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ON THE ASYMPTOTIC BEHAVIOUR OF CAUCHY-STIELTJES INTEGRALS

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The asymptotic behaviour of analytic in the unit disc function $f(z)=\int\limits_{-\pi}^{\pi}\frac{dg(t)}{(1-ze^{-it})^{\alpha}}$, where g is a complex-valued function of bounded variation, is studied.

1. Let $D=\{z:|z|<1\}, \theta\in[-\pi,\pi], \gamma\in[0,\pi)$, and $S(\theta,\gamma)$ be the closed Stolz angle having vertex $e^{i\theta}$ and opening γ. A function f defined in D is said to have a nontangential limit at $e^{i\theta}$ provided that

$$\lim_{z\to e^{i\theta}, z\in S(\theta,\gamma)} f(z)$$

exists for every $\gamma\in[0,\pi)$. For each $\alpha>0$ let $w^{\alpha}=\exp\{\alpha\ln w\}$, where $\ln w$ is the principal branch of logarithm. We consider the function

$$f(z)=\int_{-\pi}^{\pi}(1-ze^{-it})^{-\alpha}dg(t),\quad z\in D, \tag{1}$$

where g is a complex-valued function of bounded variation in $[-\pi,\pi]$. Throughout this paper we assume that every such g is extended to $(-\infty,\infty)$ by $g(t+\pi)-g(t-\pi)=g(\pi)-g(-\pi)$. Then (1) can be rewritten

$$f(z)=\int_{\theta-\pi}^{\theta+\pi}(1-z^{-it})^{-\alpha}dg(t) \tag{2}$$

for each real number θ, where g is of bounded variation on $[\theta-\pi,\theta+\pi]$.

The asymptotic behaviour of function (1) is studied under some conditions on g in [1–2]. Here we continue this investigations. With this purpose, as in [3, pp.150–151], each nonnegative in $[0,+\infty)$ function ω we call a modulus of continuity provided ω is increasing, continuous, semi-additive and $\omega(0)=0$. We remark that from the semi-additivity it follows that for each $B\geq 1$ there exists $C\geq 1$ such that $\omega(x)\leq\omega(Bx)\leq C\omega(x)$ for all $x\geq 0$.

The following theorem is the main result in this paper.

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Theorem 1. *Let $\alpha > 0$, $\theta \in [-\pi, \pi]$, g be a function of bounded variation on $[-\pi, \pi]$, and a modulus of continuity ω satisfies the condition*

$$\int_0^1 t^{-\alpha-1}\omega(t)dt = \infty. \tag{3}$$

If

$$|g(t) - g(\theta)| = o\big(\omega(|t-\theta|)\big), \qquad t \to \theta, \tag{4}$$

and f is given by (1) then

$$|f(z)| / \int_{|1-ze^{-i\theta}|}^{1} t^{-\alpha-1}\omega(t)dt \tag{5}$$

has the nontangential limit zero at $e^{i\theta}$.

A Borel set $E \subset [-\pi, \pi]$ is called of positive ω-capacity provided there exists a measure μ such that

$$\int_E d\mu = \int_{-\pi}^{\pi} d\mu = 1 \tag{6}$$

and

$$\sup_{x\in\mathbb{R}} \int_{-\pi}^{\pi} \frac{d\mu(t)}{\omega(|x-t|)} < \infty. \tag{7}$$

Otherwise, E is called of zero ω-capacity.

Theorem 2. *Let $\alpha > 0$, g be a function of bounded variation on $[-\pi, \pi]$, and a modulus of continuity ω satisfies condition (3). Then (5) has the nontangential limit zero at $e^{i\theta}$ for all θ in $[-\pi, \pi]$ excepting, possibly, a set with zero ω-capacity.*

If $\omega(t) = t^\beta$ (where $0 < \beta < 1$, $\beta \le \alpha$) and $\omega(t) = 1/\ln(1/t)$, then Theorems 1 and 2 imply the corresponding results from [2]. For proofs of Theorems 1 and 2 we use the properties of the modulus of continuity and methods from [2] and [4].

2. We begin with the proof of Theorem 1. We remark that there are positive constants $A = A(\gamma)$ and $B = B(\gamma)$ such that if $z = re^{i\theta} \in S(\theta, \gamma)$, then

$$|z - e^{i\theta}| \le A(1-r), \quad |\varphi - \theta| \le B(1-r). \tag{8}$$

By $C_j(\gamma)$ and $C_j(\gamma, \delta)$ we mean positive constants depending only on γ and on γ and δ.

