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THE FIRST BOUNDARY VALUE PROBLEM FOR COUPLED DIFFUSION SYSTEMS WITH FUNCTIONAL ARGUMENTS

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This paper discusses the existence and uniqueness theorem for the problem containing the initial boundary value problem for a parabolic system coupled with the initial problem for a system of ordinary differential equations. The system of equations contains a functional on continuous functions of space arguments.

This paper discusses the class of diffusion systems of the form

$$\begin{cases} \bar{P}_i u \equiv \dfrac{\partial \bar{u}_i}{\partial t} + L_i(t,x)\bar{u}_i + \bar{f}_i\big(t,x,u;u(t,\cdot)\big) = 0, \ (t,x) \in G^T = (0;T] \times D, \\ \hat{P}_j u \equiv \dfrac{\partial \hat{u}_j}{\partial t} + \hat{f}_j\big(t,x,u;u(t,\cdot)\big) = 0, \ (t,x) \in G^T \cup \partial_1 G^T, \end{cases} \tag{1}$$

where D is a bounded domain in $\mathbb{R}^n$, $0 < t \le T < \infty$, $i = 1,\dots,M$, $j = 1,\dots,L$, $\partial_1 G^T = (0,T] \times \partial D$; $u(t,x) = \big(\bar{u}(t,x), \hat{u}(t,x)\big)$ an unknown vector function, $\bar{u} \in \mathbb{R}^M$, $\hat{u} \in \mathbb{R}^L$;

$$L_i(t,x)\bar{u}_i \equiv -\sum_{k,l=1}^{n} a_{i,kl}(t,x)\frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_{i,k}(t,x)\frac{\partial \bar{u}_i}{\partial x_k} + c_i(t,x)\bar{u}_i$$

differential expressions with the coefficients depending on $(t,x) \in G^T$. The expressions $\bar{f}_i\big(t,x,v;u(t,\cdot)\big)$, $\hat{f}_j\big(t,x,v;u(t,\cdot)\big)$ are functionals on the continuous functions $u(t,y) = \big(\bar{u}(t,y), \hat{u}(t,y)\big)$ with respect to $y \in D$ at each point $(t,x) \in G^T$, $v \in \mathbb{R}^{M+L}$.

The coupled diffusion systems arise in many physical and mechanical problems, e.g. those describing the processes of nuclear dynamics [1], and polymerization [2]. These systems are also used in mathematical biology (the FitzHugh-Nagumo systems [3] and the Hodgkin-Huxley systems [4]).

The solution of (1) satisfies the following initial value condition

$$\bar{R}_i u \equiv \bar{u}_i(0,x) = \bar{h}_i(x), \quad \hat{R}_j u \equiv \hat{u}_j(0,x) = \hat{h}_j(x), \quad (t,x) \in \partial_0 G^T = \{0\} \times \bar{D} \tag{2}$$

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and the boundary value condition

$$\bar{\Gamma}_i u \equiv \bar{u}_i(t,x) = \bar{g}_i(t,x), \quad (t,x) \in \partial_1 G^T. \tag{3}$$

The first boundary value problems for this kind of coupled systems without functional arguments have been considered in [5,6]. The first boundary value problems for the parabolic systems with functional arguments were investigated in [7,8].

1. Comparison and uniqueness theorems

We make the following assumptions in this section.

A. The functions of problem (1)–(3) are defined and continuous in respective domains.
   At each point $(t,x) \in G^T$ and for all real n-tuples $\xi = (\xi_1, \dots, \xi_n)$ the inequality $\sum_{k,l=1}^n a_{i,kl}(t,x)\xi_k\xi_l \geq m_0 \sum_{k=1}^n \xi_k^2$ holds, where m_0 is a positive constant; and the matrix $\{a_{i,kl}\}$ for the fixed $i = 1, \dots, M$ is symmetric.

B. The functions $\bar{f}_i\big(t,x,u;v(t,\cdot)\big)$, $\hat{f}_j\big(t,x,u;v(t,\cdot)\big)$ of u are quasi-monotone non-increasing, that is if $\bar{u}_i^1 \leq \bar{u}_i^2$, $\hat{u}_j^1 \leq \hat{u}_j^2$, then

$$\bar{f}_s\big(t,x,u^1;v(t,\cdot)\big) \geq \bar{f}_s\big(t,x,u^2;v(t,\cdot)\big) \text{ whenever } \bar{u}_s^1 = \bar{u}_s^2,\ s = 1, \dots, M,$$
$$\hat{f}_r\big(t,x,u^1;v(t,\cdot)\big) \geq \hat{f}_r\big(t,x,u^2;v(t,\cdot)\big) \text{ whenever } \hat{u}_r^1 = \hat{u}_r^2,\ r = 1, \dots, L;$$

   and the non-increasing functionals of $v(t,\cdot)$, that is if $\bar{v}_i^1 \leq \bar{v}_i^2$, $\hat{v}_j^1 \leq \hat{v}_j^2$, $(t,x) \in \bar{G}^T$, then $\bar{f}_s\big(t,x,u;v^1(t,\cdot)\big) \geq \bar{f}_s\big(t,x,u;v^2(t,\cdot)\big)$, $\hat{f}_r\big(t,x,u;v^1(t,\cdot)\big) \geq \hat{f}_r\big(t,x,u;v^2(t,\cdot)\big)$ for all $s = 1, \dots, M$, $r = 1, \dots, L$.

