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ON GROUPS WITH MANY ABELIAN-BY-FINITE QUOTIENTS

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Soluble JNAF-p-groups and periodic soluble JNAF-groups with abelian Fitting subgroup are characterized. It is proved that any locally nilpotent torsion-free JNAF-group either has the trivial Fitting subgroup or is a nilpotent group with locally cyclic centre. Nilpotent torsion-free JNAF-groups with divisible centre are also characterized.

1. Introduction. If $\mathcal{X}$ is a class of groups, a group G is said to be just-non-$\mathcal{X}$ (JN$\mathcal{X}$-group for short) if it is not in $\mathcal{X}$, but all its proper quotients are $\mathcal{X}$-groups. Many authors have studied just-non-$\mathcal{X}$ groups for various classes $\mathcal{X}$ (see [1–8]).

In this paper we study the soluble groups whose proper quotients are abelian-by-finite.

We characterize soluble JNAF-p-groups and periodic soluble JNAF-groups with abelian Fitting subgroup. It is proved that any locally nilpotent torsion-free JNAF-group either has the trivial Fitting subgroup or is a nilpotent group with locally cyclic centre. We also characterize nilpotent torsion-free JNAF-groups with divisible centre.

The notation is standard and can be found, e.g., in [4, 9, 10].

2. In this section we characterize the soluble JNAF-p-groups.

Lemma 2.1. *Let G be a p-group with abelian-by-finite proper quotients. If the centre $\mathrm{Z}(G)$ of G is nontrivial then G is hypercentral.*

Proof. Let a be a central element of prime order. Then the quotient $G/\langle a\rangle$ is abelian-by-finite and, therefore, is hypercentral. As a consequence, G is hypercentral, too.

Lemma 2.2. *Any soluble p-group G with abelian-by-finite proper quotients is hypercentral.*

Proof. Let A_1 be an abelian normal subgroup of G. The subgroup $\Omega(A_1) = \{x \in A_1 \mid x \text{ is of order} \le p\}$ is normal in G. Without loss of generality we can suppose that $A_1 = \Omega(A_1)$; i.e., A_1 is an abelian group of exponent p. Further, G/A_1 is abelian-by-finite and, therefore, is hypercentral. Let xA_1 be a nontrivial central element of G/A_1. Then the subgroup $N = \langle A_1, x\rangle$ is nilpotent, by theorem of

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G. Baumslag [11]. Moreover, N is normal in G. Since the quotient G/N' is hypercentral, G is hypercentral, by theorem of A. Betten [12], and the proposition is proved.

Lemma 2.2 implies that every soluble JNAF-p-group is hypercentral.

Lemma 2.3. *Let G be a hypercentral p-group with abelian-by-finite proper quotients.*

(i) *If the centre $\mathrm{Z}(G)$ of G is not locally cyclic then G is abelian-by-finite.*

(ii) *If $\mathrm{Z}(G)$ is locally cyclic, then G is abelian-by-finite if and only if it is Černikov group.*

Proof. Remak's theorem implies (i).

For proving (ii) one can use the theorem of Černikov [9, Theorem 1.7].

Theorem 2.4. *A periodic hypercentral group G is a* JNAF*-group if and only if the following conditions hold:*

(i) *G is a p-group,*

(ii) *the centre $Z(G)$ is locally cyclic,*

(iii) *G does not satisfy the minimal condition for all subgroups,*

(iv) *G has a normal subgroup N of finite index and the derived subgroup N' of N is of order p.*

Proof. "If". Let G be a periodic hypercentral JNAF-group. Condition (i) is obvious. Lemma 2.3 implies (ii). Moreover, since G is a JNAF-group, condition (iii) holds, too. For (iv), let a be a central element of order p. Then the quotient $G/\langle a\rangle$ contains a normal abelian subgroup of finite index.

"Only if". Let G satisfy (i)–(iv). Let B be a normal subgroup of finite index with the derived subgroup B' of order p. Then B' is normal in G and, moreover, $B' \leq \mathrm{Z}(G)$.

