УДК 517.983

OPERATOR-VALUED CHARGES ON FINITE SETS

T.S. Kudryk, V.E. Lyantse

T.S. Kudryk, V.E. Lyantse. *Operator-valued charges on finite sets*, Matematychni Studii, **7**(1997) 145–156.

Investigations on applications of nonstandard methods to measure theory and operator theory (see [1] and [2]) are extended to operator-valued measures and charges. Spaces of operator-valued charges on the algebra $2^{\mathbb{T}}$ and corresponding charges on the algebra $2^{\mathbf{T}}$, where $\mathbf{T}$ is the standard filling of a finite (in the sense of IST) set $\mathbb{T}$, are introduced and examined. The notions of weak, strong and uniform nearstandardness are defined for such charges (including the technics of equipment of Hilbert space). It is proved that decomposition of the unity $\mathfrak{N}$ of discrete differentiation operator D is related to the decomposition of the unity $\mathfrak{M}$ of usual differentiation operator $\mathbf{D}$ in the same way as the operators D and $\mathbf{D}$: ${}^\circ\mathfrak{N} = \mathfrak{M}$.

This article is devoted to the investigation of operator-valued charges in hyperfinite spaces. It is a continuation of [3] and [4].

$\mathbb{T}$ denotes a finite (in the sense of IST) set such that $\operatorname{card}\mathbb{T} \approx +\infty$, and $(\mathbf{T}, Q, \lambda)$ its standard filling (see [3]). Earlier we have assumed that $\forall\, t \in \mathbb{T}\ \ \lambda Qt = h = const.$ Now this condition is substituted by a less restrictive one:

$$\forall\, t \in \mathbb{T} \quad \lambda Qt > 0. \tag{1}$$

We are to correct the embedding $Q : \mathcal{N} \to \mathcal{M}$ ($\mathcal{N}$ is the set of all charges on $\mathbb{T}$, and $\mathcal{M}$ is the set of all regular σ-additive charges on $\mathbf{T}$) as follows:

$$\forall\, \nu \in \mathcal{N} \quad \forall\, \mathcal{E} \in \Lambda \quad Q\nu\mathcal{E} := \sum_{t\in\mathbb{T}} \lambda_{\mathcal{E}}(t)\nu_t, \tag{2}$$

where $\forall\, t \in \mathbb{T} \quad \lambda_{\mathcal{E}}(t) := \lambda(\mathcal{E} \cap Qt)(\lambda Qt)^{-1}$. Let $\mathbb{H}$ be the Hilbert space of functions $x \in \mathbb{C}^{\mathbb{T}}$ with the inner product

$$(x|y) = \sum_{t\in\mathbb{T}} x(t)\overline{y(t)}\nu_t, \tag{3}$$

where $\nu := \Pi\lambda$. Note that $\nu \in {}^{nst}\mathcal{N}$ and ${}^\circ\nu = \lambda$, because the measure λ is standard.

1. Space $\tilde{\mathfrak{N}}$. Decomposition of the unity of an operator $A \in \mathcal{B}(\mathbb{H})$ is an operator-valued measure defined on the spectrum $\sigma(A)$ of A. This spectrum is a finite set ($\operatorname{card}\sigma(A) \le \operatorname{card}\mathbb{T} = \dim\,\mathbb{H}$)). Therefore, it is expedient to introduce another

1991 *Mathematics Subject Classification.* 28E05, 47S20.
Research partially supported by a grant from the Fundamental Research State Foundation.

finite (in the sense of IST) set $\hat{\mathbb{T}}$ and its standard filling $(\hat{\mathbf{T}}, \hat{Q}, \hat{\lambda})$. An operator $\mathfrak{N}_{\hat{t}} \in \mathcal{B}(\mathbb{H})$ is assigned to each $\hat{t} \in \hat{\mathbb{T}}$. Then define

$$\forall \hat{E} \in 2^{\hat{\mathbb{T}}} \quad \mathfrak{N}\hat{E} := \sum_{\hat{t}\in\hat{E}} \mathfrak{N}_{\hat{t}}. \tag{4}$$

Then $\mathfrak{N}$ is an additive function defined on the algebra $2^{\hat{\mathbb{T}}}$, with values in $\mathcal{B}(\mathbb{H})$: $\mathfrak{N} \in \mathcal{B}(\mathbb{H})^{2^{\hat{\mathbb{T}}}}$. Every such function is called an *operator-valued charge* (on $2^{\hat{\mathbb{T}}}$). Formula (4) defines the general form of an operator-valued charge. In particular, $\forall \hat{t} \in \hat{\mathbb{T}}\ \ \mathfrak{N}\{\hat{t}\} = \mathfrak{N}_{\hat{t}}$. The set of all operator-valued charges is denoted by $\tilde{\mathfrak{N}} = \tilde{\mathfrak{N}}(\hat{\mathbb{T}})$.

The variation of an operator-valued charge $\mathfrak{N} \in \tilde{\mathfrak{N}}$ is the number measure $|\mathfrak{N}| \in \mathcal{N}_+(\hat{\mathbb{T}})$ defined by

$$\forall \hat{E} \in 2^{\hat{\mathbb{T}}} \quad |\mathfrak{N}|\hat{E} := \sum_{\hat{t}\in\hat{E}} \|\mathfrak{N}_{\hat{t}}\|_{\mathcal{B}(\mathbb{H})}. \tag{5}$$

$\tilde{\mathfrak{N}}$ is a *normed* space with pointwise arithmetical operations and with the norm

$$\forall \mathfrak{N} \in \tilde{\mathfrak{N}} \quad \|\mathfrak{N}\| = |\mathfrak{N}|\hat{T}. \tag{6}$$

This normed space is *complete* because $dim\, \tilde{\mathfrak{N}} = 2^{\operatorname{card} \hat{T}} \in \mathbb{N}$.

Operator-valued charge $\mathfrak{N}$ is *absolutely continuous* with respect to its variation $|\mathfrak{N}|$ in the sense that $\forall \hat{E} \in 2^{\hat{\mathbb{T}}} \quad (|\mathfrak{N}|\hat{E} \approx 0 \implies \|\mathfrak{N}\hat{E}\|_{\mathcal{B}(\mathbb{H})} \approx 0)$. This follows from the evident inequality $\forall \hat{E} \in 2^{\hat{\mathbb{T}}} \quad \|\mathfrak{N}\hat{E}\| \le |\mathfrak{N}|\hat{E}$ ▶

$\mathfrak{N} \in \tilde{\mathfrak{N}}$ is called *real*, if $\forall \hat{E} \in 2^{\hat{\mathbb{T}}} (\mathfrak{N}\hat{E})^* = \mathfrak{N}E$. Of course, this condition is equivalent to the following: $\forall \hat{t} \in \hat{\mathbb{T}}\ (\mathfrak{N}_{\hat{t}})^* = \mathfrak{N}_{\hat{t}}$. An operator-valued charge $\mathfrak{N} \in \tilde{\mathfrak{N}}$ such that $\forall \hat{E} \in 2^{\hat{\mathbb{T}}}\ \mathfrak{N}\hat{E} \ge 0$ (i.e. $\forall x \in \mathbb{C}^{\mathbb{T}}\ (\mathfrak{N}\hat{E}x|x| \ge 0)$ is said to be an operator-valued *measure* on $2^{\hat{\mathbb{T}}}$).

To each $\mathfrak{N} \in \tilde{\mathfrak{N}}$, $\hat{E} \in 2^{\hat{T}}$, and $\hat{x} \in \mathbb{C}^{\mathbb{T}}$ assign "the discrete integral" $\mathfrak{N}_{\hat{E}}$ and the operator-valued charge $\mathfrak{N}^{\hat{x}}$ defined by $\mathfrak{N}_{\hat{E}}\hat{x} = \mathfrak{N}^{\hat{x}}\hat{E} := \sum_{\hat{t}\in\hat{E}} \hat{x}(\hat{t})\mathfrak{N}_{\hat{t}}$. Obviously, $\mathfrak{N}_{\hat{E}}$ is a linear map $\mathbb{C}^{\hat{\mathbb{T}}} \to \mathcal{B}(\mathbb{H})$, $\mathfrak{N}^{\hat{x}}$ is an additive map $2^{\hat{\mathbb{T}}} \to \mathcal{B}(\mathbb{H})$, and $\|\mathfrak{N}_{\hat{E}}\hat{x}\| \le |\mathfrak{N}|\hat{E} \cdot \|\hat{x}\|_{\hat{E}}$, where $\|\hat{x}\|_{\hat{E}} = \max_{\hat{t}\in\hat{E}} |\hat{x}(\hat{t})|$, $\|\mathfrak{N}^{\hat{x}}\hat{E} - \mathfrak{N}^{\hat{x}}\hat{F}\| \le |\mathfrak{N}|(\hat{E}\triangle\hat{F}) \cdot \|\hat{x}\|_{\hat{\mathbb{T}}}$.

