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NOETHERIAN SEMI-PERFECT RINGS OF DISTRIBUTIVE MODULE TYPE

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We prove that any noetherian semi-perfect ring of distributive module type is biserial.

§**1. Introduction.** All rings considered in the paper are associative with $1 \neq 0$, modules are right and unitary. Further, noetherian (etc.) ring means two-side noetherian (etc.) ring.

Recall that a module M is called distributive if $K \cap (L+N) = K \cap L + K \cap N$ for any submodules K, L, N. Clearly, submodules and quotient modules of a distributive module are distributive. A module is called semi-distributive if it is a direct sum of distributive modules. A ring is called right (left) semi-distributive if it is a right (left) semi-distributive module over itself. A right and left semi-distributive ring is called semi-distributive.

Theorem 1.1. [1]. *A module is distributive if and only if the socle of every its quotient module contains at most one copy of each simple module.*

Let R be the Jacobson radical of a ring A. A ring A is called semi-perfect if the quotient ring A/R is artinian and the idempotents can be lifted modulo R [2].

An idempotent $e \in A$ is called local if the ring eAe is local.

Theorem 1.2. [3]. *A ring A is semi-perfect if and only if the unity of A can be decomposed into a sum of mutually orthogonal local idempotents.*

Theorem 1.3. [4, §11.4], [5, §7]. *A ring A is semi-perfect if and only if it decomposes into direct sum of right ideals such that each one has exactly one maximal submodule.*

Denote by M^n the direct sum of n copies of a module M, $M^0 = 0$. Therefore, any semi-perfect ring A can be represented as the sum of right ideals $A = P_1^{n_1} \oplus \cdots \oplus P_s^{n_s}$, where $P_1, \dots, P_s$ are pairwise nonisomorphic modules and $U_i = P_i/P_iR$, $i = 1, \dots, s$, are simple. Modules $P_1, \dots, P_s$ exhaust up to an isomorphism all indecomposable projective A-modules, while $U_1, \dots, U_s$ exhaust all nonisomorphic simple A-modules [6].

A semi-perfect right (left) semi-distributive ring is called an $SPSDR$- ($SPSDL$-) ring and semi-perfect semi-distributive ring is called an $SPSD$-ring.

Theorem 1.4. [7] (see also [8, Theorem 4]). *A semi-perfect ring A is an $SPSDR$- ($SPSDL$-)ring if and only if for any local idempotents e and f of the ring A the set eAf is a uniserial right fAf-module (uniserial left eAe-module).*

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Corollary 1.5. [8]. *Let A be a semi-perfect ring, $1 = e_1 + \dots + e_n$ a decomposition of $1 \in A$ into the sum of mutually orthogonal local idempotents. The ring A is an $SPSDR$- ($SPSDL$-)ring if and only if for any idempotents e_i and $e_j (i \neq j$ from the decomposition above the ring $(e_i + e_j)A(e_i + e_j)$ is an $SPSDR$- ($SPSDL$-) ring.*

Corollary 1.6. [8]. *Let A be a noetherian $SPSD$-ring, $1 = e_1 + \dots + e_n$ a decomposition of the unity $1 \in A$ into the sum of mutually orthogonal local idempotents, $A_{ij} = e_i A e_j$ and R_i the Jacobson radical of the ring A_{ii}. Then $R_i A_{ij} = A_{ij} R_j$ for any $i, j = 1, \dots, n$.*

A module M over a ring A is called finitely-presented, if there exists an exact sequence $P_1 \to P_0 \to M \to 0$, where P_1 and P_0 are finitely-generated projective A-modules.

Definition 1. [9]. A ring A is called a ring of distributive module type (DMT-ring) if every finitely presented A-module M is semi-distributive.

Proposition 1.7. [9]. *Let A be a semi-perfect DMT-ring, then the ring eAe is a DMT-ring for any nonzero idempotent $e \in A$.*

An indecomposable module M is called biserial if it is distributive and contains chain submodules K_1 and K_2 (possibly zero) such that $K_1 + K_2$ is M or a maximal submodule in M and $K_1 \cap K_2$ is trivial or simple [10].