Since every normal subgroup N contains a nontrivial central element of G then $B' \leq N$ and, therefore, every proper quotient of G is abelian-by-finite. Remark, by Lemma 2.3 the group G contains no abelian subgroup of finite index.

Example 2.5.(see [9, p.39]). There exists a hypercentral periodic JNAF-group G. Indeed, let p be an odd prime. Let A (respectively, B) be an abelian group of exponent p generated by elements $a_1, \dots, a_n, \dots$ (respectively $b_1, \dots, b_n, \dots$). Denote by G the group determined by the following relations:

$$b_m^{-1} a_n b_m = a_n \quad (m \neq n), \qquad b_n^{-1} a_0 b_n = a_n a_0 \quad (n = 1, 2, \dots).$$

Clearly, this group is nilpotent of class 2 and $\mathrm{Z}(G) = \langle a_0\rangle$. Moreover, G does not satisfy the minimal condition for subgroups. Hence, the group G has no abelian subgroup of finite index.

Corollary 2.6. *A soluble p-group G is a* JNAF*-group if and only if it is a hypercentral* JNAF*-group.*

Proposition 2.7. *Let G be a hypercentral* JNAF*-p-group. Then there is an embedding of G into the unrestricted standard wreath product $G_0 \,\mathrm{wr}\, F$ of a nilpotent p-group G_0 whose proper quotients are abelian and a finite group F.*

Proof. Let G be a hypercentral JNAF-p-group. Then there is a normal subgroup B of G with a locally cyclic centre $\mathrm{Z}(B)$ and the derived subgroup B' of order p. Moreover, B does not satisfy the minimal condition for subgroups.

Let N be a maximal normal subgroup of B such that $N \cap B' = 1$. Put $G_0 = B/N$. If $N \leq M \rhd B$ then $B' \leq M$ and the quotient B/M is abelian. Then there is an embedding of G into $G_0 \operatorname{wr}(G/B)$ by (4.2) [4].

Remark 2.8. The group G_0 from (2.7) is nilpotent JNA-p-group of class 2. The JNA-groups were studied in [1,2].

3. In this section the class of periodic soluble JNAF-group with abelian Fitting subgroup is characterized.

Lemma 3.1. *Let G be a* JNAF*-group. If the Fitting subgroup $F(G)$ of G is non-abelian then G is nilpotent-by-finite. If, further, G is periodic then $F(G)$ is a p-group.*

Proof. Since the quotient $G/F(G)$ is abelian-by-finite, there exists a normal subgroup B of finite index such that the quotient $B/F(G)'$ is abelian. By the theorem of Ph. Hall (see [15, Part 1, p.117]) B is nilpotent and, therefore, G is nilpotent-by-finite.

Lemma 3.2. *Let G be a periodic* JNAF*-group. If the Fitting subgroup $F(G)$ of G is abelian then it is an elementary p-group for some prime p.*

Proof. Let S be the socle of the Fitting subgroup $F(G)$ of G. Clearly S is an elementary abelian p-group for some prime p. Assume that $S \neq F(G)$. Then G contains a normal subgroup B of finite index with abelian quotient B/S. Moreover,

$$1 = [a, x]^p = [a^p, x]$$

for each element x of B and for each element a of $F(G)$. So $\mathrm{Z}(B)$ is nontrivial. Then in G there exists a normal subgroup B_1 of finite index with abelian quotient $B_1/Z(B)$. Since $|G : B \cap B_1| < \infty$ and

$$(B \cap B_1)' \leq B_1' \leq Z(B) \leq Z(B \cap B_1),$$

the group G is nilpotent-by-finite. Hence it is abelian-by-finite, a contradiction. Thus, $S = F(G)$, as required.

