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ULTRAFILTERS ON ABELIAN GROUPS CLOSE TO BEING RAMSEY ULTRAFILTERS

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We introduce a class of ultrafilters on abelian groups that contains Ramsey ultrafilters and strongly summable ultrafilters on Boolean groups. We prove that existence of idempotent among these ultrafilters does not depend on the axioms ZFC.

Let G be an infinite abelian group and A a subset of G. Denote by $PS(A)$ the set $\{a+b : a, b \in A, a \neq b\}$ of pairwise sums of distinict elements of A. A free ultrafilter ξ on G is called a *PS-ultrafilter* (resp. *PS-idempotent*) if every 2-coloring of G gives us a set $A \in \xi$ such that $PS(A)$ (resp. $A \cup PS(A)$) is monochromatic.

Let X be an infinite set and $[X]_2$ the set of all two-element subsets of X. A free ultrafilter ξ on X is called a *Ramsey ultrafilter* if every 2-coloring of $[X]_2$ gives us a set $A \in \xi$ such that $[A]_2$ is monochromatic.

Fix any 2-coloring $\chi: G \to \{0,1\}$ and define 2-coloring $\chi^*: [G]_2 \to \{0,1\}$ by the rule $\chi^*\{a,b\} = a+b$. Given a Ramsey ultrafilter ξ on G, take $A \in \xi$ such that $[A]_2$ is χ^*-monochromatic so $PS(A)$ is χ-monochromatic. It means that every Ramsey ultrafilter on G is a PS-ultrafilter.

In this paper we investigate the relationship between PS-ultrafilters and PS-idempotents from one side and Ramsey ultrafilters and strongly summable ultrafilters from the other.

The main results concern PS-idempotents. By Theorem 1 they can live only on countable Boolean group (a Boolean group is a group of period 2). By Theorem 6 there are models of ZFC without PS-idempotents. In Theorem 4, assuming the Martin axiom, we construct a PS-ultrafilter which is neither a Ramsey ultrafilter nor a PS-idempotent. At the end of the paper we fix some gaps in our knowledge about PS-ultrafilters by raising open questions.

The paper contains also some other results not tightly connected with PS-ultrafilters. Among them: Theorem 2 stating that the known Hindman Theorem has no uncoutable counterpart, and Theorem 3 stating that strongly summable ultrafilters can live only on countable groups.

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In what follows G is an infinite abelian group. For every ultrafilter ξ on G, the family $\{PS(A) : A \in \xi\}$ forms a base of some filter on G. We denote this filter by $PS(\xi)$. The following two lemmas are paraphrases of the definitions of PS-ultrafilter and PS-idempotent.

Lemma 1. *A free ultrafilter ξ on G is a PS-ultrafilter if and only if $PS(\xi)$ is an ultrafilter.*

Lemma 2. *A free ultrafilter ξ on G is a PS-idempotent if and only if $PS(\xi) = \xi$.*

We call a subset $A \subset G$ to be a 2-independent subset if $\{a,b\} = \{c,d\}$ whenever $\{a,b\}$, $\{c,d\} \in [A]_2$ and $a+b = c+d$. For example, the subset $\{a_n \in \mathbb{Z} : a_n > 0, a_{n+1} > 2a_n\}$ is a 2-independent subset of $\mathbb{Z}$.

Lemma 3. *Let ξ be a PS-ultrafilter on G. If there exists a 2-independent subset $A \in \xi$ then ξ is a Ramsey ultrafilter.*

Proof. Fix any map $\chi\colon [G]_2 \to \{0,1\}$ and define a new map $\chi'\colon G \to \{0,1\}$. If $g \in G$ and there exists $\{a,b,\} \in [A]_2$ with $g = a+b$, we put $\chi'(g) = \chi\{a+b\}$. Otherwise, we put $\chi'(g) = 0$. Since A is a 2-independent subset, this definition is correct. Take a subset $B \in \xi$, $B \subset A$ such that $PS(B)$ is χ'-monochromatic, so ξ is a Ramsey ultrafilter.

