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SEMI-PERFECT RINGS IN WHICH EVERY IDEAL IS IDEMPOTENT

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We prove that the quiver of a right noetherian semi-perfect indecomposable ring in which every nonradical ideal is idempotent contains at most two vertices. If such a ring is semi-distributive, it is serial and two-sided notherian. Conversely, if the quiver of a serial indecomposable two-sided notherian ring contains at most two vertices then every nonradical ideal of this ring is idempotent.

Throughout this paper A denotes a semi-perfect ring, R denotes its Jacobson radical. An ideal means a two-sided ideal. Recall that an ideal J of the ring A is called idempotent if $J^2 = J$. An element $a \in A$ is called central modulo radical R if $a + R$ lies in the center of the ring $\bar{A} = A/R$. In the paper of G. Michler [3] the following statements were proved:

(a) if J an idempotent ideal of a left perfect ring A, then $J = AeA$, where e is an idempotent which is central modulo R.

(b) there exists a biunique correspondence between idempotent ideals $J \neq 0$ of a left simple perfect ring A and ideals $\bar{J} \neq \bar{0}$ of the semi-simple artinian ring $\bar{A} = A/R$.

(c) a left perfect ring A possesses exactly 2^s idempotent ideals, where s is the number of the simple components of the ring A.

Proposition 1. *Let f be an idempotent of a ring A which is central modulo R. Then AfA is an idempotent ideal of the ring A.*

Proof. Denote $g = 1 - f$ and $X = fAg, Y = gAf$. Since the idempotent f is central modulo R, we see that AfA possesses a such two-sided Peirce decomposition:

$$AfA = \begin{pmatrix} fAf & X \\ Y & YX \end{pmatrix}.$$

Consider the product

$$(AfA)(AfA) = \begin{pmatrix} fAf & X \\ Y & YX \end{pmatrix} \begin{pmatrix} fAf & X \\ Y & YX \end{pmatrix} = \begin{pmatrix} fAf & X \\ Y & YX \end{pmatrix},$$

i.e. AfA is an idempotent ideal.

Recall that a semigroup S is called a band if every element of S is an idempotent [4, 1.8].

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Proposition 2. *All idempotent ideals of an arbitrary ring form an commutative band with respect to addition.*

Proof. Let J_1 and J_2 be idempotent ideals. Then

$$(J_1 + J_2)^2 = J_1^2 + J_1J_2 + J_2J_1 + J_2^2 = J_1 + J_2.$$

Definition. We say that a ring A satisfies the left Nakayama condition for ideals if for two arbitrary ideals J_1 and J_2 of the ring A the equality $J_1 + RJ_2 = J_2$ implies $J_1 = J_2$.

Every left perfect ring satifies the left Nakayama condition for ideals [5, Theorem 11.5.5]. Every left noetherian ring satisfies the left Nakayama condition for ideals (it is a consequence from the Nakayama lemma).

Theorem 3. *Let a semi-perfect ring A satisfy the left Nakayama condition for ideals. Then every idempotent ideal J is of the form $J = AfA$, where f is a central modulo R idempotent.*

Proof. Let $A = P_1^{n_1} \oplus \cdots \oplus P_s^{n_s}$ be a decomposition of the ring A into a direct sum of indecomposable projective modules with pairwise non-isomorphic modules $P_1, \ldots, P_s$. Let $1 = f_1 + \cdots + f_s$ be a decomposition of the unity of the ring A into a sum of mutually orthogonal idempotents such that $f_iA = P_i^{n_i}$, $(i = 1, \ldots, s)$. Put down $A_{ij} = f_iAf_j$, $(i, j = 1, \ldots, s)$. Then $R = \bigoplus_{i,j=1}^{s} f_iRf_j$, where $f_iRf_j = A_{ij}, i \neq j$ and $f_iRf_i = R_i$, where R_i is the Jacobson radical of the ring A_{ii}. The ring A_{ii} is isomorphic to the ring $EndP_i^{n_i} = M_{n_i}(\mathcal{O}_i)$, where $\mathcal{O}_i = EndP_i$ is a local ring. This follows from the Muller theorem [6]. Suppose that the idempotent ideal J lies in R. Then $J \subseteq RJ$ and hence $J = RJ$. From conditions of the present theorem it follows that $J = 0$. Therefore, the idempotent ideal J is not contained in R.

Consider now the two-sided Peirce decomposition

$$J = \bigoplus_{i,j=1}^{s} f_iJf_j.$$

Note that if $f_iJ \not\subseteq P_i^{n_i}R$, $i, j = 1, \ldots, s$; then $f_iJf_j = f_iAf_j$. Since $J \not\subseteq R$, there exists a unique set of indices $1 \leq i_1 \leq i_2 \leq \ldots \leq i_t \leq s$ such that $f_{i_1}J = P_{i_1}^{n_{i_1}}, \ldots, f_{i_t}J = P_{i_t}^{n_{i_t}}$ and $f_iA \subseteq P_j^{n_j}R$ for other $1 \leq j \leq s$. Denote $f = f_{i_1} + \cdots + f_{i_t}$, $g = 1 - f$, $X = fAg$, and $Y = gAf$. Then the set

$$AfA = \begin{pmatrix} fAf & X \\ Y & gAg \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} faf & X \\ Y & gAg \end{pmatrix} = \begin{pmatrix} fAf & X \\ Y & YX \end{pmatrix}$$

is an idempotent ideal $M, M \subseteq J$ (this follows from Proposition 1).