Let $\mathfrak{N} \in \tilde{\mathfrak{N}}$. To each vector $x \in \mathbb{H}$ assign *the vector-valued charge* $\mathfrak{N}(x)$ defined by

$$\forall \hat{E} \in 2^{\hat{\mathbb{T}}} \quad \mathfrak{N}(x)\hat{E} := \sum_{\hat{t}\in\hat{E}} \mathfrak{N}_{\hat{t}}x, \tag{7}$$

with *the variation* $|\mathfrak{N}(x)|$:

$$\forall \hat{E} \in 2^{\hat{\mathbb{T}}} \quad |\mathfrak{N}(x)|\hat{E} = \sum_{\hat{t}\in\hat{E}} \|\mathfrak{N}_{\hat{t}}x\|_{\mathbb{H}}, \tag{8}$$

and with *the norm* $\|\mathfrak{N}(x)\|$:

$$\|\mathfrak{N}(x)\| := |\mathfrak{N}(x)|\hat{\mathbb{T}}. \tag{9}$$

Besides, *complex-valued charge* $\mathfrak{N}(x,y)$ corresponds to each pair $(x,y) \in \mathbb{H}^2$ according to the formula

$$\forall \hat{E} \in 2^{\hat{\mathbb{T}}} \quad \mathfrak{N}(x,y)\hat{E} := (\mathfrak{N}\hat{E}x|y)_{\mathbb{H}}, \tag{10}$$

with *the variation* $|\mathfrak{N}(x,y)|$:

$$\forall \hat{E} \in 2^{\hat{\mathbb{T}}} \quad |\mathfrak{N}(x,y)|\hat{E} = \sum_{\hat{t}\in\hat{E}} |(\mathfrak{N}_{\hat{t}}x|y)_{\mathbb{H}}|, \tag{11}$$

and with *the norm* $\|\mathfrak{N}(x,y)\|$:

$$\|\mathfrak{N}(x,y)\| = |\mathfrak{N}(x,y)|\hat{\mathbb{T}}. \tag{12}$$

To each $\hat{t} \in \hat{\mathbb{T}}$ assign *the matrix* $\mathfrak{N}_{\hat{t}}(\cdot,\cdot) \in \mathbb{C}^{\mathbb{T}^2}$ of the operator $\mathfrak{N}_{\hat{t}}$ by

$$\forall (t,s) \in \mathbb{T}^2 \quad \mathfrak{N}_{\hat{t}}(t,s) = (\mathfrak{N}_{\hat{t}}\delta_s|\delta_t),$$

where δ_t is the Dirac delta concentrated at the point $t \in \mathbb{T}$. We get from (10) that $\mathfrak{N}_{\hat{t}}(t,s) = \mathfrak{N}(\delta_s,\delta_t)\{\hat{t}\}$. This means that to each pair $(t,s) \in \mathbb{T}^2$ we assigned *the complex-valued charge* $\mathfrak{N}[t,s] \in \mathcal{N} = \mathcal{N}(\hat{\mathbb{T}})$ such that $\forall \hat{E} \in 2^{\hat{\mathbb{T}}} \quad \mathfrak{N}[t,s]\hat{E} = \sum_{\hat{t}\in\hat{E}} \mathfrak{N}_{\hat{t}}(t,s)$
It is easy to prove that $\forall \hat{E} \in 2^{\hat{\mathbb{T}}}\ \forall x,y \in \mathbb{H}\ (\mathfrak{N}\hat{E}x|y) = \sum_{(t,s)\in\mathbb{T}^2} \mathfrak{N}[t,s]\hat{E}x(s)\overline{y(t)}\nu_s\overline{\nu_t}$.

2. Space $\tilde{\mathfrak{M}}$. Let $\mathbf{H}$ be the Hilbert space of all functions $\xi \in \mathbb{C}^{\mathbf{T}}$, with the inner product

$$(\xi|\eta) = \int_{\mathbf{T}} \xi(\tau)\overline{\eta(\tau)}\lambda(d\tau).$$

Denote by $\tilde{\mathfrak{M}}$ the set of all operator-valued charges defined on the algebra $\hat{\Lambda}$ of $\hat{\lambda}$-measurable sets $\hat{\mathcal{E}} \in 2^{\mathbf{T}}$, where $(\hat{\mathbf{T}},\hat{Q},\hat{\lambda})$ is the standard filling of the set $\hat{\mathbb{T}}$. By definition, $\mathfrak{M} \in \tilde{\mathfrak{M}}$ whenever $\operatorname{dom}\mathfrak{M} = \hat{\Lambda}$, $\forall \hat{\mathcal{E}} \in \hat{\Lambda}\ \mathfrak{M}\hat{\mathcal{E}} \in \mathcal{B}(\mathbf{H})$, and $\mathfrak{M}$ is σ-additive in *the strong* operator topology.

For $\mathfrak{M} \in \tilde{\mathfrak{M}}$ by $|\mathfrak{M}|$ we denote its *variation*, i.e. the mesure $|\mathfrak{M}| \in \mathcal{M}_+(\hat{\mathbf{T}})$ such that

$$\forall \hat{\mathcal{E}} \in \hat{\Lambda} \quad |\mathfrak{M}|\hat{\mathcal{E}} := \sup \sum_k \|\mathfrak{M}\hat{\mathcal{E}}_k\|_{\mathcal{B}(\mathbf{H})},$$

where sup extends all disjoint partitions $\hat{\mathcal{E}} = \bigsqcup_k \hat{\mathcal{E}}_k$, $\hat{\mathcal{E}}_k \in \hat{\Lambda}$. $\tilde{\mathfrak{M}}$ is interpreted as *a normed* space with natural arithmetical operations and with the norm

$$\forall \mathfrak{M} \in \tilde{\mathfrak{M}} \quad \|\mathfrak{M}\| := |\mathfrak{M}|\hat{\mathbf{T}}. \tag{13}$$

Now we consider another topologies on the space $\mathfrak{N}$. Let $\mathfrak{M} \in \tilde{\mathfrak{M}}$, $\xi,\eta \in \mathbf{H}$. By $\mathfrak{M}(\xi)$ we denote *the vector-valued* (with values in $\mathbf{H}$) charge defined by $\forall \hat{\mathcal{E}} \in \hat{\Lambda} \quad \mathfrak{M}(\xi)\hat{\mathcal{E}} = (\mathfrak{M}\mathcal{E})\xi$. By $\mathfrak{M}(\xi,\eta)$ we denote *the complex-valued* charge (i.e. $\mathfrak{M}(\xi,\eta) \in \mathcal{M} := \mathcal{M}(\hat{\mathbf{T}})$), defined by $\forall \hat{\mathcal{E}} \quad \mathfrak{M}(\xi,\eta)\hat{\mathcal{E}} := ((\mathfrak{M}\hat{\mathcal{E}})\xi|\eta)_{\mathbf{H}}$. In what follows we put

$$\forall \hat{\mathcal{E}} \in \hat{\Lambda} \quad |\mathfrak{M}(\xi)|\hat{\mathcal{E}} := \sup \sum_k \|(\mathfrak{M}\hat{\mathcal{E}}_k)\xi\|_{\mathbf{H}},$$

$$\text{and } \forall \hat{\mathcal{E}} \in \hat{\Lambda} \quad |\mathfrak{M}(\xi,\eta)|\hat{\mathcal{E}} := \sup \sum_k |((\mathfrak{M}\hat{\mathcal{E}}_k)\xi|\eta)_{\mathbf{H}}|, \tag{14}$$

We also put

$$\|\mathfrak{M}(\xi)\| := |\mathfrak{M}(\xi)|\hat{\mathbf{T}},$$
$$\text{and } \|\mathfrak{M}(\xi,\eta)\| := |\mathfrak{M}(\xi,\eta)|\hat{\mathbf{T}}. \tag{15}$$

These norms define correspondingly *the strong* and *weak* topology on $\mathfrak{M}$.