An artinian ring A is called biserial if each left and right projective indecomposable A-module is biserial [10].

Definition 2. [11]. A semi-perfect ring A is called biserial if every right and every left indecomposable projective A-module is biserial.

Theorem 1.8. [12]. *An artinian DMT-ring is biserial.*

The aim of the present paper is the proof of the following main theorem:

Theorem I. *A noetherian semi-perfect DMT-ring is biserial.*

Definition 3. [13]. A finite oriented graph Q (a quiver by Gabriel) is called biserial if each point of Q is a source of at most two arrows and each point of Q is a sink of at most two arrows, and Q does not contain double arrows.

Let A be a semi-perfect ring such that the quiver $Q(A)$ is well defined (def. in [8], p.465). We define a quiver $RQ(A)$ of A that is closely connected with the structure of A-modules [14], [15].

Let $Q(A)$ contain n vertices $1, \dots, n$ and assume that there exist t_{ij} arrows from i to j. We define $RQ(A)$ as a quiver with vertices $1, \dots, n, \tau(1), \dots, \tau(n)$, where there exist t_{ij} arrows from i to $\tau(j)$. Thus $RQ(A)$ is a biparted graph such that each arrow has a source in $1, \dots, n$ and a sink in $\tau(1), \dots, \tau(n)$.

For an oriented graph Q we will denote by $\bar{Q}$ a non-oriented graph with the same set of vertices and arrows as Q. Non-oriented graph of the type: $1 \to 2 \to 3 \to \dots \to m-1 \to m$ is called a non-oriented chain. It is the Dynkin diagram A_m.

The following theorem is a corollary from the same results of [13, §5].

Theorem 1.9. *Let A be a noetherian semi-perfect DMT-ring. Then A is an $SPSD$-ring with a biseriai quiver $Q(A)$ and $\bar{R}Q(A)$ is a disjoint union of the Dynkin diagrams A_m.*

Let R be the Jacobson radical of a semi-perfect ring A.The ring A is called reduced if the quotient ring A/R is a direct product of skew fields [6].

By the theorem of Morita (see, for example, [17]) the category of modules over an arbitrary semi-perfect ring is equivalent in a natural way to the category of modules over a reduced ring. Therefore, while studying semi-perfect DMT-rings one can consider only reduced rings. Indecomposable reduced noetherian semi-perfect DMT-rings will be called "good" rings (G-rings).

§2. Minors of noetherian semi-perfect ring of a module distributive type. Following [16] we will call by minor of order n of a ring A a ring B of endomorphisms of a finite generated projective A-module which can be decomposed into a direct sum of n indecomposable modules. From Proposition 1.7 one can obtain the following result.

Proposition 2.1. *Every minor of a noetherian semi-perfect DMT-ring is a noetherian semi-perfect DMT-ring.*

Proposition 2.2. [13 §5]. *A local DMT-ring is uniserial.*

Corollary 2.3. [6]. *A noetherian uniserial ring A is either a discrete valuation ring (may be non-commutative) or a uniserial Koethe ring, i.e. a uniserial artinian ring.*

We will describe reduced minors of the second and third order of noetherian semi-perfect DMT-rings. One can use the list of all up to isomorphism biregular quivers Q with two and three vertices such that $\overline{RQ}$ is a disjoint union of the Dynkin diagrams A_m [13, §5]. Such quivers will be called admitted.