In what follows we use some well-known notions concerning the Stone-Čech compactification βG of a discrete group G. We take βG to be the set of all ultrafilters on G. Given $A \subset G$, put $\bar{A} = \{\xi \in \beta G : A \in \xi\}$. The family $\{\bar{A} : A \subset G\}$ forms an open base of topology on βG. We identify the group G with the subset of all principal ultrafilters on G. The subset $\beta G \setminus G$ of free ultrafilters is denoted by G^*.

The operation $+$ on G extends naturally to an operation on βG which will be denoted by the same symbol. We define the sum $\xi + \eta$ of ultrafilters ξ, η by the determination of all subsets $A \subset G$ which are elements of ultrafilter $\xi + \eta$,

$$A \in \xi + \eta \Leftrightarrow \{g \in G : A - g \in \eta\} \in \xi.$$

The operation $+$ on βG is associative and G^* is a closed subsemigroup of the semigroup βG. Pick $X \in \xi$ and, for each $x \in X$, take some $Y_x \in \eta$. Then the subset $A = \bigcup\{x + Y_x : x \in X\}$ is an element of the ultrafilter $\xi + \eta$. The family of such subsets forms a base of the ultrafilter $\xi + \eta$. The map $\rho_\xi(x)\colon \beta G \to \beta G$ defined by $\rho_\xi(x) = x + \xi$ is continuous for each ultrafilter $\xi \in \beta G$. Thus βG is a compact right topological semigroup.

Lemma 4. *Let ξ be a PS-ultrafilter on G. Then $PS(\xi) = \xi + \xi$.*

Proof. By the definition of operator $+$ on βG, we have $PS(\xi) \subset \xi + \xi$. Since ξ is a PS-ultrafilter, $PS(\xi) = \xi + \xi$.

This lemma explains a term "PS-idempotent". A PS-ultrafilter ξ is a PS-idempotent if and only if ξ is an idempotent of the semigroup βG.

A free ultrafiter ξ on G is called a Schur ultrafilter if, for every $A \in \xi$ there exists $\{a,b\} \in [A]_2$ with $a + b \in A$. By [3], the set of Shur ultrafilters on G is a closed subsemigroup of G^* which contains all idempotents of semigroup G^*. For every $\xi \in G^*$, the ultrafilter $(-\xi) + \xi$, $-\xi = \{-X : X \in \xi\}$ is a Schur ultrafilter.

Lemma 5. *Let ξ be a PS-ultrafilter on G. If ξ is a Schur ultrafilter then ξ is a PS-idempotent.*

Proof. Take any $A, B \in \xi$. Since $PS(A) \cap B \neq \varnothing$, $\xi = PS(\xi)$. By Lemma 2, ξ is a PS-idempotent.

Given any subset $A \subset G$, we write $2A = \{2g : g \in A\}$. For an ultrafilter $\xi \in \beta G$, we denote by 2ξ an ultrafilter with base $\{2A : A \in \xi\}$. The reader should be cautioned that $2\xi \neq \xi + \xi$ for each free ultrafilter on every group G (see [5]).

Lemma 6. *Let ξ be a PS-ultrafilter on G. Suppose $PS(\xi)$ is a PS-ultrafilter. Then $\xi + \xi + \xi + \xi = \xi + \xi + 2\xi$.*

Proof. Take any subsets $X \in \xi + \xi + \xi + \xi$, $Y \in \xi + \xi + 2\xi$. It suffices to show that $X \cap Y \neq \varnothing$.

Since $PS(\xi) = \xi + \xi$ and $PS(\xi + \xi) = \xi + \xi + \xi + \xi$, there exists $A \in \xi$ such that $PS\big(PS(A)\big) \subset X$. Take a subset $B \in \xi + \xi$ and $C_x \in \xi$, $x \in B$ with $x + 2C_x \subset Y$ for each $x \in B$. Pick $a, b \in A$, $a \neq b$ with $a + b \in B$. Fix any element $c \in A \cap C_{a+b}$ with $c \neq a$, $c \neq b$. Then $a + b + 2c \in Y$. Since $a + b + 2c = (a + c) + (b + c)$ and $(a + c) + (b + c) \in PS\big(PS(A)\big)$, we have $a + b + 2c \in X$, so $X \cap Y \neq \varnothing$.