Clearly, $AfA + R = J + R$. Therefore, $J = J^2 = (AfA + R)J \subseteq AfA + RJ \subseteq J$, i.e. $AfA + RJ = J$. It follows from the conditions of the theorem that $J = AfA$. Theorem is proved.

Definition [1, Ch.22]. A right ideal of a ring is called a radical right ideal if it is contained in the Jacobson radical of this ring. Otherwise, a right ideal is called non-radical.

The same definitions we will use for left ideals and ideals of a ring.

Recall that s stands for the number of all pairwise non-isomorphic indecomposable projective modules over a ring A.

Theorem 4. *Suppose that a semi-perfect ring A satifies the left Nakayama condition for ideals and every its non-radical ideal is idempotent. Then $s \leq 2$.*

To prove this theorem we need the following Lemma (in this Lemma the ring A is the same as in Theorem 4).

Lemma 5. *In the ring A for every idempotent f which is central modulo R we have the equality $gAfAg = gRg$, where $g = 1 - f$.*

Proof. Consider the ideal

$$J = \begin{pmatrix} fAf & fAg \\ gAf & gRg \end{pmatrix}.$$

Clearly, it is non-radical and idempotent. Then $j + AfA$ and $gAfAg = gRg$.

Let now $1 = f_1 + \cdots + f_s$ be a decomposition of the unity $1 \in A$ into a sum of mutually orthogonal idempotents such that $f_iA = P_i^{n_i}$, $i = 1, \ldots, s$. Suppose that $s > 2$. Denote by $\bar{A}_i$ the set

$$\begin{pmatrix} A_{1i} \\ A_{2i} \\ \vdots \\ A_{i-1,i} \\ A_{i+1,i} \\ \vdots \\ A_{s_i} \end{pmatrix}$$

and by $\bar{A}^{(i)}$ the set

$$(A_{i1}, \ldots, A_{i,i-1}, A_{i,i+1}, \ldots, A_{is}).$$

By Lemma 5, $\bar{A}_i\bar{A}^{(i)} = g_iRg_i$, $g_i = 1 - f_i$, $i = 1, \ldots, s$. The multiplication in the left side of this equality means the matrix multiplication of Peirce components. From this equalities it follows that $R^2 = R$ and hence $R = 0$. The obtained contradiction proves the theorem.

All necessary definitions and informations about semi-perfect semi-distributive rings is contained in [7], these rings will be called $SPSD$-rings as in [7].

Theorem 6. *Let A be a right noetherian indecomposable $SPSD$-ring. If every non-radical ideal of A is idempotent then the ring A is two-sided noetherian serial ring and the quiver of A contains at most two vertices. Conversely, if the quiver of a serial indecomposable two-sided ring contains at most two vertices then every non-radical ideal of the ring is idempotent.*

Proof. We consider the cases $s = 1$ and $s = 2$. For $s = 1$, the ring A is a right noetherian serial ring by Theorem 1.4 [7]. This ring is either discrete valuation ring or uniserial Koethe ring (see Theorem 3.2 [2, p.48]). Clearly, such rings are two-sided noetherian and every non-radical ideal of the ring A is idempotent.

Let $s = 2$. We can assume that the ring A is reduced. Denote by e_1 and e_2 local idempotents such that the unity of A is a sum of these idempotents. Now put down $\mathcal{O}_i = e_iAe_i$. Let R_i be the Jacobson radical of the ring $\mathcal{O}_i, (i = 1, 2); X = e_1Ae_2, Y = e_2Ae_1$. By Lemma 5 $XY = R_1$ and $YX = R_2$. Hence $XYX = XR_2 = R_1X$. By Theorem 1.4 [7] X is a uniserial left $\mathcal{O}_1$-module and a uniserial $\mathcal{O}_2$-module. Since $R_1X \subseteq X$ and $R_1X \neq X$, X is a finite generated left $\mathcal{O}_1$-module. Analogously, Y is a finite generated left $\mathcal{O}_2$-module. By Theorem 10.5 [8] the ring A is two-sided noetherian. Clearly, the quiver $Q(A)$ is either an arrow or a simple cycle which contains two arrows. In both cases the ring A is serial.

Conversely, if $Q(A)$ contains only one arrow then the ring A is isomorphic to the ring of upper triangle matrices over a skew field [9]. Apparently, in this ring all non-radical ideals are idempotent. In the case of the simple cycle one can assume that for non-radical ideal $J \neq A$ the equality $\mathcal{O}_1 = e_1Je_1$ holds. Then $e_1Je_2 = e_1Ae_1 * e_1Ae_2 = X$ and $e_2Je_1 = Y$, hence $YX = R_2 = e_2Je_2$. Clearly, the ideal

$$J = \begin{pmatrix} \mathcal{O}_1 & X \\ Y & R_2 \end{pmatrix}$$

is idempotent. Theorem is proved.

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