3. Maps $\mathfrak{Q}$ and $\mathfrak{P}$. Recall that $\forall A \in \mathcal{B}(\mathbb{H})$ $\mathfrak{Q}A := QA\Pi \in \mathcal{B}(\mathbf{H})$. We shall extend the map $\mathfrak{Q}$ to operator-valued charges $\mathfrak{N} \in \tilde{\mathfrak{N}}$. Put (see (2))

$$\forall \mathfrak{N} \in \tilde{\mathfrak{N}} \quad \forall \hat{\mathcal{E}} \in \hat{\Lambda} \quad (\mathfrak{Q}\mathfrak{N})\hat{\mathcal{E}} := \sum_{\hat{t}\in\hat{\mathbb{T}}} \hat{\lambda}_{\hat{\mathcal{E}}}(\hat{t})\mathfrak{Q}\mathfrak{N}_{\hat{t}}; \tag{16}$$

in particular,

$$\forall \hat{t} \in \hat{\mathbb{T}} \quad (\mathfrak{Q}\mathfrak{N})\hat{Q}\hat{t} =: \mathfrak{Q}\mathfrak{N}_{\hat{t}} = Q\mathfrak{N}_{\hat{t}}\Pi. \tag{17}$$

Proposition. *$\mathfrak{Q}$ is a linear injective unstretching (with respect to norm (13)) transformation $\tilde{\mathfrak{N}} \to \tilde{\mathfrak{M}}$. It maps real o.charges to real ones and o.measures to real ones.*

◁ It follows from $\operatorname{im}\Pi = \mathbb{H}$, $\ker Q = \{0\}$, and (17) that $\forall \hat{t} \in \hat{\mathbb{T}}$ $\mathfrak{N}_{\hat{t}} = 0$ whenever $\mathfrak{Q}\mathfrak{N} = 0$. Since the (standard) number measure $\hat{\lambda}$ is σ-additive, we obtain that $\forall \hat{t} \in \hat{\mathbb{T}}$ the map $\hat{\mathcal{E}} \mapsto \hat{\lambda}_{\hat{\mathcal{E}}}(\hat{t})$ is a σ-additive measure on the subalgebra $\{\hat{\mathcal{E}} \in \hat{\Lambda} : \hat{\mathcal{E}} \subseteq \hat{Q}\hat{t}\}$ of the algebra $\hat{\Lambda}$. We have $\hat{\lambda}_{\hat{q}\hat{t}}(\hat{t}) = 1$. Consequently, the function $\mathfrak{Q}\mathfrak{N}$ is σ-additive on $\hat{\Lambda}$. Since $\|Q\| = \|\Pi\| = 1$, we have $\|\mathfrak{Q}\mathfrak{N}\| \leq \|\mathfrak{N}\|$. Denote by $\tilde{\mathfrak{N}}_+$ and $\tilde{\mathfrak{M}}_+$ the cones of o.mesures in $\tilde{\mathfrak{N}}$ and $\tilde{\mathfrak{M}}$ respectively. By (17), we get $\forall \hat{t} \in \hat{\mathbb{T}}$ $\forall \xi \in \mathbf{H}$ $(\mathfrak{Q}\mathfrak{N}_{\hat{t}}\xi|\xi) = (\mathfrak{N}_{\hat{t}}\Pi\xi|\Pi\xi) \geq 0$, whenever $\forall \hat{t} \in \hat{\mathbb{T}}$ $\mathfrak{N}_{\hat{t}} \geq 0$. Therefore, $\mathfrak{Q}\tilde{\mathfrak{N}}_+ \subset \tilde{\mathfrak{M}}_+$ ▶

Now we extend the inductor $\mathfrak{P}$ to all of $\tilde{\mathfrak{N}}$. Recall that for $\mathbf{A} \in \mathcal{B}(\mathbf{H})$ we have $\mathfrak{P}\mathbf{A} := \Pi\mathbf{A}Q$. For this reason we define

$$\forall \mathfrak{M} \in \tilde{\mathfrak{M}} \quad \forall \hat{E} \in 2^{\hat{\mathbb{T}}} \quad (\mathfrak{P}\mathfrak{M})\hat{E} := \Pi(\mathfrak{M}\hat{Q}\hat{E})Q; \tag{18}$$

in particular,

$$\forall \hat{t} \in \hat{\mathbb{T}} \quad (\mathfrak{P}\mathfrak{M})_{\hat{t}} = \Pi(\mathfrak{M}\hat{Q}\hat{t})Q. \tag{19}$$

Proposition. *The inductor $\mathfrak{P}$ is a linear unstretching map $\tilde{\mathfrak{M}} \to \tilde{\mathfrak{N}}$ which maps real o.charges to real ones and o.measures to o.measures. It is left inverse to the embedding $\mathfrak{Q} : \tilde{\mathfrak{N}} \to \tilde{\mathfrak{M}}$. This means that*

$$\forall \mathfrak{M} \in \tilde{\mathfrak{M}} \quad \|\mathfrak{P}\mathfrak{M}\| \leq \|\mathfrak{M}\|,$$
$$\forall \mathfrak{N} \in \tilde{\mathfrak{N}} \quad \forall \hat{E} \in 2^{\hat{\mathbb{T}}} \quad \mathfrak{P}\mathfrak{Q}\mathfrak{N}\hat{E} = \mathfrak{N}\hat{E}.$$

◁ According to (6) and (19), we find $\|\mathfrak{P}\mathfrak{M}\| = \sum_{\hat{t}\in\hat{\mathbb{T}}} \|(\mathfrak{P}\mathfrak{M})_{\hat{t}}\| = \sum_{\hat{t}\in\hat{\mathbb{T}}} \|\Pi\mathfrak{M}\hat{Q}\hat{t}Q\| \leq \sum_{\hat{t}\in\hat{\mathbb{T}}} \|\mathfrak{M}\hat{Q}\hat{t}\| \leq \|\mathfrak{M}\|$. Sinse $\Pi Q = \mathbb{I}_{\mathbb{H}}$ and $\hat{\lambda}_{\hat{Q}\hat{E}}$ coincides with the characteristic function of $\hat{E}$, we have $\mathfrak{P}\mathfrak{Q}\mathfrak{N}\hat{E} = \Pi(\mathfrak{Q}\mathfrak{N})\hat{Q}\hat{E}Q = \Pi(\sum_{\hat{t}\in\hat{\mathbb{T}}} \hat{\lambda}_{\hat{Q}\hat{E}}(\hat{t})Q\mathfrak{N}_{\hat{t}}\Pi)Q = \sum_{\hat{t}\in\hat{\mathbb{T}}} \hat{\lambda}_{\hat{Q}\hat{E}}(\hat{t})\mathfrak{N}_{\hat{t}} = \mathfrak{N}\hat{E}$, by (16) and (18).

4. Nearstandard o.charges. Nearstandardness notion depends on the choice of topology.

Definition. O.charge $\mathfrak{N}$ is called *weakly*, (respectively *strongly, uniformly*) *nearstandard* if for some $\mathfrak{M} \in {}^{st}\tilde{\mathfrak{M}}$

$$\forall\, \xi, \eta \in {}^{st}\mathbf{H} \quad \|(\mathfrak{N} - \mathfrak{P}\mathfrak{M})(\Pi\xi, \Pi\eta)\| \approx 0, \tag{20}$$

(respectively

$$\forall\, \xi \in {}^{st}\mathbf{H} \quad \|(\mathfrak{N} - \mathfrak{P}\mathfrak{M})(\Pi\xi)\| \approx 0, \tag{21}$$

$$\|\mathfrak{N} - \mathfrak{P}\mathfrak{M}\| \approx 0). \tag{22}$$

Obviously, a uniformly nearstandard o.charge is strongly nearstandard, and o.charge $\mathfrak{M}$ satisfying (22) also satisfies (21). A strongly nearstandard o.charge is weakly nearstandard, and $\mathfrak{M}$ satisfying (21) also satisfies (20). This follows from the following evident inequalities:

$$\forall\, \mathfrak{N} \in \tilde{\mathfrak{N}} \quad \forall\, x \in \mathbb{H} \quad \|\mathfrak{N}(x)\| \leq \|\mathfrak{N}\| \cdot \|x\|, \tag{23}$$

$$\forall\, \mathfrak{N} \in \tilde{\mathfrak{N}} \quad \forall\, x, y \in \mathbb{H} \quad \|\mathfrak{N}(x, y)\| \leq \|\mathfrak{N}(x)\| \cdot \|y\|. \tag{24}$$

In the sequel, we assume that the inductor $\hat{\Pi}$ corresponding to the standard filling $(\hat{\mathbf{T}}, \hat{Q}, \hat{\lambda})$ of $\mathbb{T}$ is *exact*. It is easily shown that o.charge $\mathfrak{M}$ satisfying (20) is *unique*.