Theorem 1. *Let ξ be a PS-idempotent on a group G. Then there exists a countable Boolean subgroup $H \subset G$ such that $H \in \xi$.*

Proof. Put $B = \{g \in G : 2g = 0\}$. Let $f: G \to G/B$ be the quotient homomorphism. Suppose $B \notin \xi$. Since ξ is idempotent, $\bar{f}(\xi)$ is a free ultrafilter on G/B, where $\bar{f}$ is an extension of the map f onto βG. By Lemma 6, the equation $x + \bar{f}(\xi) = y + 2\bar{f}(\xi)$ has the solution $x = \bar{f}(\xi) + \bar{f}(\xi) + \bar{f}(\xi)$, $y = \bar{f}(\xi) + \bar{f}(\xi)$. By [5], this equation is nonsolvable in $\beta(G/B)$, a contradiction. Thus $B \in \xi$.

To complete the proof we use arguments due to Malykhin [2]. Since $B \in \xi$, we assume $G = B$. Suppose that every subset $X \in \xi$ is noncountable. Take the group G to be the set of finite subsets of some noncountable set X with symmetrical subtraction as a group operation. Given $x \in G$ we denote by $|x|$ the cardinality of the subset x. Let $s(x)$ be the number of multipliers 2 in the decomposition of $|x|$, $x \neq 0$ in a product of prime numbers. Put $X_1 = \{x \in G : s(x) \text{ is odd}\}$, $X_0 = G \backslash X_1$. Since ξ is an ultrafilter, $X_i \in \xi$ for some $i \in \{0, 1\}$. Since ξ is a PS-idempotent, there exists $A \in \xi$, $A \subset X_i$ with $PS\big(PS(A)\big) \subset X_i$. By Sunflower's Lemma there exists a uncountable subset $B \subset A$ disjunctive modulo intersections. This means that $x \cap y = \bigcap\{z : z \in B\}$, $|x| = |y|$ for every $\{x, y\} \in [B]_2$. Take four elements $a, b, c, d \in B$. By the choice of subset A, $a + b + c + d \in X_i$. Since $a \cap b = c \cap d$, $(a \cup b) \cap (c \cup d) = a \cap b$, we have $|a + b + c + d| = 2|a + b|$. Since $a + b \in X_i$, we have $a + b + c + d \in G \setminus X_i$, a contradiction.

Given any subset $A \subset G$, we write $FS(A) = \Sigma\{F : F \text{ is a nonempty finite subset of } A\}$. A free ultrafilter ξ on G is called a strongly summable ultrafilter if, for every $X \in \xi$, there exists $A \subset G$ with $FS(A) \subset X$, $FS(A) \in \xi$. Using Martin's Axiom it is not difficult to find a strongly summable ultrafilter on every group G. By the Theorem of Blass and Hindman [1], existence of strongly summable ultrafilter on $\mathbb{Z}$ or on countable Boolean group does not depend on the system of axioms ZFC. We show in Theorem 3 that strongly summable ultrafilters can live only on countable subgroups of a group.

Lemma 7. *Let ξ be a strongly summable ultrafilter on a Boolean group. Then ξ is a PS-idempotent.*

Proof. Fix any $X \in \xi$ and take $A \subset G$ with $FS(A) \subset X$, $FS(A) \in \xi$. Take any $x, y \in FS(A)$, $x \neq y$. Since G is Boolean group, $x + y \in FS(A)$. This means that $PS\big(FS(A)\big) \subset FS(A) \subset X$, so ξ is a PS-idempotent.

By Hindman's Theorem, for any finite partition of a group $G = A_1 \cup \dots \cup A_m$ there exist an infinite subset $A \subset G$ and the cell A_i such that $FS(A) \subset A_i$. Given a finite partition of uncountable group $G = A_1 \cup \dots \cup A_m$, one can ask whether a uncountable A with property $FS(A) \subset A_i$ can be chosen. The following statement gives a complete answer.