For fixed $\theta \in [-\pi, \pi]$ using (2) and the equation $dg(t) = d\big(g(t) - g(\theta)\big)$, and integrating by parts we have

$$f(z) = \frac{g(\theta+\pi) - g(\theta-\pi)}{(1+ze^{-i\theta})^2} + i\alpha z \int_{\theta-\pi}^{\theta+\pi} \frac{(g(t)-g(\theta))e^{-it}}{(1-ze^{-it})^{\alpha+1}} dt.$$

Since $1 + ze^{-i\theta} \to 2$ as $z \to e^{i\theta}$, $z \in S(\theta, \gamma)$, we have

$$|f(z)| \le C_1(\gamma) + \alpha \int_{-\pi}^{\pi} P(t)dt, \tag{9}$$

where

$$P(t) = P(z,\theta,t) = \frac{|g(t+\theta)-g(\theta)|}{|1-ze^{-i(\theta+t)}|^{\alpha+1}}.$$

From (4) it follows that for each $\varepsilon > 0$ there exists $\delta \in (0,\pi/3)$ such that

$$|g(\theta+t)-g(\theta)| \le \varepsilon\omega(|t|), \qquad |t|<\delta. \tag{10}$$

We suppose that $z \in S(\theta,\gamma)$ is near to $e^{i\theta}$ so that $2B(1-r) < \delta$, where B is the constant from (8). Thus $|\varphi-\theta| \le \delta/2$ and if $\delta \le |t| \le \pi$ then $|t+\theta-\varphi| \ge |t|-|\theta-\varphi| \ge \delta/2$. Therefore, $|1-ze^{-i(\theta+t)}|^2 = 1+r^2-2r\cos|t+\theta-\varphi| \ge 1-\cos^2|t+\theta-\varphi| \ge \sin^2(\delta/2)$, and

$$\Big(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\Big)P(t)dt \le C_2(\gamma,\delta). \tag{11}$$

Further, using (10), (8) and half-additivity of ω, we have

$$\begin{aligned}\int_{2B(1-r)}^{\delta} P(t)dt &\le \varepsilon\int_{2B(1-r)}^{\delta}\frac{\omega(|t|)dt}{|1-re^{-i(\theta+t-\varphi)}|^{\alpha+1}} = \varepsilon\int_{2B(1-r)+\varphi-\theta}^{\delta+\varphi-\theta}\frac{\omega(|t+\varphi-\theta|)dt}{|1-re^{-it}|^{\alpha+1}} \le \\ &\le \varepsilon\int_{B(1-r)}^{\delta+B(1-r)}\frac{\omega(|t|)+\omega(B(1-r))}{|\sin t|^{\alpha+1}}dt \le \\ &\le \varepsilon C_3(\gamma)\Big(\int_{B(1-r)}^{3\delta/2}\frac{\omega(t)}{t^{\alpha+1}}dt+\omega(1-r)\int_{B(1-r)}^{3\delta/2}\frac{dt}{t^{\alpha+1}}\Big) \le \\ &\le \varepsilon C_4(\gamma)\Big(\int_{1-r}^{1}\frac{\omega(t)}{t^{\alpha+1}}dt+\frac{\omega(1-r)}{(1-r)^{\alpha}}\Big) \le \varepsilon C_5\int_{1-r}^{1}\frac{\omega(t)}{t^{\alpha+1}}dt,\end{aligned}$$

because

$$\int_{1-r}^{1}\frac{\omega(t)}{t^{\alpha+1}}dt \ge \omega(1-r)\int_{1-r}^{1}\frac{dt}{t^{\alpha+1}} = \frac{\omega(1-r)}{\alpha}\Big(\frac{1}{(1-r)^{\alpha}}-1\Big). \tag{12}$$

The analogous estimate is correct for the integral $\int_{-\delta}^{-2B(1-r)} P(t)dt$. Therefore,

$$\Big(\int_{-\delta}^{-2B(1-r)}+\int_{2B(1-r)}^{\delta}\Big)P(t)dt \le \varepsilon C_6(\gamma)\int_{1-r}^{1}\frac{\omega(t)}{t^{\alpha+1}}dt. \tag{13}$$

Finally,

$$\int_{-2B(1-r)}^{-2B(1-r)} P(t)dt \le \varepsilon\int_{-2B(1-r)}^{2B(1-r)}\frac{\omega(|t|)}{(1-r)^{\alpha+1}}dt \le \varepsilon C_7(\gamma)\frac{\omega(1-r)}{(1-r)^{\alpha}}. \tag{14}$$

Estimates (9), (11), (13), and (14) in view of (12) yield

$$\lim_{z\to e^{i\theta},z\in S(\theta,\gamma)}\Big\{|f(z)|/\int_{1-r}^{1}t^{-\alpha-1}\omega(t)dt\Big\} = 0. \tag{15}$$