Definition. The o.charge $\mathfrak{M}$ satisfying (20) is denoted by ${}^{\circ}\mathfrak{N}$ and is called *the shadow* of $\mathfrak{N}$. *The quasikernel* of $\mathfrak{P}$ is the set $\operatorname{qker}\mathfrak{P} := \{\mathfrak{M} \in \tilde{\mathfrak{M}} : (\forall\, \hat{E} \in \hat{\mathbb{T}})\, (\|\mathfrak{P}\mathfrak{M}\hat{E}\| \approx 0)$.

Proposition. *a) The inductor $\mathfrak{P}$ is exact in the sense that ${}^{st}\operatorname{qker}\mathfrak{P} = \{0\}$. b) If $\mathfrak{N} \in \tilde{\mathfrak{N}}$, $\mathfrak{M} \in {}^{st}\tilde{\mathfrak{M}}$ and $\|\mathfrak{Q}\mathfrak{N} - \mathfrak{M}\| \approx 0$, then o.charge $\mathfrak{N}$ is uniformly nearstandard and ${}^{\circ}\mathfrak{N} = \mathfrak{M}$.*

The proof is left to the reader.

Example. Here we assume that $\hat{\mathbb{T}} = \mathbb{T}$, $(\hat{\mathbf{T}}, \hat{Q}, \hat{\lambda}) = (\mathbf{T}, Q, \lambda)$. Define the o.charges $\mathfrak{N}$ and $\mathfrak{M}$ as follows:

$$\forall\, E \in 2^{\mathbb{T}} \quad \forall\, x, y \in \mathbb{C}^{\mathbb{T}} \quad (\mathfrak{N}Ex|y) := \sum_{t \in E} x(t)\overline{y(t)}\tilde{\nu}_t, \tag{n}$$

where $\tilde{\nu}_t \in \mathcal{N}$ is some complex-valued charge, and

$$\forall\, \mathcal{E} \in \Lambda \quad \forall\, \xi, \eta \in \mathbf{H} \quad (\mathfrak{M}\mathcal{E}\xi|\eta) = \int_{\mathcal{E}} \xi(\tau)\overline{\eta(\tau)}\lambda(d\tau). \tag{m}$$

In other words, $\mathfrak{M}\mathcal{E}$ is the operator of multiplication in $\mathbf{H}$ by the characteristic function of $\mathcal{E}$. Note that $\mathfrak{M} \in {}^{st}\tilde{\mathfrak{M}}_+$.

Let $E \in 2^{\mathbb{T}}$, $x, y \in \mathbb{H}$, then $(\mathfrak{P}\mathfrak{M}Ex|y) = (\mathfrak{M}QEQx|Qy) = \int_{QE} Qx(\tau)\overline{Qy(\tau)}\lambda(d\tau) = \sum_{t \in E} x(t)\overline{y(t)}\nu_t$, where $\nu_t := \lambda Qt$. Consequently,

$$\forall\, t \in \mathbb{T} \quad \forall\, x, y \in \mathbb{H} \quad ((\mathfrak{N} - \mathfrak{P}\mathfrak{M})_t x|y) = x(t)\overline{y(t)}(\tilde{\nu}_t - \nu_t), \tag{25}$$

or $\forall t\in\mathbb{T}\quad\forall x,y\in\mathbb{H}\quad((\mathfrak{N}-\mathfrak{PM})_t x|y)=\Gamma(t)(x|\delta_t)(\delta_t|y)\nu_t$, where δ_t is a discrete Dirac delta, and $\forall t\in\mathbb{T}\quad\Gamma(t):=(\tilde\nu_t-\nu_t)\nu_t^{-1},\quad \nu_t:=\lambda Qt$. According to (11), (12), and (25), we find

$$\forall x,y\in\mathbb{H}\quad \|(\mathfrak{N}-\mathfrak{PM})(x,y)\|=\sum_{t\in\mathbb{T}}|\Gamma(t)x(t)y(t)|\nu(t). \tag{26}$$

1° Suppose that $\forall t\in\mathbb{T}\quad |\Gamma(t)|\ll\infty$ and $\Gamma(t)\approx 0$ quasieverywhere on $\mathbb{T}$, then the charge $\mathfrak{N}$ is *weakly nearstandard* and ${}^\circ\mathfrak{N}=\mathfrak{M}$.

◁ Let $\xi,\eta\in{}^{st}\mathbf{H}$, $x:=\Pi\xi$, $y:=\Pi\eta$. The function $t\mapsto\Gamma(t)x(t)\overline{y(t)}$ is ν-integrable, so that $\nu\mathbb{T}_0\approx 0$ and $\forall\, t\in\mathbb{T}\setminus\mathbb{T}_0\quad\Gamma(t)\approx 0$. According to (26), $\|(\mathfrak{N}-\mathfrak{PM})(x,y)\|=\sum\limits_{t\in\mathbb{T}_0}+\sum\limits_{t\in\mathbb{T}\setminus\mathbb{T}_0}|\Gamma(t)x(t)y(t)|\nu_t\approx 0$ because the second summand is not greater than
$\max\limits_{t\in\mathbb{T}\setminus\mathbb{T}_0}|\Gamma(t)|\cdot\|x\|\cdot\|y\|$ ▶

Taking into account (8), (9), and that $\|\delta_t\|=\nu_t^{-1/2}$, we find

$$\forall x\in\mathbb{H}\quad \|(\mathfrak{N}-\mathfrak{PM})(x)\|=\sum_{t\in\mathbb{T}}\nu_t^{-1/2}|\Gamma(t)x(t)|\nu_t. \tag{27}$$

2° Let $\sum_{t\in\mathbb{T}}|\Gamma(t)|^2\approx 0$, then $\mathfrak{N}$ is *strongly nearstandard* and ${}^\circ\mathfrak{N}=\mathfrak{M}$.

◁ It follows from (27) that

$$\|(\mathfrak{N}-\mathfrak{PM})(x)\|\le\Big(\sum_{t\in\mathbb{T}}|\Gamma(t)|^2\Big)\|x\|.$$

But for $\xi\in{}^{st}\mathbf{H}$ and $x=\Pi\xi$ we have $\|x\|\ll\infty$ ▶

Finally, since the operator $x\mapsto(x|\delta_t)\delta_t\nu_t$ is an orthoprojector $\mathbb{H}\to\mathbb{C}\delta_t$ with the unit norm, we obtain that $\forall t\in\mathbb{T}\quad\|(\mathfrak{N}-\mathfrak{PM})_t\|=|\Gamma(t)|$, hence $\|\mathfrak{N}-\mathfrak{PM}\|=\sum_{t\in\mathbb{T}}|\Gamma(t)|$.

This proves that

3° $\mathfrak{N}$ is *uniformly nearstandard* iff $\sum_{t\in\mathbb{T}}|\Gamma(t)|\approx 0$. Conversely, if $\sum_{t\in\mathbb{T}}|\Gamma(t)|\approx 0$, then ${}^\circ\mathfrak{N}=\mathfrak{M}$ and $\nu_t^{-1}\Gamma(t)\approx 0$ quasieverywhere on $\mathbb{T}$.

5. $\mathbf{H}_-$-nearstandard o.charges. A wider nearstandardness notion corresponds to a weaker operator norm. We use the equipment of $\mathbb{C}^\mathbb{T}$ by norms $\|\cdot\|_-$, $\|\cdot\|$, $\|\cdot\|_+$, induced by $\mathbf{H}$ and its equipment. Let $\mathbf{H}_-\supset\mathbf{H}\supset\mathbf{H}_+$ be a Hilbert equipment of the space $\mathbf{H}$ (see, for example, [8]). The embedding $Q:\mathbb{C}^\mathbb{T}\to\mathbf{H}$ enables us to carry the equipment of $\mathbf{H}$ over to $\mathbb{C}^\mathbb{T}$. Define

$$\forall\, x,\,y\,\in\,\mathbb{C}^\mathbb{T}\quad (x|y)_-:=(Qx|Qy)_-,\quad \|x\|_-=\|Qx\|_-.$$

The negative inner product may be represented in the following form:

$$(\forall\,\alpha,\,\beta\in\mathbf{H}_-)\quad(\alpha|\beta)_-=(\mathbf{I}\alpha|\mathbf{I}\beta)_0,$$
$$(\forall\,x,\,y\,\in\mathbb{C}^\mathbb{T})\quad(x|y)_-=(\mathbb{I}x|\mathbb{I}y)_0,$$

where $\mathbf{I}\in\mathbf{H}^{\mathbf{H}_-}$, $\mathbb{I}\in\mathbb{H}^\mathbb{H}$ are linear operators symmetric w.r.t. the inner products of $\mathbf{H}$ and $\mathbb{H}$ respectively. It is easy to prove that $\mathbb{I}^2=\Pi\mathbf{I}^2Q=\mathfrak{P}\mathbf{I}^2$ and $\mathbb{I}$ is a bijection.

