = \{a \in A \mid ax = 0 \text{ for some } 0 \neq x \in I\}$$

and setting

$$C = \{a \in A \mid ax = 0 \text{ for some } 0 \neq x \in M\},$$

we obtain $A = B \oplus C$. Since $C$ is a $DG$-submodule, the result follows.

Suppose that $I = M$. Then $AM^2 = 0$. Let us prove that $AM = 0$. If $AM \neq 0$, then $AM = B$. There exists an element $y \in D$ such that $M/M^2 = (y + M^2)D$ (see [1], Corollary 18.22). The map $\varphi\colon a \to ay$, $a \in A$, is a $DG$-homomorpism; thus $\operatorname{Im}\varphi = B$. In the same way, $\operatorname{Ker}\varphi = B$. It follows that $B \cong_{DG} A/B$, in particular, $C_G(A) = C_G(A/B)$. This is a contradiction with 2) and 3) of our lemma.

Therefore $AM = 0$ and $A$ can be viewed as an $FG$-module, where $F = D/M$ is a field. The remaining step is then to apply Lemma 1.

**Corollary 2.** *Let* X *be a formation of finite groups, $G$ a finite* XC*-nilpotent group, $D$ a Dedekind domain, and $A$ a $DG$-module. Suppose that $A$ includes a $DG$-submodule $B$ such that:*
*1) $B$ and $A/B$ are simple $DG$-modules.*
*2) $G/C_G(B) \notin \mathrm{X}$.*
*3) $G/C_G(A/B) \in \mathrm{X}$.*
*Then there exists a $DG$-submodule $C$ of $A$ such that $A = B \oplus C$.*

As in proof of Corollary 1, it can be shown that $AM = 0$ for some maximal ideal $M$ of $D$. Put $F = D/M$. Then $A$ can be viewed as an $FG$-module and Lemma 2 can be applied.

**Theorem 1.** *Let X be a formation of finite groups, $G$ a finite* XC*-nilpotent group, $D$ a Dedekind domain, and $A$ a $DG$-module. If $A$ possesses a finite $DG$-composition series, then $A$ has an* XC*-$DG$-decomposition.*

This can be proved by applying induction to the $DG$-decoposition length of $A$ and using Corollaries 1 and 2 as its basis.

The following theorem is a generalization of one of the group-theoretic versions of Fitting's theorem, also known as Fitting's lemma.

**Theorem 2.** *Let* X *be a formation of finite groups, $G$ a finite* XC*-nilpotent group, $D$ a Dedekind domain, and $A$ a $DG$-module. Suppose $A$ possesses a system of $DG$-submodules $\{A_n\}$ ($n$ runs over the set of all natural numbers) such that:*
*1) $A_n \leq A_{n+1}$ for all natural $n$.*
*2) $A = \bigcup_{n=1}^{\infty} A_n$.*
*3) $A_{n+1}/A_n$ is a simple $DG$-module for all natural $n$.*
*Then $A$ has an* XC*-$DG$-decomposition.*

*Proof.* Let $n$ be a natural number. By Theorem 3 of [6], $A_n$ has an FC-$DG$-decomposition (where $F$ is the class of all finite groups viewed as a formation of finite group). In other words,

$$A_n = \mathrm{FC}^{\infty}_{DG}(A_n) \oplus \mathrm{FC}^{*}_{DG}(A_n).$$

Put $B_n \mathrm{FC}^{\infty}_{DG}(A_n)$ and $C_n = \mathrm{FC}^{*}_{DG}(A_n)$. The module $B_n$ has a finite $DG$-composition series such that $G/C_G(Y)$ is finite for each factor $Y$ of this series; therefore the group $G/C_G(B_n)$ is finite and $B_n$ can be viewed as a $D[G/C_G(B_n)]$-module. Thus it is possible to apply Theorem to the module $B_n$. Then

$$A_n = B_n + C_n = \mathrm{XC}^{\infty}_{DG}(B_n) + \mathrm{XC}^{*}_{DG}(B_n) + \mathrm{FC}^{*}_{DG}(A_n).$$

Note that $\mathrm{XC}^{\infty}_{DG}(A_n) = \mathrm{XC}^{\infty}_{DG}(B_n)$ and $\mathrm{XC}^{*}_{DG}(A_n) = \mathrm{XC}^{*}_{DG}(B_n) + \mathrm{FC}^{\infty}_{DG}(A_n)$. This yields a decomposition $A_n = \mathrm{XC}^{\infty}_{DG}(A_n) \oplus \mathrm{XC}^{*}_{DG}(A_n)$. Finally,

$$A_n = \sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \big(\mathrm{XC}^{\infty}_{DG}(A_n) + \mathrm{XC}^{*}_{DG}(A_n)\big) = \sum_{n=1}^{\infty} \big(\mathrm{XC}^{\infty}_{DG}(A_n)\big) + \sum_{n=1}^{\infty} \big(\mathrm{XC}^{*}_{DG}(A_n)\big) =$$

$$= \mathrm{XC}^{\infty}_{DG}\Big(\sum_{n=1}^{\infty} A_n)\Big) + \mathrm{XC}^{*}_{DG}\Big(\sum_{n=1}^{\infty} A_n)\Big) = XC^{\infty}_{DG}(A) \oplus \mathrm{XC}^{*}_{DG}(A).$$

The proof is now complete.

**Acknowledgments.** In closing, the author would like to express his deep appreciation for the guidance of the research and inspiration given to him by Prof. L.A. Kurdachenko.

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*Received 10.11.96*