Proposition 2.4. *Admitted quivers Q for $n = 2$ and $n = 3$ can be defined up to isomorphism by the following matrices.*

$$n=2:\quad \alpha)\begin{pmatrix}0&1\\0&0\end{pmatrix}\ \beta)\begin{pmatrix}1&1\\0&0\end{pmatrix}\ \gamma)\begin{pmatrix}0&1\\0&1\end{pmatrix}\ \delta)\begin{pmatrix}1&1\\0&1\end{pmatrix}\ \varepsilon)\begin{pmatrix}0&1\\1&0\end{pmatrix}\ \zeta)\begin{pmatrix}1&1\\1&0\end{pmatrix}$$

$$n=3:\quad 1)\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}\ 2)\begin{pmatrix}0&1&1\\0&0&0\\0&0&0\end{pmatrix}\ 3)\begin{pmatrix}0&0&1\\0&0&1\\0&0&0\end{pmatrix}\ 4)\begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}$$

$$5)\begin{pmatrix}1&1&0\\0&0&1\\0&0&0\end{pmatrix}\ 6)\begin{pmatrix}1&0&1\\0&0&1\\0&0&0\end{pmatrix}\ 7)\begin{pmatrix}0&1&0\\0&1&1\\0&0&0\end{pmatrix}\ 8)\begin{pmatrix}0&1&1\\0&1&0\\0&0&1\end{pmatrix}\ 9)\begin{pmatrix}0&1&0\\0&0&1\\0&0&1\end{pmatrix}$$

$$10)\begin{pmatrix}1&0&1\\0&1&1\\0&0&0\end{pmatrix}\ 11)\begin{pmatrix}1&1&0\\0&1&1\\0&0&0\end{pmatrix}\ 12)\begin{pmatrix}0&1&0\\0&1&1\\0&0&1\end{pmatrix}\ 13)\begin{pmatrix}0&1&1\\0&1&0\\0&0&1\end{pmatrix}\ 14)\begin{pmatrix}1&1&0\\0&0&1\\0&0&1\end{pmatrix}$$

$$15)\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\ 16)\begin{pmatrix}0&1&1\\1&0&0\\0&0&0\end{pmatrix}\ 17)\begin{pmatrix}0&1&1\\1&0&1\\0&0&0\end{pmatrix}\ 18)\begin{pmatrix}0&1&1\\1&0&0\\0&0&1\end{pmatrix}\ 19)\begin{pmatrix}0&1&1\\1&1&0\\0&0&1\end{pmatrix}$$

$$20)\begin{pmatrix}0&1&1\\1&1&0\\0&0&1\end{pmatrix}\ 21)\begin{pmatrix}0&1&0\\0&0&1\\0&1&0\end{pmatrix}\ 22)\begin{pmatrix}0&1&1\\0&0&1\\0&1&0\end{pmatrix}\ 23)\begin{pmatrix}0&1&0\\0&0&1\\0&1&1\end{pmatrix}\ 24)\begin{pmatrix}1&1&0\\0&0&1\\0&1&1\end{pmatrix}$$

$$25)\begin{pmatrix}1&1&0\\0&0&1\\0&1&1\end{pmatrix}\ 26)\begin{pmatrix}0&1&0\\0&0&1\\0&1&0\end{pmatrix}\ 27)\begin{pmatrix}0&1&0\\1&0&1\\1&0&0\end{pmatrix}\ 28)\begin{pmatrix}1&1&0\\0&0&1\\1&0&0\end{pmatrix}\ 29)\begin{pmatrix}1&1&0\\0&0&1\\1&0&1\end{pmatrix}$$

$$30)\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}\ 31)\begin{pmatrix}0&1&1\\0&0&1\\1&1&0\end{pmatrix}\ 32)\begin{pmatrix}0&0&1\\0&1&1\\1&1&0\end{pmatrix}.$$

Denote by $M_n(R)$ the set of all real matrices of order n.

Definition 4. A matrix $B \in M_n(R)$ is called permutational reducible if there exists a permutation matrix P such that $P^TBP = \left(\begin{smallmatrix} B_1 & B_{12} \\ 0 & B_2 \end{smallmatrix}\right)$, where B_1 and B_2 are square matrices of order less than n. Otherwise, the matrix is permutation irreducible.

Definition 5. A finite oriented graph is called strongly connected if there is an oriented path between any two of its vertices.

Let $[Q]$ denotes adjacent matrix of quiver Q.