C. The functions $\bar{f}_i$, $\hat{f}_j$ are Lipschitz, continuous with respect to u and $u(t,\cdot)$:

$$\big|f\big(t,x,u^1;v^1(t,\cdot)\big) - f\big(t,x,u^2;v^2(t,\cdot)\big)\big| \leq m|u^1 - u^2| + m_1 \sup_{y \in D}\big|v^1(t,y) - v^2(t,y)\big|. \tag{4}$$

Lemma. *Suppose that the functions $u^1, u^2 \in \mathbb{U}$, where $\mathbb{U} = \big\{v(t,x) = (\bar{v}(t,x), \hat{v}(t,x)) : \bar{v} \in \mathbb{R}^M, \hat{v} \in \mathbb{R}^L, \bar{v} \in C^{1,2}(G^T) \cap C(\bar{G}^T),\ \hat{v} \in C^{1,0}(G^T \cup \partial_1 G^T) \cap C(\bar{G}^T)\big\}$. Suppose further that conditions A, B and the inequalities:*

$$\bar{P}_i u^1 < \bar{P}_i u^2,\ \hat{P}_j u^1 < \hat{P}_j u^2, \quad (t,x) \in G^T; \tag{i}$$
$$\bar{R}_i u^1 < \bar{R}_i u^2,\ \hat{R}_j u^1 < \hat{R}_j u^2, \quad (t,x) \in \partial_0 G^T; \tag{ii}$$
$$\bar{\Gamma}_i u^1 < \bar{\Gamma}_i u^2, \quad (t,x) \in \partial_1 G^T \tag{iii}$$

are valid. Then $\bar{u}_i^1 < \bar{u}_i^2$, $\hat{u}_j^1 < \hat{u}_j^2$ in $\bar{G}^T$.

Proof. Let $t^* < T$ be the largest number with the property that $\bar{u}_i^1 < \bar{u}_i^2$ and $\hat{u}_j^1 < \hat{u}_j^2$ in $[0;t^*)$, then there exists $(t^*,x^*) \in \bar{G}^T$, that is at this point $\bar{u}_s^1 = \bar{u}_s^2$, where $s \in \{1, \dots, M\}$ or $\hat{u}_r^1 = \hat{u}_r^2$, where $r \in \{1, \dots, L\}$. At the first and the second cases $t^* > 0$, because (ii) are valid.

Let $\bar{u}_s^1 = \bar{u}_s^2$ in $(t^*,x^*) \in \bar{G}^T$, then $x \notin \partial D$, because (iii) are valid. The difference $\bar{u}_s^1 - \bar{u}_s^2$ would vanish at this point and attains the maximum in the $\bar{G}^{t^*}$, consequently

$\partial(\bar{u}_s^1-\bar{u}_s^2)/\partial t \geq 0$, $\partial(\bar{u}_s^1-\bar{u}_s^2)/\partial x_k = 0$ and $\sum_{k,l=1}^n a_{i,kl}(t^*,x^*)\partial^2(\bar{u}_s^1-\bar{u}_s^2)/\partial x_k \partial x_l \leq 0$ in (t^*,x^*). Then

$$\bar{P}_s u^1 - \bar{P}_s u^2 = \frac{\partial}{\partial t}(\bar{u}_s^1 - \bar{u}_s^2) + L_s(t^*,x^*)(\bar{u}_s^1 - \bar{u}_s^2) + \bar{f}_s\big(t^*,x^*,u^1;u^1(t^*,\cdot)\big) - \\ - \bar{f}_s\big(t^*,x^*,u^2;u^2(t^*,\cdot)\big) \geq \Big[\bar{f}_s\big(t^*,x^*,u^1;u^1(t^*,\cdot)\big) - \bar{f}_s\big(t^*,x^*,u^2;u^1(t^*,\cdot)\big)\Big] + \\ + \Big[\bar{f}_s\big(t^*,x^*,u^2;u^1(t^*,\cdot)\big) - \bar{f}_s\big(t^*,x^*,u^2;u^2(t^*,\cdot)\big)\Big] \geq 0,$$

at this point, thus contradicting inequality (i).

