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TRIPLE SUMS OF ABELIAN LIE ALGEBRAS

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For a Lie algebra L (over an arbitrary field) of the form $L = A + B = A + N = B + N$ with abelian subalgebras A, B and an abelian ideal N. We construct an associative algebra R over the same field such that the adjoint Lie algebra $R^{(-)}$ is isomorphic to L. Applying the structure results for associative algebras we prove that in finite dimensional case such a Lie algebra L contains a nilpotent ideal I such that L/I is the direct product of nonabelian two-dimensional Lie algebras.

Analogously to triple factorizations of groups (see, for example [1]) one can study triple sums of Lie algebras: a Lie algebra L is decomposable into a triple sum of its subalgebras A, B, C if $L = A + B = A + C = B + C$. Triple sums of abelian Lie algebras appear in investigations of ideals of a Lie algebra L which can be decomposed into a sum $L = A + B$ of nilpotent or close to nilpotent subalgebras A and B.

In the present paper for a triple sum $L = A + B = A + N = B + N$ of abelian subalgebras A, B and an abelian ideal N (over an arbitrary field) an associative algebra R over the same field is constructed such that the adjoint Lie algebra $R^{(-)}$ is isomorphic to L. In the finite dimensional case this associative algebra is easier to study and a return to Lie algebras gives a description of such sums. In contrast to the group theory (in finite case) this triple sum can be non-nilpotent.

All notations in the paper are standard, Lie products are left-normed. For an arbitrary linear algebra A over a field (x, y, z) denotes the associator of the elements x, y, z, i.e. $(x, y, z) = (xy)z - x(yz)$, $A^{(-)}$ denotes a linear algebra on the same vector space A with multiplication rule: $[x, y] = xy - yx$; $A^{(+)}$ denotes the linear algebra on the vector space A with zero multiplication.

Let A be an arbitrary linear algebra over a field K. Define the multiplication $(a_1, b_1)\cdot \cdot(a_2, b_2) = (a_1a_2, b_1a_2)$ on the vector spase $A \bigoplus A$. It is easy to see that $A \bigoplus A$ with this multiplication is a linear algebra; we will denote it by A^*. Clearly, the subspace $A^{(+)} = \{(0, b) \mid b \in A\}$ is a two-sided ideal of the algebra A^* and $(A^{(+)})^2 = 0$, the subspace $\{(a, 0) \mid a \in A\}$ is a subalgebra in A^* which is isomorphic

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to the algebra A. We will also denote $A^* = A \leftthreetimes A^{(+)}$ (the semi-direct product of two linear algebras).

Lemma 1. *Let L be a Lie algebra over an arbitrary field K which is decomposed into a triple sum $L = A + B = A + N = B + N$, where A, B are subalgebras of L and N is an abelian ideal of algebra L. If $A \cap B = A \cap N = B \cap N = 0$ then a multiplication $(\cdot)$ on the vector space B can be defined in a such way that $B_0 = B(\cdot)$ is a linear algebra which satisfies the identity $(x, y, z) = (x, z, y)$, $B_0^{(-)}$ is a Lie algebra which is isomorphic to the Lie algebra B and the adjoint algebra $B^{*(-)}$ is isomorphic to the Lie algebra L.*

Proof. Let b_1 and b_2 be two arbitrary elements from the subalgebra B. It is easy to see that there exist uniquely defined elements $n_1, n_2 \in N$ and $a_1, a_2 \in A$ such that $n_1 = a_1 + b_1, n_2 = a_2 + b_2$. Define the multiplication on B by the rule: $b_1 b_2$ is the B-component of the element $[n_1, b_2]$, i.e. $[n_1, b_2] = b_1 b_2 + a$ for some element $a \in A$. Clearly, B_0 is a linear algebra over K (with nonzero multiplication if N is not in the centre of L). From the identity

$$[n_1, n_2] = [a_1 + b_1, a_2 + b_2] = [a_1, a_2] - [b_1, b_2] + [n_1, b_2] - [n_2, b_1]$$

and the equality $[n_1, n_2] = 0$ it follows that $b_1 b_2 - b_2 b_1 = [b_1, b_2]$, where $[b_1, b_2]$ is the product of elements in the Lie algebra L. Hence, $B_0^{(-)} = B$.

Further, from the identity

$$[n_1, b_2, b_3] - [n_1, b_3, b_2] = [n_1, [b_2, b_3]]$$

it follows that

$$(b_1 b_2)b_3 - (b_1 b_3)b_2 = b_1(b_2 b_3 - b_3 b_2), \text{ i.e. } (b_1 b_2)b_3 - b_1(b_2 b_3) = (b_1 b_3)b_2 - b_1(b_3 b_2).$$

This means the equality $(b_1, b_2, b_3) = (b_1, b_3, b_2)$ for every elements $b_1, b_2, b_3 \in B_0$. Prove that the algebra $B^* = B_0 \leftthreetimes B_0^{(+)}$ also satisfies the identity $(x, y, z) = (x, z, y)$. For arbitrary elements $(b_i, b_i'), b_i \in B_0, b_i' \in B_0^{(+)}, i = 1, 2, 3$ we have

$$\{(b_1, b_1')(b_2, b_2')\}(b_3, b_3') - (b_1, b_1')\{(b_2, b_2')(b_3, b_3')\} = (b_1 b_2, b_1' b_2)(b_3, b_3') - (b_1, b_1')(b_2 b_3, b_2' b_3)$$
$$((b_1 b_2)b_3, (b_1' b_2)b_3) - (b_1(b_2 b_3), b_1'(b_2 b_3)) = ((b_1, b_2, b_3), (b_1', b_2, b_3)).$$

Analogously, one can show that

$$\{(b_1, b_1')(b_3, b_3')\}(b_2, b_2') - (b_1, b_1')\{(b_3, b_3')(b_2, b_2')\} = ((b_1, b_3, b_2), (b_1', b_3, b_2)).$$

From the last two equalities it follows that

$$((b_1, b_1'), (b_2, b_2'), (b_3, b_3')) = ((b_1, b_1'), (b_3, b_3'), (b_2, b_2'))$$

and hence the linear algebra B^* satisfies the identity $(x, y, z) = (x, z, y)$. It is also easy to see that the adjoint Lie algebra $B^{*(-)}$ is isomorphic to the Lie algebra L. The lemma is proved.