Proposition 2.5. [18]. *A quiver Q is strongly connected if and only if the matrix $[Q]$ is permutation irreducible.*

Note that a renumeration of vertices of quiver Q transforms the matrix $[Q]$ into the matrix $P^T[Q]P$.

Proposition 2.6. [18]. *There exists a permutation matrix P such that*

$$P^T[Q]P = \begin{pmatrix} B_1 & B_{12} & \cdots & B_{1t} \\ 0 & B_2 & \cdots & B_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_t \end{pmatrix},$$

where matrices $B_1, \dots, B_t$ are permutation irreducible.

Theorem 2.7. [18]. *Let A be a noetherian semi-perfect ring and a matrix $[Q]$ block upper triangular, where the diagonal matrices $B_1, \dots, B_t$ are permutaional irreducible. Then there exists a decomposition of $1 \in A$ into a sum of mutually orthogonal idempotents: $1 = g_1 + \cdots + g_t$ such that $A = \bigoplus_{i,,j=1}^{t} g_iAg_j$ is a two-sided Pierce decomposition with $g_iAg_j = 0$, $i < j$, and the incidence matrices of quivers $Q(A_i)$ of rings $A_i = g_iAg_i$ coincide with B_i, $i = 1, \dots, t$.*

While describing reduced minors of order 2 of noetherian semi-perfect DMT-rings we will use the same notation as in [19, n.2]; $1 = e_1 + e_2$ is a decomposition of $1 \in B$ into a sum of two local idempotents, $B_i = e_iBe_i$, R_i is the Jacobson radical of the ring B_i, $i = 1, 2$, R is the Jacobson radical of the ring B, $X = e_1Be_2$, $Y = e_2Be_1$, $R = \left(\begin{smallmatrix} R_1 & X \\ Y & R_2 \end{smallmatrix}\right)$, and $XR_2 = R_1X$, $YR_1 = R_2Y$ by Theorem 1.9 and Corollary 1.6.

Proposition 2.8. *The following list contains all reduced minors of order two in noetherian semi-perfect DMT-rings B. Conversely, all rings from the list are noetherian semi-perfect DMT-rings. (The numeration is the same as in Proposition 2.4)*

(α) $B = T_2(D)$, the ring of upper triangle matrices of order two over skew field D.

(β) $B = \left(\begin{smallmatrix} B_1 & X \\ 0 & B_2 \end{smallmatrix}\right)$, $B_2 = D$ is a skew field, B_1 is either discrete valuation ring, or an uniserial Koethe ring, $R_1X = XR_2 = 0, X$ is a one-dimensional right D-space and a one-dimensional left B_1/R_1-space.

(γ) $B = \left(\begin{smallmatrix} B_1 & X \\ 0 & B_2 \end{smallmatrix}\right)$, $B_1 = D$ is a skew field, B_2 is either discrete valuation ring, or an uniserial Koethe ring, $R_1X = XR_2 = 0$, X is a one-dimensional right B_2/R_2-space and a one-dimensional left D-space.

(δ) $B = \left(\begin{smallmatrix} B_1 & X \\ 0 & B_2 \end{smallmatrix}\right)$, B_1 and B_2 are either discrete valuation rings, or uniserial Koethe rings, $R_1X = XR_2 = 0$, X is a one-dimensional right B_2/R_2-space and a one-dimensional left B_1/R_1-space.

(ε) B is a serial ring. In the non-artinian case $B = H_2(\mathcal{O}) = \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{M} \end{smallmatrix}\right)$, where $\mathcal{O}$ is

a discrete valuation ring, $\mathcal{M}$ is a unique maximal ideal.
(ζ) $B = \begin{pmatrix} B_1 & X \\ Y & B_2 \end{pmatrix}$, B_1 and B_2 are either discrete valuation rings, or uniserial Koethe rings; $YX = R_2$, $R_1X = XR_2$, $YR_1 = R_2Y$, $R_2^2 = 0$; X is a one-dimensional right B_2/R_2-space and a one-dimensional left B_1/R_1-space. Y is a one-dimensional right B_1/R_1-space and a one-dimensional left B_2/R_2-space.