The positive inner product is introduced by

$$\forall x,\, y \in \mathbb{C}^{\mathbb{T}} \quad (x|y)_+ = (\mathbb{I}^{-1}x|\mathbb{I}^{-1}y)_0, \quad \|x\|_+ := \|\mathbb{I}^{-1}x\|_0.$$

$\mathbb{C}^{\mathbb{T}}$ with the positive inner product by $\mathbb{H}_+$, and with the negative inner product is denoted by $\mathbb{H}_-$. One can find more details on the properties of $\mathbb{H}_+$ and $\mathbb{H}_-$ in [7]. The notions of $\mathbf{H}_-$-nearstandardness for functions $x \in \mathbb{C}^{\mathbb{T}}$ and operators $A \in \mathcal{B}(\mathbb{H})$ were considered in [8], too.

For $A \in \mathcal{B}(\mathbb{H})$ denote $\|A\|' = \sup\{\|Ax\|_- : x \in \mathbb{H}, \|x\|_+ = 1\}$. We have $\forall x \in \mathbb{H}\ \|x\|_- \le \|x\| \le \|x\|_+$, hence

$$\forall A \in \mathcal{B}(\mathbb{H}) \quad \|A\|' \le \|A\|. \tag{28}$$

Definition. For $\mathfrak{N} \in \tilde{\mathfrak{N}}$ put

$$\|n\|' = \sum_{t \in \mathbb{T}} \|\mathfrak{N}_t\|'. \tag{29}$$

$\mathfrak{N}$ is called $\mathbf{H}_-$-*nearstandard* if for some $\mathfrak{M} \in {}^{st}\tilde{\mathfrak{M}}$

$$\|\mathfrak{N} - \mathfrak{P}\mathfrak{M}\|' \approx 0. \tag{30}$$

Proposition. *If $\mathfrak{N}$ is uniformly neastandard, then it is $\mathbf{H}_-$-nearstandard, and its shadow $^\circ\mathfrak{N}$ coincides with $\mathfrak{M}$ which satisfies condition (30).*

◁ By (28), (29), and (6), $\forall \mathfrak{N} \in \tilde{\mathfrak{N}} \quad \|\mathfrak{N}\|' \le \|\mathfrak{N}\|$. So, (30) follows from (22). Uniqueness of $\mathfrak{M}$ follows from the following propositions ▶

Proposition. *If $\|\mathfrak{N}\|' \approx 0$, then $\mathfrak{N}$ is weakly nearstandard and $^\circ\mathfrak{N} = 0$.*

◁ Obviously, $\|\mathfrak{N}(x,y)\| = \sum_{t\in\mathbb{T}} |(\mathfrak{N}x|y)| \le \sum_{t\in\mathbb{T}} \|\mathfrak{N}_t\|'\|x\|_+\|y\|_+$, hence $\|\mathfrak{N}(x,y)\| \le \|\mathfrak{N}\|'\|x\|_+\|y\|_+$. Our statement follows from definition (20) because the chain $\mathbf{H}_- \supset \mathbf{H} \supset \mathbf{H}_+$ is standard, and $\forall \xi \in {}^{st}\mathbf{H} \ \ \|\Pi\xi\|_+ \ll \infty$ ▶

Corollary. *If $\mathfrak{N}$ is $\mathbf{H}_-$-nearstandard, then it is weakly nearstandard, and $\mathfrak{M} \in {}^{st}\tilde{\mathfrak{M}}$ satisfying (30) coincides with the shadow $^\circ\mathfrak{N}$.*

Example. Let the conditions of the previous example be satisfied. By (25), we conclude that $\|\mathfrak{N} - \mathfrak{P}\mathfrak{M}\|' \le \sum_{t\in\mathbb{T}} |\Gamma(t)|\|\delta_t\|_-^2\nu_t$. So if $\sum_{t\in\mathbb{T}} |\Gamma(t)|\|\delta_t\|_-^2\nu_t \approx 0$, then $\mathfrak{N}$ defined in (n) is $\mathbf{H}_-$-nearstandard and it has the shadow $^\circ\mathfrak{N} = \mathfrak{M}$ defined in (m). Note that $\|\delta_t\|_- \ll \infty$ since the chain $\mathbf{H}_- \supset \mathbf{H} \supset \mathbf{H}_+$ is standard. Therefore, the multiplicator $\|\delta_t\|_-^2$ is not essential.

6. Spectral decompositions of differential operators. Here we use the terminology and notation of [7], where it was proved that the discrete differentiation operator D is nearstandard and its shadow $\mathbf{D} = {}^\circ D$ is the "usual" differentiation operator.

O.charge $\mathfrak{N} = \mathfrak{N}_D$. Recall that $\operatorname{dom} D = \mathbb{C}^{\mathbb{T}} = \mathbb{H}$, where $\mathbb{T} = \overset{\circ\!-}{2\ell}$ is a discrete 2ℓ-periodic discrete axis, $\overset{\circ\!-}{2\ell} = \{-\ell, -\ell+h, \dots, \ell-h\}$, $\ell > 0$, $h > 0$, $h \approx 0$. D takes the form: $\forall x \in \mathbb{H}, \quad \forall t \in \mathbb{T} \quad Dx(t) = \frac{1}{ih}[x(t+h) - x(t)]$ in view of condition of periodicity $(\ell - h) + h = -\ell$. It has eigenfunctions $e_{\hat{t}}$, where $e_{\hat{t}}(t) = e^{i\hat{t}t}$, $t \in \mathbb{T}$,

$\hat{t} \in \overset{\circ\!-}{2\hat{\ell}}$, $\overset{\circ\!-}{2\hat{\ell}} = \{-\hat{\ell}, -\hat{\ell}+\hat{h}, \dots \hat{\ell}-\hat{h}\}$, $\hat{\ell} := \frac{\pi}{h}$, $\hat{h} := \frac{\pi}{\ell}$. As $\hat{\mathbb{T}}$ we take the spectrum of D:

$$\hat{\mathbb{T}} = \sigma(D) = \{z_{\hat{t}}\}_{\hat{t} \in \overset{\circ\!-}{2\hat{\ell}}}, \quad z_{\hat{t}} := \frac{e^{i\hat{t}}h - 1}{ih}. \tag{31}$$

We have $\forall \hat{t} \in \overset{\circ\!-}{2\hat{\ell}}\ De_{\hat{t}} = z_{\hat{t}} e_{\hat{t}}$. Since $\|e_{\hat{t}}\| = \sqrt{2\ell}$, the decomposition of the unity $\mathfrak{N} = \mathfrak{N}_D$ of D acts according to the formula

$$\forall \hat{E} \in 2^{\hat{\mathbb{T}}} \quad \forall x \in \mathbb{H} \quad \mathfrak{N}\hat{E}x = \frac{1}{2\ell} \sum_{z_{\hat{t}} \in \hat{E}} (x|e_{\hat{t}}) e_{\hat{t}}. \tag{32}$$

O.measure $\mathfrak{M} = \mathfrak{M}_{\mathbf{D}}$. At first we examine the case $\ell \ll \infty$. To avoid technical difficulties we assume that $\ell \in {}^{st}\mathbb{R}$. In this case $\mathbf{D} = {}^{\circ}D$ is an operator in $\mathbf{H} := L_2(\mathbf{T}, \lambda)$, where $\mathbf{T} = [-\ell, \ell[$, $\lambda = d\tau$ is the Lebesgue measure on $\mathbf{T}$. $\mathbf{D}$ is defined on functions $\xi \in \mathbf{H}$ such that $\frac{d\xi}{d\tau} \in \mathbf{H}$ and $\xi(-\ell) = \xi(\ell)$; $\mathbf{D}\xi = \frac{1}{i}\frac{d\xi}{d\tau}$. According to our plan we take the spectrum of $\mathbf{D}$ as $\hat{\mathbf{T}}$:

$$\hat{\mathbf{T}} := \sigma(\mathbf{D}) = \hat{h}\mathbb{Z} := \{\dots, -2\hat{h}, -\hat{h}, 0, \hat{h}, 2\hat{h}, \dots\}.$$