If $\hat{u}_r^1 = \hat{u}_r^2$ in (t^*,x^*), then at this point $\partial(\hat{u}_r^1 - \hat{u}_r^2)/\partial t \geq 0$ and

$$\hat{P}_r u^1 - \hat{P}_r u^2 = \frac{\partial}{\partial t}(\hat{u}_r^1 - \hat{u}_r^2) + \hat{f}_r\big(t^*,x^*,u^1;u^1(t^*,\cdot)\big) - \hat{f}_r\big(t^*,x^*,u^2;u^2(t^*,\cdot)\big) \geq \\ \geq \Big[\hat{f}_r\big(t^*,x^*,u^1;u^1(t^*,\cdot)\big) - \hat{f}_r\big(t^*,x^*,u^2;u^1(t^*,\cdot)\big)\Big] + \\ + \Big[\hat{f}_r\big(t^*,x^*,u^2;u^1(t^*,\cdot)\big) - \hat{f}_r\big(t^*,x^*,u^2;u^2(t^*,\cdot)\big)\Big] \geq 0.$$

We obtain a contradiction with (i). The lemma is proved.

Theorem 1. *Assume that all the conditions of lemma are satisfied, but the inequalities (i)–(iii) are not strong and suppose further that C is valid. Then $\bar{u}_i^1 \leq \bar{u}_i^2$, $\hat{u}_j^1 \leq \hat{u}_j^2$ in $\bar{G}^T$.*

Proof. We consider a function $u_\lambda^2 = u^2 + \lambda e^{\mu t}$, $\lambda > 0$. If $\mu > (M+L)\big[m + m_1 + \sup\big\{|c_i(t,x)| : (t,x) \in G^T, i = 1,\dots,M\big\}\big]$, then (i)–(iii) are satisfied for u^2, u_λ^2, hence they are also satisfied for u^1, u_λ^2. Hence, $\bar{u}_i^1 < \bar{u}_{\lambda\ i}^2$, $\hat{u}_j^1 < \hat{u}_{\lambda\ j}^2$ are satisfied in $(t,x) \in \bar{G}^T$, $\lambda > 0$. To prove the theorem we use the continuity of functions u_λ^2, where $\lambda \geq 0$.

Theorem 2. *Let conditions A–C be satisfied, then problem (1)–(3) has at most one solution in the space of functions $\mathbb{U}$.*

Proof. Let u^1 and u^2 be two solutions of the problem, then $\bar{P}_i u^1 = \bar{P}_i u^2$, $\hat{P}_j u^1 = \hat{P}_j u^2$, $(t,x) \in G^T$; $\bar{R}_i u^1 = \bar{R}_i u^2$, $\hat{R}_j u^1 = \hat{R}_j u^2$, $(t,x) \in \partial_0 G^T$; $\bar{\Gamma}_i u^1 = \bar{\Gamma}_i u^2$, $(t,x) \in \partial_1 G^T$. Using Theorem 1, we obtain $\bar{u}_i^1 \leq \bar{u}_i^2 \leq \bar{u}_i^1$, $\hat{u}_j^1 \leq \hat{u}_j^2 \leq \hat{u}_j^1$, $(t,x) \in \bar{G}^T$. Then problem (1)–(3) has at most one solution. Hence, the theorem is proved.

2. Existence theorem

We shall study the existence of solution in the Hölder spaces with the following norms [9]

$$|u|_0^G = \sup_{(t,x)\in G} |u(t,x)|, |u|_\alpha^G = |u|_0^G + H_\alpha^G[u], H_\alpha^G[u] = \sup_{(t,x),(t^1,x^1)\in G} \frac{|u(t,x) - u(t^1,x^1)|}{d(P,P^1)^\alpha},$$

the Hölder continuity of functions of t, x is to be understood in the sense of the parabolic metric

$$d\big(P(t,x),P^1(t^1,x^1)\big)=\Big(\sum_{k=1}^{n}(x_k-x_k^1)^2+|t-t^1|\Big)^{1/2};$$

$$|u|_{1+\alpha}^G=|u|_\alpha^G+\sum_{k=1}^{n}\Big|\frac{\partial u}{\partial x_k}\Big|_\alpha^G,\quad |u|_{2+\alpha}^G=|u|_{1+\alpha}^G+\sum_{k=1}^{n}\Big|\frac{\partial u}{\partial x_k}\Big|_{1+\alpha}^G+\Big|\frac{\partial u}{\partial t}\Big|_\alpha^G,$$

$$|u|_2^G=|u|_0^G+\sum_{k=1}^{n}\Big|\frac{\partial u}{\partial x_k}\Big|_0^G+\Big|\frac{\partial u}{\partial t}\Big|_0^G+\sum_{k,l=1}^{n}\Big|\frac{\partial^2 u}{\partial x_k\partial x_l}\Big|_0^G.$$

We denote by C^0, C^α, $C^{1+\alpha}$, $C^{2+\alpha}$, C^2 in G the spaces of functions with the finite norms $|u|_0^G$, $|u|_\alpha^G$, $|u|_{1+\alpha}^G$, $|u|_{2+\alpha}^G$, $|u|_2^G$ respectively.

We make the following assumption in this section.