Theorem 2. *Let G be an uncountable group. There exists a partition $G = X_0 \cup X$ such that $FS(A) \cap X_0 \neq \varnothing$, $FS(A) \cap X_1 \neq \varnothing$ for every uncountable subset $A \subset G$.*

Proof. Every abelian group is a subgroup of some divisible group. Every divisible group is a direct sum of countable groups. In views of this facts, we take G to be the direct sum $G = \bigoplus\{G_\alpha : \alpha < \gamma\}$ of countable groups G_α. Given any element $g \in G$, $g = \{g_\alpha : \alpha < \gamma\}$, $g_\alpha \in G_\alpha$, we write $\operatorname{supp} g = \{\alpha < \gamma : g_\alpha \neq 0\}$. It should be noticed that each $\operatorname{supp} g$ is a finite subset. For each $g \in G$, $g \neq 0$, take $s(g) \in \mathbb{Z}$ such that $2^{s(g)} \leq |\operatorname{supp} g| < 2^{s(g)+1}$.

Put $X_1 = \{g \in G : s(g) \text{ is odd}\}$, $X_0 = G \setminus X_1$. Fix any uncountable subset $A \subset G$. Since all summands G_α are countable groups, we consider $\operatorname{supp} g = \operatorname{supp} h$ for every $\{g, h\} \in [A]_2$. By Sunflower's Lemma there exists an uncountable subset $B \subset A$ with

$$\operatorname{supp} g \cap \operatorname{supp} h = \bigcap\{\operatorname{supp} x : x \in B\}, \quad |\operatorname{supp} g| = |\operatorname{supp} h|, \quad \{g, h\} \in [B]_2$$

Take any sequence $\langle a_n \rangle_{n=1}^{\infty}$ in B and put $d = |\operatorname{supp} a_n|$, $c = |\operatorname{supp} a_i \cap \operatorname{supp} a_j|$, where $i \neq j$. If $b_n = a_1 + \dots + a_n$ then $n(d-c) - c \leq |\operatorname{supp} b_n| \leq n(d-c) + c$. By these inequalities, one can choose b_n, b_m with $b_n \in X_0$, $b_m \in X_1$. Since $b_n, b_m \in FS(A)$, it proves $FS(A) \cap X_0 \neq \varnothing$, $FS(A) \cap X_1 \neq \varnothing$.

In fact, for every uncountable group G, there exists a countable partition $G = \bigcup_{n=1}^{\infty} G_n$ such that $g + FS(A) \cap X_n \neq \varnothing$ for every uncountable subset $A \subset G$, $g \in G$ and for each n. See [6] for construction and application of this partition to topological groups.

Theorem 3. *Let ξ be a strongly summable ultrafilter on a group G. Then there exists a countable subset $A \subset G$ with $A \in \xi$.*

Proof. Suppose the contrary and use a partition of G given by Theorem 2.

Let X be a countable set. A free ultrafilter ξ on X is called a P-point in X^* if, given any partition $X = \bigcup_{n=1}^{\infty} X_n$ with $X_n \notin \xi$, there exists $Y \in \xi$ such that $|y \cap X_n| < \aleph_0$ for each n. By Shelah's Theorem there are models of ZFC without P-point in $\mathbb{N}^*$.

A free ultrafilter ξ on X is called a Q-point if, given any partition $X = \bigcup_{n=1}^{\infty} X_n$ with $|X_n| < \aleph_0$, there exists $Y \in \xi$ with $|Y \cap X_n| \leq 1$ for each n.

An ultrafilter ξ on X is a Ramsey ultrafilter if and only if ξ is both P-point and Q-point.

We add a topological characterization to a list of equivalent definitions of P-point. We do not unify the following two lemmas in criterium, because the first of them belongs to folklore.

Lemma 8. *Let X be a countable set and ξ a P-point in X^*. Given any Hausdorff topology τ on X, there exists a subset $A \in \xi$ with at most one accumulation point in the topology τ.*

Proof. Denote by Y the set of limit points of the ultrafilter ξ in the topology τ. Since τ is a Hausdorff topology, $|Y| \leq 1$. For every $x \in X \setminus Y$ take a neighborhood $\mathcal{U}_x$ of x with $\mathcal{U}_x \notin \xi$. Since ξ is a P-point, there exists $A \in \xi$ with $|A \cap \mathcal{U}_x| < \aleph_0$. This means that Y is a set of accumulation points of subset A in topology τ.