The eigenfunction $\epsilon_{\hat{\tau}}$ such that $\forall \tau \in \mathbf{T}\ \epsilon_{\hat{\tau}}(\tau) = e^{i\hat{\tau}\tau}$ corresponds to the eigenvalue $\hat{\tau} \in \hat{\mathbf{T}}$. Since $\|\epsilon_{\hat{\tau}}\|^2 = 2\ell$, we get that the decomposition of the unity $\mathfrak{M} = \mathfrak{M}_{\mathbf{D}}$ of $\mathbf{D}$ operates as follows

$$\forall \hat{\mathcal{E}} \in 2^{\hat{\mathbf{T}}} \quad \forall \xi \in \mathbf{H} \quad \mathfrak{M}\hat{\mathcal{E}}\xi = \frac{1}{2\ell} \sum_{\hat{\tau} \in \hat{\mathcal{E}}} (\xi|\epsilon_{\hat{\tau}}) \epsilon_{\hat{\tau}};$$

in particular,

$$\forall \hat{\tau} \in \hat{\mathbf{T}} \quad \forall \xi \in \mathbf{H} \quad \mathfrak{M}\{\hat{\tau}\}\xi = \frac{1}{2\ell} (\xi|\epsilon_{\hat{\tau}}) \epsilon_{\hat{\tau}}. \tag{33}$$

Exactness of $\hat{\Pi}$. Recall that the embedding $Q : 2^{\mathbb{T}} \to \Lambda \subset 2^{\mathbf{T}}$ is defined by $\forall t \in \mathbb{T} \quad Qt = [t, t+h[$. The embedding $\hat{Q} : 2^{\hat{\mathbb{T}}} \to 2^{\hat{\mathbf{T}}}$ will be defined by

$$\forall \hat{\tau} \in \overset{\circ\!-}{2\hat{\ell}} \quad \hat{Q} z_{\hat{\tau}} = \{\hat{\tau}\}, \quad z_{\hat{\tau}} = \frac{e^{i\hat{\tau}h} - 1}{ih}, \tag{34}$$

what is natural for reason of $z_{\hat{\tau}} \approx \hat{\tau}$ whenever $|\hat{\tau}| \ll \infty$. The measure $\hat{\lambda}$ is defined on $\hat{\Lambda} = 2^{\hat{\mathbf{T}}}$ by $\forall \hat{\tau} \in \hat{\mathbf{T}} \quad \hat{\lambda}_{\hat{\tau}} := \hat{\lambda}\{\hat{\tau}\} = \hat{h}$. Note that $\hat{\mathbf{T}}$, $\hat{\Lambda}$, $\hat{\lambda}$ are standard because $\hat{h} = \frac{\pi}{\ell} \in {}^{st}\mathbb{R}$. The inductor $\hat{\Pi}$ corresponding to the standard filling $(\hat{\mathbf{T}}, \hat{Q}, \hat{\lambda})$ of $\hat{\mathbb{T}}$ is *exact*. Really, let $\mu \in {}^{st}\operatorname{qker}\hat{\Pi}$ (that is, μ is a standard complex-valued charge defined on $\hat{\Lambda}$) be such that $\forall \hat{E} \in 2^{\hat{\mathbb{T}}}\ \hat{\Pi}\mu\hat{E} \approx 0$, then $\forall \hat{\tau} \in \hat{\mathbf{T}} = \hat{h}\mathbb{Z}\ \mu\hat{Q}\{z_{\hat{\tau}}\} = \hat{\Pi}\mu\{z_{\hat{\tau}}\} \approx 0$, hence by (34), $\mu\{\hat{\tau}\} \approx 0$. But for $\hat{\tau} \in {}^{st}\hat{h}\mathbb{Z}$ the number $\mu\{\hat{\tau}\}$ is standard, therefore $\mu = 0$.

Proposition. *The decomposition of the unity $\mathfrak{N} = \mathfrak{N}_D$ is weakly nearstandard and its shadow is $\mathfrak{M} = \mathfrak{M}_{\mathbf{D}}$.*

◁ Recall that $e_{\hat{t}}$ are related to $\epsilon_{\hat{\tau}}$ through $\Pi : \mathbf{H} \to \mathbb{H}$ by

$$\forall \hat{t} \in \hat{\mathbb{T}} \quad e_{\hat{t}} = \gamma_{\hat{t}} \Pi \epsilon_{\hat{t}}, \quad \text{where } \gamma_{\hat{t}} := \frac{i\hat{t}h}{e^{i\hat{t}h} - 1}.$$

By (34), $(\mathfrak{P}\mathfrak{M})_{z_{\hat\tau}} := (\mathfrak{P}\mathfrak{M})\{z_{\hat\tau}\} = \Pi(\mathfrak{M}\hat{Q}\{z_{\hat\tau}\})Q = \Pi\mathfrak{M}\{\hat\tau\}Q$. Consequently, by (33), $\forall x \in \mathbb{H}$ $(\mathfrak{P}\mathfrak{M})_{z_{\hat\tau}}x = \Pi\mathfrak{M}\{\hat\tau\}Qx = \frac{1}{2\ell}(Qx|\epsilon_{\hat\tau})\Pi\epsilon_{\hat\tau}$. Taking into account $(Qx|\epsilon_{\hat\tau}) = (x|\Pi\epsilon_{\hat\tau})$, we find

$$\forall x \in \mathbb{H} \quad \forall \hat t \in \overset{\circ\!-}{2\hat\ell} \quad (\mathfrak{P}\mathfrak{M})_{z_{\hat t}}x = \frac{1}{2\ell}\frac{1}{|\gamma_{\hat t}|^2}(x|e_{\hat t})e_{\hat t}. \tag{35}$$

Define the complex-valued charge $(\mathfrak{N}-\mathfrak{P}\mathfrak{M})(x,y)$ on $2^{\hat{\mathbb{T}}}$ by $(\mathfrak{N}-\mathfrak{P}\mathfrak{M})(x,y)\{z_{\hat\tau}\} = ((\mathfrak{N}-\mathfrak{P}\mathfrak{M})_{z_{\hat t}}x|y)_{\mathbb{H}}$. According to (32) and (35), we have $(\mathfrak{N}-\mathfrak{P}\mathfrak{M})(x,y)\{z_{\hat t}\} = \frac{1}{2\ell}\left(1-\frac{1}{|\gamma_{\hat t}|^2}\right)(x|e_{\hat t})(e_{\hat t}|y)$. So, (see (12))

$$\|(\mathfrak{N}-\mathfrak{P}\mathfrak{M})(x,y)\| = \frac{1}{2\ell}\sum_{z_{\hat t}\in\hat{\mathbb{T}}}\left(1-\frac{1}{|\gamma_{\hat t}|^2}\right)|(x|e_{\hat t})(e_{\hat t}|x)|.$$

Note that $\forall \hat t \in {}^{st}(\hat h\mathbb{Z})$ $\hat t h \approx 0$. Thus $\forall \hat t \in {}^{st}(\hat h\mathbb{Z})$ $\gamma_{\hat t} \approx 1$. By the Robinson lemma, $\gamma_{\hat t} \approx 1$ up to some $\hat t_0 \in \hat h\mathbb{Z}_+ \setminus {}^{st}(\hat h\mathbb{Z})$, i.e. $\gamma_{\hat t} \approx 1$ whenever $|\hat t| \le t_0$. We conclude that

$$\|(\mathfrak{N}-\mathfrak{P}\mathfrak{M})(x,y)\| \le \frac{1}{2\ell}\sum_{|\hat t|\le\hat t_0}\left(1-\frac{1}{|\gamma_{\hat t}|^2}\right)|(x|e_{\hat t})(e_{\hat t}|y)| + \frac{1}{\ell}\Big(\sum_{|\hat t|>\hat t_0}|(x|e_{\hat t})|^2\Big)^{1/2}\Big(\sum_{|\hat t|>\hat t_0}|(y|e_{\hat t})|^2\Big)^{1/2},$$

because $|\gamma_{\hat t}| \ge 1$ and $|1-\frac{1}{|\gamma_{\hat t}|^2}| \le 2$. Let $x = \Pi\xi$, $y = \Pi\eta$ with $\xi,\eta \in {}^{st}\mathbf{H}$. Then $x, y \in {}^{nst}\mathbb{H}$ and, by nearstandardness criterion 6.6.1 [7], the addend in the last sum is infinitesimal. The augend is also infinitesimal, because it is not greater than $\|x\|\cdot\|y\|\max_{|\hat t|<\hat t_0}|1-\frac{1}{|\gamma_{\hat t}|^2}|$, a $\|x\|, \|y\| \ll \infty$ ▶