Lemma 3.3. *Let G be a periodic soluble* JNAF*-group with abelian Fitting subgroup $F(G)$. Then in G there exists a normal subgroup B of finite index with abelian quotient $B/F(G)$ and every nontrivial element of $B/F(G)$ acts fixed-point-freely on $F(G)$.*

Proof. Let $xF(G)$ be a nontrivial element of $B/F(G)$, where B is a normal subgroup of finite index in G with abelian quotient $B/F(G)$. The map

$$\tau \colon F(G) \longrightarrow F(G)$$

given by the rule $\tau(a) = [a, x]$ for $a \in F(G)$ is a G-homomorphism, so that $\operatorname{Ker} \tau$ and $\operatorname{Im} \tau$ are normal subgroups of G. Assume that $\operatorname{Ker} \tau$ is nontrivial. Then the quotient $G/\operatorname{Ker} \tau$ is abelian-by-finite and so in G there exists a normal subgroup B_1

of finite index with abelian quotient $B_1/\operatorname{Ker}\tau$. Then $F(G)/\operatorname{Ker}\tau$ is a polytrivial B-module and $\operatorname{Im}\tau$ is also a polytrivial B-module. Clearly, $\operatorname{Im}\tau$ is nontrivial since $C_G(F(G)) = F(G)$, and therefore $B/\operatorname{Im}\tau$ is abelian-by-finite. Thus in G there exists a normal subgroup B_2 of finite index such that $B_2/\operatorname{Im}\tau$ is abelian. Hence B_2 is abelian and G is abelian-by-finite. This contradiction shows that $\operatorname{Ker}\tau$ is trivial, and this means that x acts fixed-point-freely on $F(G)$, as required.

Lemma 3.4. *Let G be a periodic soluble* JNAF*-group with abelian Fitting subgroup $F(G)$ and let B be a normal subgroup of finite index of G. If B contains $F(G)$ and $B/F(G)$ is abelian, then*
(i) *the quotient $B/F(G)$ has no elements of order p;*
(ii) *B splits over $F(G)$ and $F(G)$ is the only minimal normal subgroup of G.*

Proof. (i) Suppose $xF(G)$ is an element of $B/F(G)$ whose order is a power of p. Since $F(G)$ is an elementary abelian p-group, we see that the normal subgroup $\langle F(G), x\rangle$ of B is nilpotent [11]. But $F(G) = F(B)$ and hence $\langle F(G), x\rangle = F(G)$. Consequently $xF(G) = \overline{1}$.
(ii) Let $zF(G)$ be an element of $B/F(G)$ which has a prime order q and $q \neq p$. From Lemma 2.3 it follows that z acts fixed-point-freely on $F(G)$, and thus $C_{F(G)}(z) = 1$. Therefore,

$$F(G) = [F(G), z] \times C_{F(G)}(z) = [F(G), z] \qquad (*)$$

(see, e.g., [13, Lemma 1]). For every G-invariant subgroup N of $F(G)$ the quotient B/N is abelian-by-finite. Hence there exists a subgroup B_1 such that B_1/N is abelian and $|B : B_1| < \infty$. From $(*)$ it follows that $F(G) = [F(G), B_1]$ and hence $N = F(G)$. Since $B/F(G)$ is abelian, by [14, Corollary 1] B splits over $F(G)$ and this completes the proof.

Using Lemma 3.2 and Lemma 3.4, we can easily prove the following result.

Theorem 3.5. *Let G be a periodic soluble group with abelian Fitting subgroup $F(G)$. Then the following conditions are equivalent.*
(i) *G is a* JNAF*-group.*
(ii) *G is a finite extension of a normal subgroup $B = F(G) \rtimes L$, where $F(G)$ is an infinite elementary abelian p-group and L is an infinite abelian group without elements of order p, acting faithfully and irreducibly on $F(G)$.*

Proof. Lemma 3.2 and Lemma 3.4 imply (i)$\Rightarrow$(ii).

Conversely, since $F(G)$ is minimal normal and $C_G(F(G)) = F(G)$, we obtain that every proper quotient of G is abelian-by-finite. But G is not abelian-by-finite since $[L, F(G)] \neq 1$.