Lemma 9. *Let X be a countable set and let ξ be a free ultrafilter on X. Assume that, given any Hausdorff topology τ on X, there exists a subset $A \in \xi$ with at most one accumulation point in the topology τ. Then ξ is a P-point in X^*.*

Proof. Suppose on the contrary there exists a partition $X = \bigcup_{n=1}^{\infty} X_n$, $X_n \notin \xi$ such that, for every $A \in \xi$, one can find X_m with $|A \cap X_m| = \aleph_0$. Fix any $x_n \in X_n$ and endow the set X_n with a topology τ_n. We say that a subset $\mathcal{U} \subset X_n$ is open if and only if either $x_n \notin \mathcal{U}$ or $x_n \in \mathcal{U}$, $|X_n \setminus Y| < \aleph_0$. Let (X, τ) be the union of the topological spaces (X_n, τ_n). Then the subset A has an infinite number of accumulation points in the topology τ, a contradiction.

Lemma 10. *Let G be a countable group and ξ an idempotent in G^*. Then ξ is not P-point in G^*.*

Proof. For every $x \in G$, put $\mathcal{T}_x = \{x + \{P \cup \{0\}\} : P \in \xi\}$. By [4], there exists a topology τ on G such that $\mathcal{T}_x$ is a base of neighborhoods at point x. Since ξ is an idempotent, the set of accumulation points of any subset $A \in \xi$ is an element of the ultrafilter ξ. By Lemma 8, ξ is not P-point.

Theorem 4. *Assume the Martin Axiom. Let G be a countable Boolean group. There exists a PS-ultrafilter on G which is neither Ramsey ultrafilter nor PS-idempotent.*

Proof. We use the Martin Axiom just to fix some strongly summable ultrafilter ξ on G. By Lemma 7, ξ is a PS-idempotent. By Lemma 1, $g + \xi$ is a PS-ultrafilter for each $g \in G$. Suppose that $g+\xi$ is an idempotent. Then $2g+\xi = g+\xi$, so $g = 0$. By Lemma 10, $g + \xi$ is not a P-point. It follows that $g + \xi$, $g \neq 0$ is a required ultrafilter.

Theorem 5. *Let G be a countable torsion group and ξ a PS-ultrafilter on G. If ξ is a Q-point then ξ is a Ramsey ultrafilter.*

Proof. We consider a group G as the union of the increasing chain of finite subgroups $G = \bigcup_{n=1}^{\infty} G_n$, $G_n \subset G_{n+1}$. Put $X_1 = G_1$, $X_n = G_n \setminus G_{n+1}$. Since $G = \bigcup_{n=1}^{\infty} G_n$ and ξ is a Q-point, there exists $A \in \xi$ with $|A \cap X_n| \leq 1$ for each n. It is clear that A is a 2-independent subset of G. By Lemma 3, ξ is a Ramsey ultrafilter.

Theorem 6. *Existence of PS-idempotent implies existence of P-point in $\mathbb{N}^*$.*

Proof. Let ξ be a PS-idempotent on a group G. By Theorem 1, we can take G to be a countable Boolean group. We write an element of G as $x = (\alpha_1, \dots, \alpha_n, \dots)$, where $\alpha_i \in \{0,1\}$ and $\alpha_n = 0$ for almost all n. For $x \neq 0$, denote by $\min x$ and $\max x$ the number of the first and the number of the last nonzero coordinates of x. Let $\min \xi$ and $\max \xi$ be ultrafilters on $\mathbb{N}$ with bases $\{\min X : X \in \xi\}$ and $\{\max X : X \in \xi\}$. In [4] it was shown that $\max \xi$ is a P-point. Using the method of Blass and Hindman [1], it is easy to show that $\min \xi$ is also a P-point.

In [4] our interest to PS-idempotents was motivated by the following observation. Let (G, τ) be a topological group with only one free ultrafilter ξ that converges to zero. Then ξ is a PS-idempotent.

Question 1. *Is it possible to prove the existence of PS-ultrafilter on some group G without making any special set theoretic assumptions?*

Question 2. *Let ξ be a PS-ultrafilter on $\mathbb{Z}$. Is ξ a Ramsey ultrafilter?*

Question 3. *Let ξ be a PS-idempotent on a countable Boolean group. Is ξ astrongly summable ultrafilter?*

Question 4. *Let ξ be a strongly summable ultrafilter on a group G. Is ξ an idempotent of the semigroup G^*?*

Question 5. *Does existence of strongly summable ultrafilter on a group imply existence of P-point in $\mathbb{N}^*$?*

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