The proof of this Proposition in cases (ε) and (ζ) follows from Theorem 5.9 [13]. In the other cases the proof follows from Theorem 2.7 ($Y = 0$) and is analogous to the proof of Theorem 5.9.

The ring of endomorphisms of an indecomposable projective module over a semi-perfect ring will be called a principal ring of endomorphisms.

Proposition 2.9. *If all principal rings of endomorphisms of a G-ring A are artinian, then A is an artinian biserial ring.*

Proof. Let $1 = e_1 + \cdots + e_s$ be a decomposition of $1 \in A$ into the sum of mutually orthogonal local idempotents $A_{ij} = e_iAej(i, j = 1, \ldots, s)$. We will prove this Proposition by induction on s. For $s = 1$ the statement follows from Corollary 2.3. In the general case put $e = e_1 + \cdots + e_{s-1}$, $f = 1 - e$, $X = eAf_1$, $Y = fAe_1$, $A_1 = eAe$, $A_2 = fAf$; R_i is the Jacobson radical of a ring A_i ($i = 1, 2$). From the description of minors of order two it follows that X is an artinian right A_2-module and Y is an artinian A_1-module. Therefore one can conclude that the Jacobson radical R of the ring A is nilpotent. Since the ring A is a noetherian $SPSD$-ring, A is artinian, by Theorem 1.1. Thus A is an artinian biserial ring, by Theorem 1.8.

Proposition 2.10. *Let all principal rings of endomorphisms of a G-ring A be discrete valuation rings. Then A is a prime serial hereditary ring which is isomorphic to the ring $H_s(\mathcal{O})$, where $H_s(\mathcal{O})$ is a matrix ring of order m:* $H_s(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{M} & \mathcal{O} & \cdots & \mathcal{O} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \end{pmatrix}$. *Here $\mathcal{O}$ is a discrete valuation ring, and $\mathcal{M}$ is its single maximal ideal.*

Proof follows from Proposition 2.8 and Michler's Theorem [20].

Therefore one can assume while proving Theorem I that among principal ring of endomorphisms there are discrete valuation rings and uniserial Koethe rings. Recall that a ring is called semi-prime if it possesses no nonzero nilpotent ideals. A ring is called prime if the product of two its arbitrary nonzero ideals is nonzero. A ring will be called weakly prime if the product of its two arbitrary two-sided ideals which do not lie in the Jacobson radical R of the ring A is nonzero. Clearly, every prime ring is weakly prime.

Proposition 2.11. [13]. *Let A be a semi-perfect ring, $1 = e_1 + \ldots + e_n$ be a decomposition of the unity of a ring A into the sum of mutually orthogonal local idempotents, $e_iAe_j = A_{ij}$ ($i, j = 1 \ldots, n$). A ring A is weakly prime iff the sets A_{ij} are nonzero for $i, j = 1, \ldots, n$.*

Theorem 2.12. [21]. *Any weakly prime two-sided noetherian semi-perfect DMT-ring is biserial.*

Thus we obtain a weakly prime ring for $A_{ij} \neq 0$ and Theorem 1 is proved. Therefore one can study G-rings A with $A_{ij} = 0$.

In connection with Propositions 2.9 and 2.10 one can assume that there are discrete valuation rings and uniserial Koethe rings among principal rings of endomorphisms. Analogously as in Proposition 2.8 minors of order three of G-rings were investigated. It was proved that all these minors are biserial.

§**3. General case.** Let I be the prime radical of a noetherian semi-perfect ring A of the distributive module type. Then $\bar{A} = A/I = \bar{A}_1 \times \cdots \times \bar{A}_t$ is the direct product of prime rings which are Morita equivalent either to a skew field D or to the ring $H_s(\mathcal{O})$.

Let $\bar{1} = \bar{f}_1 + \cdots + \bar{f}_t$ be a decomposition of the unit of the ring $\bar{A}$ into the sum of mutually orthogonal idempotents. Denote $V = I/I^2$. Put into correspondence the points $1, \ldots, t$ to the idempotents $\bar{f}_1, \ldots, \bar{f}_t$ and construct an arrow from point i to point j if $\bar{f}_i V \bar{f}_j \neq 0$. We obtain the prime quiver of the ring A.

