УДК 517.988

ON FIXED POINTS OF ONE CLASS OF OPERATORS

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Yu. Korobeinik. *On fixed points of one class of operators*, Matematychni Studii, **7**(1997) 187–192.

Let $T\colon Q \to Q$ be a selfmapping of some subset Q of a Banach space E. The operator T does not increase the distance between Q and the origin (the null element E) if

$$\forall x \in Q \quad \|Tx\| \le \|x\|$$

(such operator is called a NID operator). In this paper some sufficient conditions for the existence of a non-trivial fixed point of a NID operator are obtained. These conditions enable in turn to prove some new results on fixed points of nonexpanding operators.

Let Q be a subset of a linear normed space $(E, \|\cdot\|)$. An operator $T : Q \to E$ is said to be *nonexpanding* or *NE operator* if

$$\forall x_1, x_2 \in Q \quad \|Tx_1 - Tx_2\| \le \|x_1 - x_2\|.$$

It is evident that each nonexpanding mapping of Q is continuous in Q.

The problem of existence of fixed points of nonexpanding operators was investigated in many papers (see [1]–[9]). The monograph [10] contains a comparatively detailed survey of essential results on fixed points of NE operators obtained in the beginning of the seventies. In particular, it was proved that each NE operator in a closed bounded convex subset of a uniformly convex space has at least one fixed point in Q (for the definition of a uniformly convex space see [10], ch.2, §4 or [11], ch.V, §12).

We shall say that a Banach space E has the FPNE *property* if each NE operator $T : Q \to Q$, where Q is an arbitrary weakly compact convex subset of E has a fixed point in Q. Since any uniformly convex space is reflexive (cf. [10], [11]) the above mentioned result can be reformulated as follows: each uniformly convex space has the FPNE property. However it is not yet known whether each strictly convex space has the FPNE property. Let us recall that a linear normed space E is called *strictly convex* if its unit sphere $S = \{x \in E : \|x\| = 1\}$ contains no segment. It is known

1991 *Mathematics Subject Classification.* 46B99, 47B99, 54H25.

This work has been supported by Russian Fund of Fundamental Investigations (project 96-01-01041).

(cf. [10], [11], [12]) that the class of strictly convex spaces contains all uniformly convex spaces, including Hilbert spaces.

In this paper we introduce one class of operators and investigate their connection with nonexpanding operators and the problem of fixed points.

Let Q be a subset of linear normed space $(E, \|\cdot\|)$. Let us fix an arbitrary element $x_0 \in E$.

Definition. An operator $T: Q \to E$ does not increase distance from the point x_0 in Q if $\forall x \in Q$ $\|Tx - x_0\| \leq \|x - x_0\|$.

Such operator is called below a *NID operator for the pair Q, x_0*. It is evident that if $x_0 \in Q$ and T is a NID operator for the pair Q, x_0, then T is continuous in x_0 and the latter is the fixed point of T in Q. A NID operator for the pair Q, ϑ will be called simply a NID operator in Q (here ϑ denotes the origin of E). Remark that T is continuous at x_0, but can be discontinuous at each other point.

We put

$$d(Q, x_0) = \inf\{\|x - x_0\| : x \in Q\},$$
$$\mathrm{Min}(Q, x_0) = \{x \in Q : \|x - x_0\| = d(Q, x_0)\}.$$

The set $\mathrm{Min}(Q, x_0)$ can be void. Nevertheless, we have the following simple

Proposition 1. *If Q is a weakly compact convex subset of a Banach space E then* $\mathrm{Min}(Q, x_0)$ *is nonvoid, convex, and weakly compact.*

Proof. Denote by $B(x_0, r) = \{x \in E : \|x - x_0\| \leq r\}$ the closed r-ball around x_0 and observe that

$$Min(Q, x_0) = Q \cap B(x_0, d(Q, x_0)) = \bigcap_{r > d(Q, x_0)} Q \cap B(x_0, r).$$

Then $\mathrm{Min}(Q, x_0)$, being the intersection of the weakly compact convex set Q and the weakly closed convex ball $B(x_0, d(Q, x_0))$, is convex and weakly compact.

To see that $\mathrm{Min}(Q, x_0)$ is not void notice that $\mathrm{Min}(Q, x_0)$ is the intersection of the centered collection $\{Q \cap B(x_0, r)\}_{r > d(Q, x_0)}$ of weakly compact sets. □

We have also the following evident

Proposition 2. *Let Q be a convex subset of a normed space E, $x_0 \in E$, and $T: Q \to Q$ be a NID operator for the pair Q, x_0. Then T maps* $\mathrm{Min}(Q, x_0)$ *in* $\mathrm{Min}(Q, x_0)$.

It follows from Proposition 2 that if the set $\mathrm{Min}(Q, x_0)$ consists of a unique element $\{x_1\}$ then this element is a fixed point for each NID operator for the pair Q, x_0. If the space E is strictly convex then $\mathrm{Min}(Q, x_0)$, being a convex subset of the sphere $\{x \in E : \|x - x_0\| = d(Q, x_0)\}$, can not contain more than one point. These remarks lead us to the following simple result.