D. The functions $a_{i,kl}$, $b_{i,k}$, c_i, f are Hölder continuous in G^T i.e. $|a_{i,kl}|_\alpha^{G^T}<A$, $|b_{i,k}|_\alpha^{G^T}<A$, $|c_i|_\alpha^{G^T}<A$, $|f|_\alpha^{G^T}<\infty$, moreover, $a_{i,kl}$ are Lipschitz continuous with respect to t,x: $\big|a_{i,kl}(t,x)-a_{i,kl}(t^1,x^1)\big|\le A_1\big(\sum_{k=1}^{n}(x_k-x_k^1)^2+(t-t^1)^2\big)^{1/2}$.

E. The $\partial_1 G^T$ (lateral surface) is of the class $C^{2+\alpha}$. There exists a vector function $\bar G(t,x)=\big(\bar G_1(t,x),\dots,\bar G_M(t,x)\big)\in C^{2+\alpha}$ and $\bar G(t,x)$ satisfies conditions (2), (3), moreover, the vector-function $\hat h(x)$ is Hölder continuous (exponent α) in $\bar D$.

Theorem 3. *Suppose that conditions A–E are satisfied. Suppose also that there exist two functions $\varphi(t,x)$, $\psi(t,x)\in\mathbb{U}$ and the vector-function $\psi(t,x)$ is Hölder continuous (exponent α) in G^T. Suppose further that the functions φ and ψ satisfy the system of inequalities*

$$\bar P_i\varphi\le 0\le \bar P_i\psi,\ \hat P_j\varphi\le 0\le\hat P_j\psi,(t,x)\in G^T;$$
$$\bar R_i\varphi\le\bar h_i\le\bar R_i\psi,\ \hat R_j\varphi\le\hat h_j\le\hat R_j\psi,(t,x)\in\partial_0G^T;\mathit{\Gamma}_i\varphi\le\bar g_i\le\mathit{\Gamma}_i\psi,(t,x)\in\partial_1G^T.$$

Under these assumptions there exists a unique solution u, moreover, the function $\bar u$ belongs to $C^{2+\gamma}$ ($0<\gamma<\alpha<1$) in G^T and the functions $\hat u$ and $\partial\hat u/\partial t$ belong to C^γ in G^T.

Proof. The proof of existence will be carried out with the aid of the method of iterations as in [5,6].

1. The iteration scheme. Consider the sequence of functions u^p ($p=1,2,\dots$) defined by the equations

$$\begin{cases}\dfrac{\partial\bar u_i^p}{\partial t}+L_i(t,x)\bar u_i^p+m\bar u_i^p=-\bar f_i\big(t,x,u^{p-1};u^{p-1}(t,\cdot)\big)+m\bar u_i^{p-1}, & (t,x)\in G^T,\\ \dfrac{\partial\hat u_j^p}{\partial t}+m\hat u_j^p=-\hat f_j\big(t,x,u^{p-1};u^{p-1}(t,\cdot)\big)+m\hat u_j^{p-1}, & (t,x)\in G^T\cup\partial_1G^T,\\ \bar R_iu^p=\bar h_i(x),\quad \hat R_ju^p=\hat h_j(x), & (t,x)\in\partial_0G^T,\\ \mathit{\Gamma}_iu^p=\bar g_i(t,x), & (t,x)\in\partial_1G^T,\\ u^0=\psi, & (t,x)\in\bar G^T.\end{cases}\tag{5}$$

If u^{p-1} are known to be Hölder continuous (exponent α) in G^T, then so are the right hand sides of equations (5). From the existence theorem on the initial boundary value problem (see [9,10]) it readily follows that the function $\bar{u}^p$ can be determined uniquely and is Hölder continuous (exponent $2+\alpha$) in G^T.

The existence and the Hölder continuity of $\hat{u}^p$ are implied by the explicit formulae

$$e^{mt}\hat{u}_j^p(t,x) = \hat{h}_j(x) + \int_0^t e^{m\tau}\left[-\hat{f}_j\big(\tau,x,u^{p-1}(\tau,x);u^{p-1}(\tau,\cdot)\big) + m\hat{u}_j^{p-1}(\tau,x)\right]d\tau. \tag{6}$$

2. The boundness and the monotonicity of the sequence $\{u^p\}$ $(p=1,2,\dots)$. An induction with the aid of Theorem 1 establishes that

$$\bar{\varphi}_i \le \bar{u}_i^{p+1} \le \bar{u}_i^p \le \bar{\psi}_i, \quad \hat{\varphi}_j \le \hat{u}_j^{p+1} \le \hat{u}_j^p \le \hat{\psi}_j \quad \text{in } \bar{G}^T,\ p\in\mathbb{N}.$$

a) The proof of $\bar{u}_i^1 \le \bar{\psi}_i$, $\hat{u}_j^1 \le \hat{\psi}_j$ in G^T will be obtained from Theorem 1 and from the following inequalities