Since for all $z \in S(\theta, \gamma)$ enough near to $e^{i\theta}$

$$|1 - e^{-i\theta}z|^2 \le (1-r)^2 + 2(\varphi - \theta)^2 \le (1+2B^2)(1-r)^2,$$

we have

$$\int_{|1-e^{-i\theta}z|}^{1} \frac{\omega(t)}{t^{\alpha+1}}dt \ge \int_{(1-r)\sqrt{1+2B^2}}^{1} \frac{\omega(t)}{t^{\alpha+1}}dt \ge (1+2B^2)^{-\alpha/2}\Big(C_8(\gamma) + \int_{1-r}^{1} \frac{\omega(t)}{t^{-\alpha-1}}dt\Big)$$

and hence, from (15) it follows that the nontangential limit of (5) at $e^{i\theta}$ is equal to 0.

3. We use the following lemma for proving Theorem 2.

Lemma 1. *Let g be a nondecreasing function on $[-\pi, \pi]$ extended to $(-\infty, \infty)$ by $g(t+\pi) - g(t-\pi) = g(\pi) - g(-\pi)$, and ω a modulus of continuity. Then (4) is correct for all θ in $[-\pi, \pi]$ excepting, possibly, a set whose ω-capacity is zero.*

Indeed, let $E = \{\theta \in [-\pi, \pi] : J(\theta) = +\infty\}$, where $J(\theta) = \int_{-\pi}^{\pi} 1/\omega(|\theta - t|)dg(t)$. We show that this Borel set E has zero ω-capacity. Actually, if E has positive ω-capacity, then (6) and (7) are correct. Thus in view of Fubini theorem

$$\infty > \int_{-\pi}^{\pi}\Big(\int_{-\pi}^{\pi} \frac{d\mu(\theta)}{\omega(|t-\theta|)}\Big)dg(t) = \int_{-\pi}^{\pi}\Big(\int_{-\pi}^{\pi} \frac{dg(t)}{\omega(|\theta - t|)}\Big)d\mu(\theta).$$

This contradicts $J(\theta) = \infty$ on E. We remark that the set of points of discontinuity of g belong to E.

If $\theta \notin E$ that is $J(\theta) < \infty$, and therefore $\int_{-\pi}^{\pi} 1/\omega(|t|)dg(t+\theta) < \infty$, then for each $\varepsilon > 0$ and enough small $x > 0$

$$\varepsilon > \int_{-2x}^{2x} \frac{dg(t+\theta)}{\omega(|t|)} \ge \frac{g(\theta+x) - g(\theta - x)}{\omega(2x)}$$

that is $g(\theta+x) - g(\theta-x) \le 2\varepsilon\omega(x)$. Since the function g is nondecreasing, Lemma 1 is proved.

Theorem 2 is a direct consequence of Theorem 1 and Lemma 1.

4. We remark that if $\int_0^1 dt/\omega(t) < \infty$ and a set $E \subset [-\pi, \pi]$ has zero ω-capacity, then E has zero Lebesque measure. Indeed, if m is the Lebesque measure on $[-\pi, \pi]$ and $mE > 0$, let $d\mu(x) = (\chi_E(x)/m(E))dm(x)$, where $\chi_E(x)$ is the characteristic function of E. Thus

$$\int_{-\pi}^{\pi} \frac{d\mu(x)}{\omega(|x-t|)} = \frac{1}{m(E)}\int_{-\pi}^{\pi} \frac{\chi_E(t)dm(t)}{\omega(|x-t|)} \le \frac{1}{m(E)}\int_{-\pi}^{\pi} \frac{dt}{\omega(|x-t|)} \le \frac{2}{m(E)}\int_{0}^{2\pi} \frac{dt}{\omega(t)} < \infty,$$

that is the ω-capacity of E is positive.

We also remark that Theorems 1 and 2 are valid with $\alpha = 0$ for the function

$$f(z) = \int_{-\pi}^{\pi} \ln\left(1 - ze^{-it}\right) dg(t).$$

REFERENCES

1. Hallenbeck D.J., MacGregor T.H. *Radial limits and radial growth of Cauchy-Stieltjes transforms* // Complex Variables. 1993. V.21. P.219–229.
2. Hallenbeck D.J., MacGregor T.H. *Radial growth and exceptional sets for Cauchy-Stieltjes integrals* // Proc. Edinburgh Math. Soc. 1993. V.37. P.73–89.
3. Дзядык В.К. Введение в теорию равномерной аппроксимации функций полиномами. — М.: Наука. 1977. 512с.
4. Twomey J.B. *Tangential boundary behaviour of the integral* // J. London Math. Soc. 1988. V.37. P.447–454.

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