Theorem 1. *Let Q be a weakly compact convex subset of a strictly convex Banach space E. Let $T: Q \to Q$ be a NID operator for the pair Q, x_0. Then the operator T has at least one fixed point x_1 in Q, namely the (single) point x_1 for which $\|x_1 - x_0\| = d(Q, x_0)$.*

Let us return to a general case of a multipoint $\mathrm{Min}(Q, x_0)$.

Let us say that a Banach space E has *the FPNID property* if each continuous NID operator $T: Q \to Q$ for the pair Q, x_0 where Q is an arbitrary convex weakly compact subset of E and x_0 is arbitrary element of E has a fixed point in Q.

A Banach space E is said to be *almost Kadec-Klee space* (AKK-space) if each weakly compact convex subset of the unit sphere $S_h = \{x : \|x\| = 1\}$ is compact.

Theorem 2. *Every AKK-space has the FDNID property.*

Proof. Let Q be a convex weakly compact subset of an AKK-space E, $x_0 \in E$, and $T : Q \to Q$ a continous NID operator for the pair Q, x_0. According to Proposition 1 $\mathrm{Min}(Q, x_0)$ is a convex weakly compact subset of the sphere $\{x \in E : \|x - x_0\| = d(Q, x_0)\}$. Since E is an AKK-space the set $\mathrm{Min}(Q, x_0)$ is a convex compactum. Then the restriction $T_{\mathrm{Min}(Q,x_0)}$ of the operator T on $\mathrm{Min}(Q, x_0)$ is a continuous selfmapping of $\mathrm{Min}(Q, x_0)$. Due to theorem of Schauder-Tikhonov ([11], ch.V, §10) $T_{\mathrm{Min}(Q,x_0)}$ has a fixed point in $\mathrm{Min}(Q, x_0)$.

Let us remark that the class of AKK-spaces includes both the class of all strictly convex spaces and that of KK-spaces (recall that a Banach space has the Kadec-Klee property (or is a KK-space) if the strong and the weak topologies coincide on its unit sphere). Since there are strictly convex Banach spaces without the Kadec-Klee property (e.g. l_∞ with any equivalent strictly convex norm) the class of AKK-space is bigger than the class of KK-spaces.

Now we are going to find out a Banach space failing to satisfy the FPNID property. In the paper [13] for the space $E = L_1[0,1]$ the convex set Q_0 of all functions $f \in L_1[0,1]$ such that $\int_0^1 f(t)dt = 1$ and almost everywhere on $[0,1]$ $0 \le f(t) \le 2$ was thoroughly investigated. It was shown in [13] that Q_0 is weakly compact and the operator T_0:

$$(T_0 f)(t) = 2\min\{f(2t), 1\}, \qquad 0 \le t \le \frac{1}{2};$$
$$(T_0 f)(t) = 2\max\{0, f(2t-1) - 1\}, \qquad \frac{1}{2} < t \le 1$$

is a nonexpanding operator in Q_0 without fixed points.

Simple calculations show that $\forall f \in Q_0$ $\|L_0 f\| = \|f\|$. Therefore T_0 is a continuous NID operator in Q_0 free of fixed points. So we may say that the following result is actually contained in [13].

Proposition 3. $L_1[0,1]$ *does not satisfy the FPNID property.*

It is worth mentioning that the operator T_0 constructed in [13] is both NE and NID operator. Let us estblish some relationships between these two classes of operators. At first notice that a NE-operator on a subset $Q \subset E$ containing the origin ϑ is a NID-operator if and only if it does not move the origin. If $\vartheta \notin Q$ then one can easily prove that NE operator $T : Q \to E$ is a NID in Q iff $\forall x \in Q$ $\|x\| \ge \inf\{\|Ty\| + \|x - y\| : y \in Q\}$. On the other hand, there are NID operators which are not NE (e.g. the function x^2 on the segment $[0,1] \subset \mathbb{R}$).

These simple examples show that in general, NE and NID operators are different classes of operators which need their own specific methods for their investigation. Nevertheless, it is possible under additional assumptions to reduce one of these operators to the other and obtain in such a manner a new information on fixed points. Here is one example of such a reduction, connected with the following problem. Let Q be some subset of a Banach space E. Let T be a NE operator in Q:

$$T : Q \to E;\ \forall x, y \in Q \quad \|Tx - Ty\| \leq \|x - y\|.$$

Suppose that one fixed point x_0 of T in Q is known but we need to find in Q fixed points of T different from x_0. For example, if $\vartheta \in Q$ and $T\vartheta = \vartheta$ (the operator T may be nonlinear) then we usually look for other solutions of the equation $Ty = y$ in Q, namely, nontrivial solutions $y \neq \vartheta$.