$$\begin{aligned}
&\left[\frac{\partial \bar{u}_i^1}{\partial t} + L_i(t,x)\bar{u}_i^1 + m\bar{u}_i^1\right] - \left[\frac{\partial \bar{\psi}_i}{\partial t} + L_i(t,x)\bar{\psi}_i + m\bar{\psi}_i\right] = -\bar{f}_i\big(t,x,\psi;\psi(t,\cdot)\big) + m\bar{\psi}_i - \\
&\qquad - \left[\frac{\partial \bar{\psi}_i}{\partial t} + L_i(t,x)\bar{\psi}_i + m\bar{\psi}_i\right] = -\left[\frac{\partial \bar{\psi}_i}{\partial t} + L_i(t,x)\bar{\psi}_i + \bar{f}_i\big(t,x,\psi;\psi(t,\cdot)\big)\right] \le 0,\\
&\left[\frac{\partial \hat{u}_j^1}{\partial t} + m\hat{u}_j^1\right] - \left[\frac{\partial \hat{\psi}_j}{\partial t} + m\hat{\psi}_j\right] = -\hat{f}_j\big(t,x,\psi;\psi(t,\cdot)\big) + m\hat{\psi}_j - \left[\frac{\partial \hat{\psi}_j}{\partial t} + m\hat{\psi}_j\right] = \\
&\qquad = -\left[\frac{\partial \hat{\psi}_j}{\partial t} + \hat{f}_j\big(t,x,\psi;\psi(t,\cdot)\big)\right] \le 0, \quad (t,x)\in G^T,\\
&\bar{R}_i u^1 \le \bar{R}_i\psi,\ \hat{R}_j u^1 \le \hat{R}_j\psi,\ (t,x)\in\partial_0 G^T, \quad \bar{\varGamma}_i u^1 \le \bar{\varGamma}_i\psi,\ (t,x)\in\partial_1 G^T.
\end{aligned}$$

b) Since $\bar{u}_i^p \le \bar{u}_i^{p-1}$, $\hat{u}_j^p \le \hat{u}_j^{p-1}$ in $\bar{G}^T$, we obtain $\bar{u}_i^{p+1} \le \bar{u}_i^p$, $\hat{u}_j^{p+1} \le \hat{u}_j^p$ in $\bar{G}^T$ using Theorem 1 and the following inequalities

$$\begin{aligned}
&\bar{R}_i u^{p+1} = \bar{R}_i u^p,\ \hat{R}_j u^{p+1} = \hat{R}_j u^p,\ (t,x)\in\partial_0 G^T, \quad \bar{\varGamma}_i u^{p+1} = \bar{\varGamma}_i u^p,\ (t,x)\in\partial_1 G^T,\\
&\left[\frac{\partial \bar{u}_i^{p+1}}{\partial t} + L_i(t,x)\bar{u}_i^{p+1} + m\bar{u}_i^{p+1}\right] - \left[\frac{\partial \bar{u}_i^p}{\partial t} + L_i(t,x)\bar{u}_i^p + m\bar{u}_i^p\right] = \\
&\qquad = m(\bar{u}_i^p - \bar{u}_i^{p-1}) - \left[\bar{f}_i\big(t,x,u^p;u^p(t,\cdot)\big) - \bar{f}_i\big(t,x,u^{p-1};u^{p-1}(t,\cdot)\big)\right] = \\
&\qquad = m(\bar{u}_i^p - \bar{u}_i^{p-1}) - \left[\bar{f}_i\big(t,x,u^p;u^p(t,\cdot)\big) - \bar{f}_i\big(t,x,u^p;u^{p-1}(t,\cdot)\big)\right] - \\
&\qquad - \left[\bar{f}_i\big(t,x,u^p;u^{p-1}(t,\cdot)\big) - \bar{f}_i\big(t,x,\{\bar{u}_s^{p-1} - \delta_{s,i}(\bar{u}_s^{p-1} - \bar{u}_i^p)\}_{s=1}^M,\hat{u}^p;u^{p-1}(t,\cdot)\big)\right] - \\
&\qquad - \left[\bar{f}_i\big(t,x,\{\bar{u}_s^{p-1} - \delta_{s,i}(\bar{u}_s^{p-1} - \bar{u}_i^p)\}_{s=1}^M,\hat{u}^p;u^{p-1}(t,\cdot)\big) - \bar{f}_i\big(t,x,u^{p-1};u^{p-1}(t,\cdot)\big)\right] \le \\
&\qquad \le m(\bar{u}_i^p - \bar{u}_i^{p-1}) + m|\bar{u}_i^p - \bar{u}_i^{p-1}| = 0, \quad (t,x)\in G^T,
\end{aligned}$$