4. In this section we characterize the torsion-free locally nilpotent $JNAF$-groups with divisible centre.

Lemma 4.1. *Let G be a non-nilpotent locally nilpotent torsion-free group with abelian-by-finite proper quotients. If its Fitting subgroup $F(G)$ is non-trivial then $G/F(G)$ acts fixed-point-freely on $F(G)$.*

Proof. Since $F(G)$ coincides with its isolator $I_G(F(G))$ in G, the quotient $G/F(G)$ is torsion-free and hence abelian. Let $xF(G)$ be a non-trivial element of $G/F(G)$. Consider the mapping $\tau : F(G) \longrightarrow F(G)$ defined by the rule $\tau(a) = [a, x]$, where

$a \in F(G)$. Clearly, τ is a G-homomorphism and the subgroups $\operatorname{Ker}\tau$ and $\operatorname{Im}\tau$ are normal in G.

Assume that $\operatorname{Ker}\tau$ is non-trivial. Ther $G/\operatorname{Ker}\tau$ is abelian and $[G, F(G)] \leq \operatorname{Ker}\tau$. We obtain that $\operatorname{Im}\tau$ is a polytrivial G-module. Since $\operatorname{Im}\tau$ is non-trivial, the quotient $G/\operatorname{Im}\tau$ is abelian and thus G is nilpotent, a contradiction.

Lemma 4.2. *Let G be a locally nilpotent torsion-free group with abelian-by-finite proper quotients. If the Fitting subgroup $F(G)$ of G is nonabelian then G is nilpotent.*

Proof. Let N be a nonabelian nilpotent subgroup of G. Then the quotient $G/I_G(N')$ is abelian, where $I_G(N')$ is an isolator of N in G. Moreover, $I_G(N') = I_G(N)'$ and thus $I_G(N)$ is nilpotent. Now we can apply a well-known result of Ph. Hall.

Lemma 4.3. *Let G be a locally nilpotent torsion-free group with abelian-by-finite proper quotients. If G contains a nontrivial normal nilpotent subgroup then G is nilpotent.*

Proof. Assume that G is non-nilpotent. By Lemma 4.2 the Fitting subgroup $F(G)$ is abelian. Let $xF(G)$ be a nontrivial element of $G/F(G)$ and let a be a nontrivial element of $F(G)$. Since $F(G) \cap \langle a, x\rangle$ is a non-trivial normal subgroup of the nilpotent group $\langle a, x\rangle$, $F(G) \cap \mathrm{Z}(\langle a, x\rangle)$ is nontrivial, a contradiction with Lemma 4.1. Hence $x \in F(G)$ and $G = F(G)$, and this completes the proof.

Corollary 4.4. *Let G be a locally nilpotent torsion-free group with abelian-by-finite proper quotients. Then either G is a nilpotent group or its Fitting subgroup is trivial. Moreover, if G is a* JNAF*-group then its centre $\mathrm{Z}(G)$ is a locally cyclic subgroup.*

In connection with the following proposition, let us remark that the JNCF-groups were studied in [6]. Recall that a group G is said to be a JNCF-group if it is not central-by-finite, but all its proper quotients are central-by-finite groups.

Proposition 4.5. *Let G be a nilpotent torsion-free group with divisible centre $\mathrm{Z}(G)$. Then G is a* JNAF*-group if and only if G is a finite extension of a* JNCF*-group B with $\mathrm{Z}(G) \leq B$.*

We need two auxiliary results.

Lemma 4.6. *Let G be a nilpotent torsion-free* JNAF*-group and a is a nontrivial central element of G. If $B = \max\{N \mid N \lhd G, |G : N| < \infty$ and the quotient $N/\langle a\rangle$ is abelian$\}$, then $\mathrm{Z}(B) = \mathrm{Z}(G)$.*

Proof. Assume that $\mathrm{Z}(G) \neq \mathrm{Z}(B)$. Then $\mathrm{Z}(B) = \mathrm{Z}(G) \times A$ for some central subgroup A of B.

Show that A is isolated in G. Assume that there exists an element $x \notin A$ such that $x^n \in A$ for some positive integer n.

Clearly, $C_{G/\langle a\rangle}(B/\langle a\rangle) = B/\langle a\rangle$. Moreover, $C_G(B)/\langle a\rangle \leq C_{G/\langle a\rangle}(B/\langle a\rangle)$. Hence $C_G(B)/\langle a\rangle \leq B/\langle a\rangle$ and consequently $C_G(B) \leq B$.