The following result is actually a direct corollary of Schauder-Tikhonov theorem.

Proposition 4. *Let T be a NE operator in some set Q of a Banach space E and let T have a fixed point x_0 in Q. The operator T has a fixed point $x_1 \neq x_0$ in Q iff there exists a convex compact subset Q_1 of the set $Q \setminus x_0$ such that $x_1 \in Q_1$ and the restriction of T on Q_1 is a selfmapping of Q_1.*

It is interesting to investigate the possibility of replacing in this proposition of the compact subset Q_1 by weakly compact one. We have such a possibility at least for the class of Banach spaces satisfying the FPNID property.

Theorem 3. *Suppose that a Banach space E has the FPNID property. Let T be a NE operator in the set $Q \subset E$ such that T has a fixed point x_0 and T is a selfmapping of some convex weakly compact subset Q_1 of the set $Q \setminus x_0$. Then there exists a fixed point x_1 of T such that $\|x_1 - x_0\| = d(Q, x_0)$. Moreover, if the space E is strictly convex then any point of the segment $[x_0, x_1]$ is a fixed point of T.*

Proof. By Proposition 1, the set $Min(Q_1, x_0)$ is convex, weakly compact, and not empty. Then, by the definition of the FPNID-property, there is a fixed point of the map T in Q_1. So the first part of the theorem is proven.

Now suppose the space E is strictly convex and let x be any point of the segment $[x_0, x_1]$. Let $r_0 = \|x - x_0\|$ and $r_1 = \|x - x_1\|$ and notice that

$$r_0 + r_1 = \|x_0 + x_1\|$$

(this follows from the linearity of the norm $\|\cdot\|$). Recall that $B(x, r)$ and $S(x, r)$ denote respectively the r-ball and r-sphere about x. From the triangle inequality and the equality $r_0 + r_1 = \|x_0 - x_1\|$ it follows that $B(x_0, r_0) \cap B(x_1, r_1)$ is a convex set coinciding with the intersection $S(x_0, r_0) \cap S(x_1, r_1)$. Since E is strictly convex, this intersection consists of a unique point x. Since T is a NE-operator,

$$T(B(x_i, r_i)) \subset B(x_i, r_i) \quad \text{for} \quad i = 0, 1.$$

Then

$$T(x) \in T(B(x_0, r_0)) \cap T(B(x_1, r_1)) \subset B(x_0, r_0) \cap B(x_1, r_1) = \{x\},$$

i.e. $T(x) = x$. $\square$

Corollary. *Let T be a NE operator in some closed convex set Q of a strictly convex space E. Suppose T has a fixed point x_0 in Q and maps into itself some convex weakly compact subset Q_1 of the set $Q \setminus x_0$. Then there exists in Q_1 a unique fixed point x_1 such that $\|x_1 - x_0\| = d(Q, x_0)$ and all points of the segment $[x_0, x_1]$ are fixed ones for T.*

Fixed points of every operator $T : Q \to E$ are solutions of the equation

$$Tx = x. \tag{1}$$

By analogy with generalized solutions of equations in partial derivatives one can introduce generalized solutions of the equation (1). The element v from Q is called a *weak generalized solution* of the equation (1) if there exists a sequence $(x_k)_{k=1}^{\infty}$ of the elements x_k from Q such that x_k tends weakly to v in E and $\lim_{k\to\infty} \|Tx_k - x_k\| = 0$. The point v is also said to be a *quasifixed point* of the operator T.

Theorem 4. *Let Q be a weakly compact subset of a Banach space E which is starlike with respect to a certain point v from E. Let $T\colon Q \to Q$ be a NE operator in Q. Then T has at least one quasifixed point in Q.*

It is worth reminding that the set Q is starlike with respect to a point $v \in Q$ if $\forall x \in Q$ the segment $[v, x]$ belongs to Q. Theorem 4 was known earlier for convex sets Q.

In order to prove theorem 4 we introduce for $q \in (0, 1)$ the operator

$$T_q x := qTx + (1 - q)v.$$

This operator is a selfmapping of Q with the following property:

$$\forall x_1, x_2 \in Q \quad \|T_q x_1 - T_q x_2\| \le q\|x_1 - x_2\|.$$

Due to Theorem of Banach T_q has a unique fixed point x_q in Q. If $0 < q_n \uparrow 1$ and $y_n := x_{q_n}$ then

$$\lim_{n\to\infty} \|Ty_n - y_n\| = 0.$$

The following result is analogous.

Theorem 5. *Let Q be a compact subset of a Banach space E which is starlike with respect to some point $v \in Q$. Let $T\colon Q \to Q$ be a NE operator in Q. Then T has a fixed point in Q.*

For convex sets Q this result is a very special case of well known theorem of Schauder-Tikhonov.

The results of this paper can be applied to investigation of the solvability of various linear and nonlinear operator equations (for example, equations with Urysohn operators or with convolution-type operators).

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Received 17.04.1996