A weight of a point i of a prime quiver is called a ring of endomorphisms of indecomposable projective $\bar{A}_i$-module i.e. weights are D and $H_s(\mathcal{O})$.

If weights are only D then the ring A is artinian and this Theorem was proved by Colby and Fuller [12].

Renumerate all vertices of the prime quiver of a ring A in such way that the vertices $1, \ldots, m$ are of weight $H_s(\mathcal{O})$ and weights of points $m+1, \ldots, t$ are skew fields (may be $m = t$). Further we numerate vertices weights which are skew fields in the following way:

1) at first there are vertices with $R_k^2 \neq 0$,
2) further are vertices with $R_k^2 = 0$ and $R_k \neq 0$,
3) $R_k = 0$. Obtain the two-sided Pierce decomposition of the ring A: $A = \begin{pmatrix} H & \cdots & \cdots & \cdots \\ \cdots & B_3 & \cdots & \cdots \\ \cdots & \cdots & B_2 & \cdots \\ \cdots & \cdots & \cdots & B_1 \end{pmatrix}$.

The ring H correspondes to the points with weights $H_s(\mathcal{O})$, i.e. to M first points. It possesses the two-sided Pierce decomposition

$$H = \begin{pmatrix} H_{s_1} & A_{12} & \cdots & A_{1m} \\ A_{21} & H_{s_2} & \cdots & A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & H_{s_m} \end{pmatrix}$$

The following Lemma 3.1 is a consequence of Proposition 2.8.

Lemma 3.1. *Let e and f be local idempotents of a ring A and $eAf \subseteq I$. Then eAf is either zero or one-dimensional right vector space over a skew field of endomorphisms of a right simple module which corresponds to the idempotent f and either zero or one-dimensional left vector space over a skew field of endomorphisms of left simple module which corresponds to idempotent e.*

Theorem 3.2. [8]. *The skew felds of endomorphisms of all simple modules over a noetherian indecomposable semi-perfect semi-distributive ring are all isomorphic.*

Corollary 3.3. *All components A_{ij} in Pierce decomposition of the ring H are finite dimensional vector spaces. Besides, it is easy to show that in the prime quiver of the ring H there are no oriented cycles. In other case the weight of one point of this cycle was the skew field D. Thus the ring H is of the block triangle form:*

$$H = \begin{pmatrix} H_{s_1} & A_{12} & \cdots & A_{1m} \\ 0 & H_{s_2} & \cdots & A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & H_{s_m} \end{pmatrix}.$$

From Proposition 2.8 and description of minors of the order three it follows that there exists $k \in \mathbb{N}$ such that $A_{ij} R_j^k = R_i^k A_{ij} = 0$, where R_i is the Jacobson radical of the ring H_{s_i}. Analogously, for rings B_3, B_2, B_1 it holds $A_{ij}R_j^k = R_i A_{ij}^k = 0$. Therefore, the two-sided Pierce decomposition for G-ring is of the form

$$A = \begin{pmatrix} H_{s_1} & A_{12} & \cdots & A_{1m} & A_{1,m+1} & A_{1,m+2} & A_{1,m+3} \\ 0 & H_{s_2} & \cdots & A_{2m} & A_{2,m+1} & A_{2,m+2} & A_{2,m+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & H_{s_m} & A_{m,m+1} & A_{m,m+2} & A_{m,m+3} \\ 0 & \cdots & 0 & 0 & B_3 & A_{m+1,m+2} & A_{m+1,m+3} \\ A_{m+2,1} & \cdots & \cdots & A_{m+2,m} & A_{m+2,m+1} & B_2 & A_{m+2,m+3} \\ A_{m+31} & \cdots & \cdots & A_{m+3,m} & A_{m+3,m+1} & A_{m+3,m+2} & B_3 \end{pmatrix},$$

where $A_{ij}R_j^k = R_i^k A_{ij} = 0$. Consider now the set

$$I_n = \begin{pmatrix} R_1^n & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & R_2^n & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & R_m^n & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix},$$

It easy to verify that I_n is a two-sided ideal of the ring A for each $n \geq k$. Therefore, the quotient ring $A_n = A/I_n$ is an artinian ring of distributive module type which is biserial by Theorem 1.8 for every natural number $n \geq k$.