If $x^n \in A$, then $x^n \in Z(B)$ and consequently $x \in Z(B)$. Hence, there are $z \in Z(G)$ and $a \in A$ such that $x = za$. Therefore, $x^n = z^n a^n \in A$ and as consequence $z^n = a_1 a^{-n} \in Z(G) \cap A = 1$. Thus, $z = 1$ and $x \in A$, a contradiction. This means that A is an isolated subgroup of G. Moreover, the normalizer $\mathrm{N}_G(A)$

is also isolated in G. Since $|G : \mathrm{N}_G(A)| < \infty$, we see that $G = \mathrm{N}_G(A)$ and hence A is normal in G. As a consequence, G is abelian-by-finite, a contradiction. Thus $A = 1$ and $\mathrm{Z}(G) = \mathrm{Z}(B)$.

Lemma 4.7. *Any nilpotent abelian-by-finite* FC-*group G is a finite extension of its centre $Z(G)$.*

The proof is immediate.

Proof of 4.5. "If". Let a be a nontrivial central element of G and B a maximal normal subgroup of finite index in G with abelian quotient $B/\langle a\rangle$. By Lemma 4.6, $\mathrm{Z}(B) = \mathrm{Z}(G)$. If N is a nontrivial normal subgroup of B, then $N \cap Z(G) \neq 1$ and thus $N \cap \langle a\rangle \neq 1$. Thus $(B/N)' \cong B'N/N \cong \langle a^m\rangle/(\langle a^m\rangle \cap N)$ for some positive integer m, and therefore $(B/N)'$ is finite. Thus every proper quotient of B has a finite derived subgroup. Clearly, B/N is abelian-by-finite. By Lemma 4.7, B is a JNCF-group.

"Only if". If G is a finite extension of JNCF-group B with $Z(G) \leq B$, then, obviously, G is a JNAF-group. The proposition is proved.

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REFERENCES

[1] Newman M.F., *On a class of metabelian groups*, Proc. London Math. Soc. **3** (1960), no. 10, 354–364.

[2] Newman M.F., *On a class of nilpotent groups*, Proc. London Math. Soc. **3** (1960), no. 10, 365–375.

[3] Wilson J.S., *Groups with every proper quotient finite*, Math. Proc. Cambridge Phil. Soc. **69** (1971), 373–391.

[4] Robinson D.J.S., Wilson J.S., *Soluble groups with many polycyclic quotients*, Proc. London. Math. Soc. **3** (1984), no. 48, 193–229.

[5] Franciosi S, de Giovanni F., *Soluble groups with many Černikov quotients*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **79** (1985), no. 8, 19–24.

[6] Robinson D.J.S., Zhirang Zhang, *Groups whose proper quotients have finite derived subgroups*, J. Algebra **118** (1988), 346–368.

[7] S. Franciosi, F. Giovanni, L.A. Kurdachenko, *Groups whose proper quotients are* FC-*groups*, Universitá degli di Napoli, preprint (1995), no. 64.

[8] Курдаченко Л.А., Пылаев В.В., *О группах с минимаксными фактор-группами*, Укр. мат. ж. **42** (1990), no. 5, 620–625.

[9] С.Н. Черников, *Группы с заданными свойствами системы подгрупп*, М.: Наука, 1980.

[10] D.J.S. Robinson, *A course in the theory of group*, Springer, New York e.a., 1982.

[11] G .Baumslag, *Wreath products and p-groups*, Proc. Cambridge Phil. Soc. **55** (1959), 224–231.

[12] A. Betten, *Hinreichende Kriterien fur die Hyperzentralität einer Gruppen*, Arch. Math. **20** (1969), 471–480.

[13] Franciosi S., de Giovanni F., *On torsion groups with nilpotent automorphis groups*, Commut. Algebra **14** (1986), 1909–1935.

[14] D.J.S. Robinson, *Splitting theorem for infinite groups*, Symposia Math. **17** (1976), 441–470.

[15] D.J.S. Robinson, *Finiteness conditions and generalized soluble groups*, Springer, Berlin.

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