Case $\ell \approx +\infty$. In this case the description of spectrum $\sigma(D) =: \hat{\mathbb{T}}$ and decomposition of the unity $\mathfrak{N} = \mathfrak{N}_D$ of D remains the same as for $\ell \in {}^{st}\mathbb{R}$. Note that $\hat h = \frac{\pi}{\ell} \approx 0$, so $\forall \hat h \in \overset{\circ\!-}{2\hat\ell}$ the eigenvalues $z_{\hat t}$ and $z_{\hat t+\hat h}$ are infinitely close: $z_{\hat t} \approx z_{\hat t+\hat h}$. Since now ${}^{\circ}D = \mathbf{D}$ is the differentiation on whole axis, it is natural to take $\mathbf{T} = \mathbb{R}$, λ be the Lebesgue measure on $\mathbb{R}$, and $\forall\ t \in \mathbb{T}$ $Qt = [t, t+h[$ as the standard filling $(\mathbf{T}, Q, \lambda)$ of $\mathbb{T}$. Besides, the standard filling $(\hat{\mathbf{T}}, \hat Q, \hat\lambda)$ of $\hat{\mathbb{T}} = \sigma(D)$ is defined as follows: $\hat{\mathbf{T}} = \sigma(\mathbf{D}) = \mathbb{R}$, $\hat\lambda$ is the Lebesgue measure on $\mathbb{R}$, and

$$\forall z_{\hat t} \in \hat{\mathbb{T}} \quad \hat Q\{z_{\hat t}\} = [\hat t, \hat t + \hat h[. \tag{36}$$

As it was noted, the condition $\hat h \approx 0$ implies the exactness of $\hat\Pi$.

It is a well-known fact that decomposition of the unity $\mathfrak{M} = \mathfrak{M}_{\mathbf{D}}$ of $\mathbf{D}$ takes the form

$$\forall \xi \in \mathbf{H} \quad \forall \hat{\mathcal{E}} \in \hat\Lambda \quad \mathfrak{M}\hat{\mathcal{E}}\xi(\tau) = \frac{1}{\sqrt{2\pi}}\int_{\hat{\mathcal{E}}}\hat\xi(\hat\tau)e^{i\hat\tau\tau}d\tau,$$

where $\hat\Lambda$ is the algebra of Lebesgue measurable sets $\hat{\mathcal{E}} \subset \mathbb{R}$, $\hat\xi = \mathfrak{F}\xi$ is the Fourier transform of $\xi \in \mathbf{H}$:

$$\hat\xi(\hat\tau) = \mathfrak{F}\xi(\hat\tau) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\xi(\tau)e^{-i\hat\tau\tau}d\tau.$$

In particular, according to (36) we have

$$\forall\, z_{\hat t} \in \hat{\mathbb{T}} \quad \mathfrak{M}\hat{Q}\{z_{\hat t}\}\xi(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\hat t}^{\hat t+\hat h} \hat\xi(\hat\tau) e^{i\hat\tau\tau} d\tau.$$

It is easily shown that, for $x \in \mathbb{H}$, the "usual" Fourier transform $\hat{Q}x := \mathfrak{F}Qx$ is related to the discrete Fourier transform $\hat{x} := \mathfrak{C}x$ by

$$\forall\, \hat\tau \in \mathbb{R} \quad \hat{Q}x(\hat\tau) = \frac{1}{\gamma_{\hat t}} \hat{x}(\hat\tau), \quad \gamma_{\hat\tau} := \frac{i\hat\tau h}{e^{i\hat\tau h} - 1}. \tag{38}$$

For any $x, y \in \mathbb{H}$ form the complex-valued charge $\mathfrak{P}\mathfrak{M}(x,y)$ as follows:

$$\forall\, \hat{\mathcal{E}} \in \hat\Lambda \quad \mathfrak{P}\mathfrak{M}(x,y)\hat{\mathcal{E}} = \sum_{z_{\hat t} \in \hat{\mathcal{E}}} \Big(\Pi \mathfrak{M}\hat{Q}\{z_{\hat t}\} Qx | y \Big)_{\mathbb{H}}.$$

In particular, taking into account that $\Pi^* = Q$ and by (37), we get

$$\forall\, z_{\hat t} \in \hat{\mathbb{T}} \quad \mathfrak{P}\mathfrak{M}(x,y)\{z_{\hat t}\} = \Big(\mathfrak{M}\hat{Q}\{z_{\hat t}\} Qx | Qy \Big)_{\mathbf{H}} =$$

$$\int_{\mathbb{R}} d\tau \, \frac{1}{\sqrt{2\pi}} \int_{\hat t}^{\hat t+\hat h} \hat{Q}x(\hat\tau) e^{i\hat\tau\tau} \, d\hat\tau \, \overline{Qy(\tau)} = \int_{\hat t}^{\hat t+\hat h} \hat{Q}x\,(\hat\tau)\, \overline{\hat{Q}y(\hat\tau)}\, d\hat\tau.$$

On the other hand, by (32),

$$\mathfrak{N}(x,y)\{z_{\hat t}\} = \frac{\pi}{\ell} \hat{x}(\hat t)\overline{\hat{y}(\hat t)} = \int_{\hat t}^{\hat t+\hat h} d\hat\tau \, \hat{x}(\hat t)\overline{\hat{y}(\hat t)}.$$

Thus according to (12) and (38),

$$\|(\mathfrak{N} - \mathfrak{P}\mathfrak{M}(x,y)\| = \sum_{z_{\hat t} \in \hat{\mathbb{T}}} \Big| (\mathfrak{N} - \mathfrak{P}\mathfrak{M}(x,y)\{z_{\hat t}\} \Big| = \sum_{\hat t \in \hat{\mathbb{T}}} \Big| \int_{\hat t}^{\hat t+\hat h} (|\gamma_{\hat\tau}|^2 - 1) \hat{Q}x(\hat\tau)\overline{\hat{Q}y(\hat\tau)}\, d\tau \Big|.$$

If $|\hat t| \ll \infty$, then $\gamma_{\hat t} \approx 1$, hence there exists a $\hat{t}_0 \approx +\infty$ such that $\gamma := \max\limits_{|\hat t| < \hat t_0} \big| |\gamma_{\hat t}|^2 - 1 \big| \approx 0$. Taking the condition $|\gamma_{\hat\tau}| < \frac{\pi}{2}$ into account, we find that

$$\|(\mathfrak{N} - \mathfrak{P}\mathfrak{M}(x,y)\| \le \gamma \|\hat{Q}x\} \cdot \|\hat{Q}y\| + C \Big| \int_{|\hat\tau| > t_0} \hat{Q}x(\hat\tau)\overline{\hat{Q}y(\hat\tau)}\, d\hat\tau \Big|,$$

where C is independent of x and y, $0 < C \ll \infty$. If $x, y \in {}^{nst}\mathbb{H}$, then the augend at the right is infinitesimal, because $\|\hat{Q}x\| = \|Qx\| = \|x\| \ll \infty$ and $\|\hat{Q}y\| = \|Qy\| = \|y\| \ll \infty$. The addend is also infinitesimal, because $Qx, Qy \in {}^{nst}\mathbf{H}$ ▶

Remark. Formula (38) may be interpreted in the following way: $\mathfrak{F}\mathbf{D}'Q = \mathfrak{C}D^*$, where $\mathbf{D}' = \frac{1}{i}\frac{d}{d\tau}$ is the differentiation operator on $\mathbb{R}$ in the sense of distributions, D^* is adjoint to D: $\forall\, x \in \mathbb{C}^{\mathbb{T}} \quad D^*x(t) = (Dx)(t-h) = \frac{1}{ih}[x(t) - x(t-h)]$.

◁ Since $\gamma_{\hat\tau} = \frac{\hat\tau}{z_{\hat\tau}}$, relationship (38) may be represented in the form $\hat\tau\hat{Q}x(\hat\tau) = \overline{z_{\hat\tau}}\hat{x}(\hat\tau)$. It remains to take into account that differentiation converts to multiplication by independent variable under the Fourier transrorm ▶

Appendix. Nearstandardness criterion for a scalar measure.

1. Let X be a standard set, $\mathfrak{X}$ a standard σ-algebra of sets $A \subset X$. *A measure* (on X) is a σ-additive function $\mu \in \mathbb{R}_+^{\mathfrak{X}}$. By definition, it is *nearstandard* if $\mathrm{var}(\mu - \nu) \approx 0$ for some standard measure ν on X. Then ν is denoted by ${}^\circ\mu$ and called *the shadow* of μ.

Definition. A measure μ is said to be *S-continuous* iff $\mu\mathfrak{X} \ll \infty$ and for any standard decreasing sequence $A_n \in \mathfrak{X}$ such that $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$ the following condition holds:
$n \approx \infty \quad \Longrightarrow \quad \mu(A_n) \approx 0.$

Remark. Each nearstandard measure is S-continuous.