Remark 1. One can show that every linear algebra A which satisfies the identity $(x, y, z) = (x, z, y)$ is Lie admitted, i.e. $A^{(-)}$ is a Lie algebra.

Lemma 2. *Let R be a linear algebra which satisfies the identity $(x, y, z) = (x, z, y)$. If the algebra R is commutative then R is associative.*

Proof. By the condition of the lemma

$$(xy)z - x(yz) - (xz)y + (x(zy) = 0.$$

Then

$$(xy)z - (xz)y = x(yz - zy) = 0$$

because of commutativity of the algebra R. Thus the algebra R satisfies the identity $(xy)z = (xz)y$. Then $(xy)z = (yx)z = (yz)x = x(yz)$, i.e. the algebra R is associative.

Proposition 1. *Let L be a Lie algebra over an arbitrary field K which is decomposed into a triple sum $L = A + B = A + N = B + N$ of abelian subalgebras A, B and an abelian ideal N of algebra L. If $A \cap B = A \cap N = B \cap N = 0$ then a structure of an associative-commutative algebra on the vector space B over the same field can be defined such that the adjoint Lie algebra $B^{*(-)}$ to the associative algebra $B^* = B \leftthreetimes B^{(+)}$ is isomorphic to the Lie algebra L. Further, to every ideal I of algebra L of the form*

$$I = A_I + B_I = N_I + A_I = N_I + B_I,\ A_I \subseteq A,\ B_I \subseteq B,\ N_I \subseteq N$$

corresponds an ideal I^* of the algebra B^* such that $I^{*(-)} \simeq I$.

Proof. Define a multiplication on the vector K-space B accordingly to Lemma 1. Then B is a commutative linear algebra which satisfies the identity $(x, y, z) = (x, z, y)$ and by Lemma 2 the algebra B is associative. It is easy to check that the linear algebra $B^* = B \leftthreetimes B^{(+)}$ is also associative (not necessarily commutative). Then $B^{*(-)} \simeq L$ by Lemma 1.

Let now I be an ideal of algebra L which is decomposed into a triple sum

$$I = A_I + B_I = A_I + N_I = B_I + N_I,\ \ A_I \subseteq A, B_I \subseteq B, N_I \subseteq N$$

(N_I is an ideal in L). Show that the K-space B_I which is a projection of I onto B is an ideal of associative-commutative algebra B. Let $b_1 \in B_I, b \in B$. Then there exists an element $n_1 \in N_I$ such that $n_1 = a_1 + b_1$ for some element $a_1 \in A_I$. By the definition the element $b_1 b$ is a B-component of the element $[n_1, b]$. Clearly, $N_I = N \cap I$ is an ideal of the algebra L and hence $[n_1, b] \in N_I$. Then $b_1 b \in B_I$ and B_I is a right ideal of the algebra B. Since the algebra B is commutative, B_I is an ideal of the algebra B. It is easy to see that $B_I^* = B_I \leftthreetimes B_I^{(+)}$ is an ideal of the algebra B^* and the adjoint Lie algebra $B_I^{*(-)}$ is isomorphic to Lie algebra I. The proposition is proved.

Theorem. *Let L be a non-nilpotent finite dimensional Lie algebra over an algebraic closed field which is decomposed into a triple sum*

$$L = A + B = A + N = B + N$$

of abelian subalgebras A, B and an abelian ideal N such that $A \cap B = A \cap N = B \cap N = 0$. Then L contains a nilpotent ideal I (may be $I = 0$) which is decomposed into a triple sum $I = A_I + B_I = A_I + N_I = B_I + N_I$ of some subalgebras $A_I \subseteq A$, $B_I \subseteq B$ and an ideal $N_I \subseteq N$ and some subalgebra D which is decomposed into the direct product of non-abelian two-dimensional Lie algebras and

$$L = D + I, \quad D \cap I = 0.$$

Proof. Let B be the associative-commutative algebra which is constructed on the K-space B accordingly to Lemma 1. Denote by S the Jacobson radical of algebra B. Then S is a nilpotent ideal and the ideal $S^* = S \leftthreetimes S^{(+)}$ of the associative algebra B^* is also nilpotent. The semi-simple commutative quotient-algebra $\overline{B} = B/S$ can be lifted modulo the ideal S by the Maltsev-Weddeburne theorem (see for example [2, §6.2]). Thus there exists a subalgebra B_0 of the algebra B with a basis $\{e_1, \dots, e_n\}$ and the rule of multiplication $e_i^2 = e_i, e_i e_j = 0, i \neq j$. Let B_0^* be the corresponding subalgebra of the algebra B^*. Clearly, $B^* = S^* + B^*$ and the adjoint Lie algebra $B_0^{*(-)}$ is the direct product of non-abelian two-dimensional Lie algebras. Turning to Lie algebra $B^{*(-)} \simeq L$ we get the statement of Theorem.

Remark 2. In investigations of triple factorizable groups an approach is used which is connected with consideration of the adjoint group of radical ring (see [3]).

The results of this paper were announced in [4].

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