where $\delta_{l,k}=0$ if $l\neq k$ and $\delta_{l,k}=1$ if $l=k$, and similarly

$$\Big[\frac{\partial \hat u_j^{p+1}}{\partial t}+m\hat u_j^{p+1}\Big]-\Big[\frac{\partial \hat u_j^{p}}{\partial t}+m\hat u_j^{p}\Big]=$$
$$=m(\hat u_j^p-\hat u_j^{p-1})-\big[\hat f_j\big(t,x,u^p;u^p(t,\cdot)\big)-\hat f_j\big(t,x,u^{p-1};u^{p-1}(t,\cdot)\big)\big]=$$
$$=m(\hat u_j^p-\hat u_j^{p-1})-\big[\hat f_j\big(t,x,u^p;u^p(t,\cdot)\big)-\hat f_j\big(t,x,u^{p};u^{p-1}(t,\cdot)\big)\big]-$$
$$-\big[\hat f_j\big(t,x,u^p;u^{p-1}(t,\cdot)\big)-\hat f_j\big(t,x,\bar u^p,\{\hat u_r^{p-1}-\delta_{r,j}(\hat u_r^{p-1}-\hat u_j^p)\}_{r=1}^L;u^{p-1}(t,\cdot)\big)\big]-$$
$$-\big[\hat f_j\big(t,x,\bar u^p,\{\hat u_r^{p-1}-\delta_{r,j}(\hat u_r^{p-1}-\hat u_j^p)\}_{r=1}^L;u^{p-1}(t,\cdot)\big)-\hat f_j\big(t,x,u^{p-1};u^{p-1}(t,\cdot)\big)\big]\le$$
$$\le m(\hat u_j^p-\hat u_j^{p-1})+m|\hat u_j^p-\hat u_j^{p-1}|=0,\quad (t,x)\in G^T.$$

c) Since $\bar\varphi_i\le \bar u_i^{p-1}$, $\hat\varphi_j\le\hat u_j^{p-1}$ in $\bar G^T$, we obtain $\bar\varphi_i\le\bar u_i^p$, $\hat\varphi_j\le\hat u_j^p$ in $\bar G^T$ using Theorem 1 and the following inequalities

$$\bar R_i\varphi=\bar R_i u^p,\ \ \hat R_j\varphi=\hat R_j u^p,\quad (t,x)\in\partial_0 G^T,\quad \bar\Gamma_i\varphi=\bar\Gamma_i u^p,\quad (t,x)\in\partial_1 G^T,$$
$$\Big[\frac{\partial\bar\varphi_i}{\partial t}+L_i(t,x)\bar\varphi_i+m\bar\varphi_i\Big]-\Big[\frac{\partial \bar u_i^p}{\partial t}+L_i(t,x)\bar u_i^p+m\bar u_i^p\Big]=$$
$$=\Big[\frac{\partial\bar\varphi_i}{\partial t}+L_i(t,x)\bar\varphi_i+m\bar\varphi_i\Big]-\big[m\bar u_i^{p-1}-\bar f_i\big(t,x,u^{p-1};u^{p-1}(t,\cdot)\big)\big]\le$$
$$\le m(\bar\varphi_i-\bar u_i^{p-1})-\big[\bar f_i\big(t,x,\varphi;\varphi(t,\cdot)\big)-\bar f_i\big(t,x,u^{p-1};u^{p-1}(t,\cdot)\big)\big]=$$
$$=m(\bar\varphi_i-\bar u_i^{p-1})-\big[\bar f_i\big(t,x,\varphi;\varphi(t,\cdot)\big)-\bar f_i\big(t,x,\varphi;u^{p-1}(t,\cdot)\big)\big]-$$
$$-\big[\bar f_i\big(t,x,\varphi;u^{p-1}(t,\cdot)\big)-\bar f_i\big(t,x,\{\bar u_s^{p-1}-\delta_{s,i}(\bar u_s^{p-1}-\bar\varphi_i)\}_{s=1}^M,\hat u^p;u^{p-1}(t,\cdot)\big)\big]-$$
$$-\big[\bar f_i\big(t,x,\{\bar u_s^{p-1}-\delta_{s,i}(\bar u_s^{p-1}-\bar\varphi_i)\}_{s=1}^M,\hat u^p;u^{p-1}(t,\cdot)\big)-\bar f_i\big(t,x,u^{p-1};u^{p-1}(t,\cdot)\big)\big]\le$$
$$\le m(\bar\varphi_i-\bar u_i^{p-1})+m|\bar\varphi_i-\bar u_i^{p-1}|=0,\quad (t,x)\in G^T,$$

similarly

$$\Big[\frac{\partial\hat\varphi_j}{\partial t}+m\hat\varphi_j\Big]-\Big[\frac{\partial\hat u_j^p}{\partial t}+m\hat u_j^p\Big]=\Big[\frac{\partial\hat\varphi_j}{\partial t}+m\hat\varphi_j\Big]-\big[m\hat u_j^{p-1}-\hat f_j\big(t,x,u^{p-1};u^{p-1}(t,\cdot)\big)\big]\le$$