Proof. First suppose that μ is standard. Let $(A_n)_{n\in\mathbb{N}}$ be a standard decreasing sequence in $\mathfrak{X}$ with $\cap_{n\in\mathbb{N}} = \varnothing$. The σ-additivity of μ implies $\mu(A_1) = \sum_{n\in\mathbb{N}} \mu(A_n \setminus A_{n+1})$. The series above is standard and convergent, hence $\mu(A_n) = \sum_{k\geq n} \mu(A_k \setminus A_{k+1}) \approx 0$ for any $n \approx \infty$. Now consider a nearstandard measure μ on X. By definition, there exists (a unique) standard measure ${}^\circ\mu$ on X such that $\mathrm{var}(\mu - {}^\circ\mu) \approx 0$. Since $\forall\, A \in \mathfrak{X}\;\; (\mu)(A) \leq (\mathrm{var}(\mu - {}^\circ\mu))(X) + ({}^\circ\mu)(A)$, from $({}^\circ\mu)(A) \approx 0$ it follows that $(\mu)(A) \approx 0$. Hence μ is S-continuous, too.

Definition. A measure μ_0 on X is said to be the *shadow* of μ in *the weak sense* iff
(i) μ_0 is standard,
(ii) for any *standard* $A \in \mathfrak{X}\;\; \mu_0(A) \approx \mu(A)$. Obviously, if μ has the shadow in the weak sense, then it is unique. As above it will be denoted by ${}^\circ\mu$ (without ambiguity).

Proposition. *A S-continuous measure μ has the shadow in the weak sense.*

Proof. For $A \in {}^{st}\mathfrak{X}$ put $\mu_1(A) = {}^\circ[\mu(A)]$ and let μ_0 be the *standard extension* of μ_1 from ${}^{st}\mathfrak{X}$ to the whole $\mathfrak{X}$. Consider an arbitrary standard partition $A = \sqcup_{n\in\mathbb{N}} A_n$ of an arbitrary standard $A \in \mathfrak{X}$. For $n \ll \infty\;\; A_n$ is standard. Hence $\forall\, n \ll \infty$ $\mu_0(A) = \sum_{k\leq n} \mu_0(A_k) + {}^\circ[\mu(B_n)]$, where $B_n := \sqcup_{k>n} A_k$. Therefore, for $n \ll \infty$ $\mu_0(A) - \sum_{k\leq n} \mu_0(A_k) - \mu(B_n) \approx 0$. By the Robinson lemma, this holds up to some $n = n_0 \approx \infty$. But the B_n's form a standard decreasing sequence with empty intersection. Therefore, $\forall n \approx \infty\;\; \mu(B_n) \approx 0$. Hence $\mu_0(A) \approx \sum_{k\leq n_0} \mu_0(A_k)$. The number $\mu_0(A)$ and the series $\sum_{n\in\mathbb{N}} \mu_0(A_n)$ are standard. Besides $\forall n\;\; \mu_0(A_n) \geq 0$. Therefore $\mu_0(A) = \sum_{n\in\mathbb{N}} \mu_0(A_n)$. By transfer, μ_0 is σ-additive, hence $\forall\, A \in {}^{st}\mathfrak{X}$ $\mu(A) \approx \mu_0(A)$, $\mu_0 = {}^\circ\mu$ ▶

2. Now we consider another approach. Let X be a standard set, $\mathfrak{X}$ a standard σ-algebra of subsets $A \subset X$, m a σ-additive measure on X. We interpret $\mathfrak{X}$ as a *metric space* with *equality* and *distance* defined as follows: $\forall\; A, B \in \mathfrak{X}$

$$\begin{aligned} (A = B \;&\equiv\; m(A \bigtriangleup B) = 0) \\ d(A,B) \;&:=\; m(A,B), \end{aligned}$$

where $A \bigtriangleup B := (A \setminus B) \cup (B \setminus A)$.

Remark. Let $f \in Y^X$ (here Y is an internal set), then f is a function on $(\mathfrak{X}, d)$ iff

$$m(A \bigtriangleup B) = 0 \quad \Longrightarrow \quad f(A) = f(B). \tag{1}$$

But for an *additive* function $\mu \in \mathcal{C}^{\mathfrak{X}}$ this is simpler:

$$m(A) = 0 \quad \Longrightarrow \quad \mu(A) = 0. \tag{2}$$

◁ Let (1) hold. Put $B = \varnothing$. The additivity of μ implies $\mu(B) = 0$ and we see that (2) is satisfied. The additivity of μ implies $|\mu(A) - \mu(B)| \leq \mu(A \bigtriangleup B)$.

Suppose that (1) holds for $f = \mu$ and $m(A \triangle B) = 0$. Then $\mu(A \triangle B) = 0$, hence $\mu(A) = \mu(B)$ ▶

Now suppose that a measure space $(X, \mathfrak{X}, m)$ is standard. Arguing as above, we find that for an additive function $\mu \in \mathcal{C}^{\mathfrak{X}}$ the <nst>-condition of graph-nearstandardness is

$$m(A) \approx 0 \quad \Longrightarrow \quad \mu(A) \approx 0. \tag{3}$$

We emphasize that (2) is exactly the condition of *absolute continuity* of μ with respect to m.

Definition. A measure μ on X is said to be *absolutely continuous* with respect to m iff $\forall A \in \mathfrak{X} \;\; \mu(A) \ll \infty$ and μ satisfies (2) and (3).

Let μ be an absolutely S-continuous (with respect to m) measure. Since $\forall A \in \mathfrak{X}$ $\mu(A) \ll \infty$, ${}^\circ\mu$ considered as the shadow of a graph-nearstandard function is defined everywhere: $dom({}^\circ\mu) = \mathfrak{X}$. Besides $\forall A \in {}^{st}\mathfrak{X} \;\; ({}^\circ\mu)(A) = {}^\circ[\mu(A)]$. We see that ${}^\circ\mu$ is the standard extension of ${}^{st}\mathfrak{X} \ni A \mapsto {}^\circ[\mu(A)]$ to the whole $\mathfrak{X}$. Note that ${}^\circ\mu$ is *S-continuous*. Indeed, let $(A_n)_{n \in \mathbb{N}}$ be a standard decreasing sequence in $\mathfrak{X}$ such that $\cap_{n \in \mathbb{N}} A_n = \varnothing$. Then $\forall n \approx \infty \;\; m(A_n) \approx 0$ (since m is standard). Therefore, by (3), $\forall n \approx \infty \;\; \mu(A_n) \approx 0$. Hence μ has the shadow in the weak sense. Thus the following statement holds.

Proposition. *The shadow of a graph-nearstandard measure coincides with the one in the weak sense. In particular, it is σ-additive.*

Many other approaches to the problems which are discussed in this paper are possible. For instance, see [1, Ch.5].

REFERENCES

1. S. Albeverio, J. Fenstad, T. Hoegh-Krohn, T. Lindström, Nonstandard methods in stochastic analysis and mathematical physics. — New York: Academic Press, 1986.
2. N.J. Cutland, *Nonstandard measure theory and its applications* // Bull. Amer. Math. Soc. 1983, V.15, №6. P.529–589.
3. T.S. Kudryk, V.E. Lyantse, G.I. Chuiko, *Nearstandardness on finite set* // Математичні студії, **2**, 1993, С.25–34.
4. T.S. Kudryk, V.E. Lyantse, G.I. Chuiko, *Nearstandard operators* // Математичні студії, **3**, 1994, С.29–40.
5. Ю.М. Березанский, Г.Ф. Ус, З.Г. Шефтель, Функциональный анализ. — Киев.: Вища школа. 1990.
6. Т.С. Кудрик, В.Э. Лянце, Г.И. Чуйко. *Нестандартные методы в теории операторов. 4,5. Околостандартные операторы* / Львов. гос.ун-т. 1992. 19с. Деп. в ГНТБ Украины 28.06.93г., №1248-Ук93.
7. Т.С.Кудрик, В.Э.Лянце,Г.И.Чуйко. *Нестандартные методы в теории операторов. 6. Дискретное преобразование Фурье* / Львов. гос.ун-т. 1992. 15с. Деп. в ГНТБ Украины 28.10.93г., №2138–Ук93.
8. Т.С.Кудрик, В.Э.Лянце,Г.И.Чуйко. *Нестандартные методы в теории операторов. 7. Околостандартность и оснащение* / Львов. гос.ун-т. 1994. 11с. Деп. в ГНТБ Украины 25.08.94г., №1788-Ук94.

Ukraine, 290602, Lviv, 1, Universytetska str,

Lviv State University, Department of Mathematics

Received 27.12.1995