Show that every indecomposable projective A-module is biserial. Consider, for example, the module $P_1 = (\mathcal{O}, \mathcal{O}, \cdots, \mathcal{O}, A_{12}, \cdots, A_{1m+3})$. The largest submodule is $P_1R = (\mathcal{M}, \mathcal{O}, \cdots, \mathcal{O}, A_{12}, \cdots, A_{1m+3})$. Clearly, $K_1 = (\mathcal{M}, \mathcal{O}, \cdots, \mathcal{O}, 0, \cdots, 0)$ is a serial module of the ring A. The A_n-module $V_n = K_1/K_1R_1^n$, $(n \in \mathbb{N})$ is also serial while $U_1, \ldots, U_{s_1}$ exhaust all nonisomorphic simple factor-modules [6]. Put $K_2 = (0, \ldots, 0, A_{12}, \ldots, A_{1m+3})$. It is artinian by Lemma 3.1 and is A_n-module for every natural number $n \geq k$. Let $P_1^{(n)}$ be the first indecomposable projective A_n-module. Since A_n is a biserial ring, the unique maximal submodule in $P_1^{(n)}$ is of the form $K_1^{(n)} + K_2^{(n)}$, where $K_i^{(n)} (i = 1, 2)$ are serial. Since $P_1^{(n)}$ is distributive, we have

$$(K_1^{(n)}+K_2^{(n)})\cap V_n = K_1^{(n)}\cap V_n+K_2^{(n)}\cap V_n, \quad (K_1^{(n)}+K_2^{(n)})\cap K_2 = K_1^{(n)}\cap K_2+K_2^{(n)}\cap K_2 \quad (*)$$

In the composition series of the module K_2 there are no quotient modules isomorphic to $U_1, \ldots, U_{s_1}$ From the equalities $(*)$ it follows that $K_1^{(n)} \cap V_n$ and $K_2^{(n)} \cap V_n$ are submodules of the serial module V_n and therefore one can assume that $K_1^{(n)} \cap V_n \supseteq K_2^{(n)} \cap V_n$, i.e. $V_n = K_1^{(n)} \cap V_n$ and $V_n \subseteq K_1^{(n)}$. We obtained that the socle of the module $K_1^{(n)}$ coincides with the socle of the module V_n and is equal to one from simple $U_1, \ldots, U_{s_1}$. In the decomposition of the socle of the module K_2 there is one from simple modules $U_{s_1+1}, \ldots, U_{m+3}$. Let the modules $K_1^{(n)} \cap K_2$ and $K_1^{(n)} \cap K_2$ be simultaneously non zero. Since the socle $K_1^{(n)}$ contains at least one from simple modules $U_k, (k = 1, \ldots, s_1)$, we see that K_2 contains this module. Contradiction. Therefore, $K_1^{(n)} \cap K_2 = 0$ and $K_2^{(n)} \cap K_2 = K_2$, i.e. K_2 is a serial module as a submodule of the serial module

$K_2^{(n)}$. Thus $P_1R = (\mathcal{M}, \mathcal{O}, \cdots, \mathcal{O}) \oplus (0, \cdots, 0, A_{12}, \cdots, A_{1m+3}) = K_1 \oplus K_2$. Analogously one can prove that every indecomposable projective A-module is biserial. Note that the indecomposable projective A-modules $P_{m+1}, P_{m+2}, P_{m+3}$ are right biserial modules and $Q_{m+1}, Q_{m+2}, Q_{m+3}$ are left biserial modules automatically, because they are also A_n-modules. Theorem 1 is proved.

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