$$\le m(\hat\varphi_j-\hat u_j^{p-1})-\big[\hat f_j\big(t,x,\varphi;\varphi(t,\cdot)\big)-\hat f_j\big(t,x,u^{p-1};u^{p-1}(t,\cdot)\big)\big]=$$
$$=m(\hat\varphi_j-\hat u_j^{p-1})-\big[\hat f_j\big(t,x,\varphi;\varphi(t,\cdot)\big)-\hat f_j\big(t,x,\varphi;u^{p-1}(t,\cdot)\big)\big]-$$
$$-\big[\hat f_j\big(t,x,\varphi;u^{p-1}(t,\cdot)\big)-\hat f_j\big(t,x,\bar\varphi,\{\hat u_r^{p-1}-\delta_{r,j}(\hat u_r^{p-1}-\hat\varphi_j)\}_{r=1}^L;u^{p-1}(t,\cdot)\big)\big]-$$
$$-\big[\hat f_j\big(t,x,\bar\varphi,\{\hat u_r^{p-1}-\delta_{r,j}(\hat u_r^{p-1}-\hat\varphi_j)\}_{r=1}^L;u^{p-1}(t,\cdot)\big)-\hat f_j\big(t,x,u^{p-1};u^{p-1}(t,\cdot)\big)\big]\le$$
$$\le m(\hat\varphi_j-\hat u_j^{p-1})+m|\hat\varphi_j-\hat u_j^{p-1}|=0,\quad (t,x)\in G^T.$$

Hence, the sequence u^p converges to a bounded limit in $\bar G^T$.
3. a) The solution $\bar u^p$ of problem (5) satisfies the following estimation

$$|\bar u_i^p|_{1+\delta}^{G^T}\le C_1\Big(\big|\bar f_i\big(t,x,u^{p-1}(t,x);u^{p-1}(t,\cdot)\big)\big|_0^{G^T}+m|\bar u_i^{p-1}|_0^{G^T}+|\bar G_i|_2^{G^T}\Big)\le C_2,$$

where C_1 depends on $A,\ A_1,\ m_0,\ m,\ m_1,\ \alpha$ and on the domain $G^T,\ 0<\delta<\alpha$ [10], but $C_1,\ C_2$ are independent on p, because the sequence $\{u^p\}$ is uniformly bounded and the conditions C, D are valid.

b) We will show that $\{\hat{u}^p\}$ is uniformly Hölder bounded (exponent α) in G^T.

$$\left|\hat{u}_j^p(t_1,x_1)-\hat{u}_j^p(t_2,x_2)\right| \le \left|\hat{u}_j^p(t_1,x_1)-\hat{u}_j^p(t_1,x_2)\right| + \left|\hat{u}_j^p(t_1,x_2)-\hat{u}_j^p(t_2,x_2)\right|$$

From the uniform boundness $\{u^p\}$ in G^T and from (6) we get

$$\left|\hat{u}_j^p(t_1,x)-\hat{u}_j^p(t_2,x)\right| \le K|t_1-t_2|$$

in G^T, where K is independent of p, hence $\{\hat{u}^p\}$ are uniformly Lipschitz continuous with respect to t.

Let us introduce the following notation

$$\bar{Q}_i^p(t)=\left|\bar{u}_i^p(t,x_1)-\bar{u}_i^p(t,x_2)\right|,\quad \bar{Q}^p(t)=\sum_{i=1}^{M}\bar{Q}_i^p(t),\quad \bar{Q}^p[t]=\max\{\bar{Q}^p(\tau):\ 0\le\tau\le t\},$$

$$\hat{Q}_j^p(t)=\left|\hat{u}_j^p(t,x_1)-\hat{u}_j^p(t,x_2)\right|,\quad \hat{Q}^p(t)=\sum_{j=1}^{L}\hat{Q}_j^p(t),\quad \hat{Q}^p[t]=\max\{\hat{Q}^p(\tau):\ 0\le\tau\le t\},$$

$$\Delta x=|x_1-x_2|,\qquad \lambda_p=e^{mt}-\sum_{l=0}^{p-1}\frac{(mt)^l}{l!}\qquad \Big(\int_0^t m\lambda_p d\tau=\lambda_{p+1}\Big).$$

From (4), (6) we obtain

$$e^{mt}\hat{Q}^p(t)\le\int_0^t 2me^{m\tau}\hat{Q}^{p-1}(\tau)d\tau+\int_0^t me^{m\tau}\bar{Q}^{p-1}(\tau)d\tau+\Delta x^\alpha\Big[\hat{h}+\hat{f}\frac{1}{m}(e^{mt}-1)\Big],$$

where $\hat{h}$ i $\hat{f}$ are the Hölder constants in $\bar{G}^T$ for $\hat{h}(x)$ and $\hat{f}\big(t,x,u;u(t,\cdot)\big)$ respectively.

Using an iteration we prove the following estimation

$$e^{mt}\hat{Q}^p(t)\le 2^p\lambda_p\hat{Q}^0[t]+\sum_{l=1}^{p}2^{l-1}\lambda_l\bar{Q}^{p-l}[t]+\Delta x^\alpha\Big[\hat{f}\frac{1}{m}\sum_{l=1}^{p}2^{l-1}\lambda_l+\hat{h}\sum_{l=1}^{p}\frac{(2m)^{l-1}t^{l-1}}{(l-1)!}\Big]. \tag{7}$$

For $p=1$ we obtain

$$\begin{aligned}e^{mt}\hat{Q}^1(t)&\le\int_0^t 2me^{m\tau}\hat{Q}^0(\tau)d\tau+\int_0^t me^{m\tau}\bar{Q}^0(\tau)d\tau+\Delta x^\alpha\Big[\hat{h}+\hat{f}\frac{1}{m}(e^{mt}-1)\Big]\le\\ &\le 2\lambda_1\hat{Q}^0[t]+\lambda_1\bar{Q}^0[t]+\Delta x^\alpha\Big[\hat{f}\frac{1}{m}\lambda_1+\hat{h}\Big].\end{aligned}$$

Let us assume that (7) is true for $p=q-1$, then for $p=q$ we obtain

$$\begin{aligned}e^{mt}\hat{Q}^q(t)&\le\int_0^t 2me^{m\tau}\hat{Q}^{q-1}(\tau)d\tau+\int_0^t me^{m\tau}\bar{Q}^{q-1}(\tau)d\tau+\Delta x^\alpha\Big[\hat{h}+\hat{f}\frac{1}{m}(e^{mt}-1)\Big]\le\\ &\le\int_0^t 2^q m\hat{Q}^0[t]\lambda_{q-1}d\tau+\int_0^t m\sum_{k=1}^{q-1}2^k\lambda_k\bar{Q}^{q-1-k}[t]d\tau+\int_0^t me^{m\tau}\bar{Q}^{q-1}(\tau)d\tau+\\ &+\Delta x^\alpha\Big[\hat{h}+\hat{f}\frac{(e^{mt}-1)}{m}+\int_0^t 2m\Delta x^\alpha\Big(\hat{h}\sum_{l=1}^{q-1}\frac{(2m)^{l-1}t^{l-1}}{(l-1)!}+\frac{\hat{f}}{m}\sum_{k=1}^{p-1}2^{k-1}\lambda_k\Big)d\tau\Big]\le\\ &\le 2^q\lambda_q\hat{Q}^0[t]+\sum_{l=1}^{q}2^{l-1}\lambda_l\bar{Q}^{q-l}[t]+\Delta x^\alpha\Big[\hat{f}\frac{1}{m}\sum_{l=1}^{q}2^{l-1}\lambda_l+\hat{h}\sum_{l=1}^{q}(2m)^{l-1}\frac{t^{l-1}}{(l-1)!}\Big].\end{aligned}$$

Hence (7) is proved.

From (7) we obtain

$$\hat{Q}^p[T] \le e^{mT}\big[\Delta x^\alpha \hat{h} + \hat{Q}^0[T]\big] + 2e^{3mT}\Delta x^\alpha \Big[C_2 + \frac{\hat{f}}{m}\Big] \le C_3 \Delta x^\alpha,$$

where C_3 is independed of p.

Hence, the sequence $\{u^p\}$ is uniformly Hölder bounded (exponent α) in G^T.

c) From assumptions C, D and from $u^p \in C^\alpha$ we get $\big|\bar{f}_i\big(t,x,u^p(t,x);u^p(t,\cdot)\big)\big|_\alpha^{G^T} \le C_4$, where C_4 is independent of p, hence [9] we obtain the uniform on p estimation

$$|\bar{u}_i^p|_{2+\alpha}^{G^T} \le C_5\Big(\big|\bar{f}_i\big(t,x,u^{p-1}(t,x);u^{p-1}(t,\cdot)\big)\big|_\alpha^{G^T} + m|\bar{u}_i^{p-1}|_\alpha^{G^T} + |\bar{G}_i|_{2+\alpha}^{G^T}\Big) \le C_6,$$

where C_5 depends on A, m_0, and the domain G^T.

d) Using compactness of Hölder spaces C^α and $C^{2+\alpha}$ in C^γ and $C^{2+\gamma}$ respectively ($0<\gamma<\alpha<1$) [9], we choose the convergent subsequence $\{u^{p_1}\}$ from $\{u^p\}$, such that the sequense $\{u^{p_1}\}$ converges to a limit in C^γ ($0<\gamma<\alpha$) and the sequense $\{\bar{u}^{p_1}\}$ converges to a limit in $C^{2+\gamma}$.

We set $u(t,x) = \lim_{p_1\to\infty} u^{p_1}(t,x)$, then $\bar{u} \in C^{2+\gamma}$ in G^T and $\hat{u}$, $\partial\hat{u}/\partial t \in C^\gamma$ in G^T. The theorem is proved.

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