УДК 512.544+519.46

SEMIPERFECT RINGS WITH PERIODIC LOCALLY NILPOTENT GROUP OF UNITS

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Yu.B. Ishchuk. *Semiperfect rings with periodic locally nilpotent group of units*, Matematychni Studii, **7**(1997) 125–128.

The semiperfect rings with periodic locally nilpotent group of units are characterized.

1. Let us recall that a ring R is semiperfect if the quotient $R/\mathcal{J}(R)$ is right Artinian and all idempotents of $R/\mathcal{J}(R)$ can be lifted to idempotents of R.

In this note we are going to characterize the semiperfect rings with periodic locally nilpotent group of units.

Notation:
p is a prime, $U(R)$ is the group of units of ring R,
$R_\circ$ is the subring generated by identity of R,
$\langle R_\circ, U(R)\rangle$ is the subring of R generated by $R_\circ$ and $U(R)$,
$\mathcal{J}(R)$ is the Jacobson radical of R,
$R[G]$ is the group ring of group G over ring R,
$1+\mathcal{J}(R)$ is the unipotent group of R,
T^* is the multiplicative group of a skew field T,
$\omega_{R[G]}(H)=\sum_{h\in H}(1-h)R[G]$ is the right ideal of $R[G]$ generated by the elements $(1-h)$ $(h\in H)$, where H is a subgroup of G.

2. The following lemma is a consequence of [4].

Lemma 2.1. *Let T be a skew field. Then the following statements are equivalent:*
1) T^* is locally nilpotent.
2) T^* is hypercentral.
3) T^* is nilpotent.
4) T^* is abelian.
5) T^* is radical.
6) T^* is locally solvable.
7) T^* is solvable.
8) T^* is a RN^*-group.
9) T^* is hypocentral.

1991 *Mathematics Subject Classification.* 16L30, 20F19.

Lemma 2.2. *Let R be a semiperfect ring. Then $R = \langle R_\circ, U(R)\rangle$ if and only if at most one simple component of the quotient ring $R/\mathcal{J}(R)$ is isomorphic to the field $GF(2)$.*

Proof. ($\Rightarrow$) Let $R = \langle R_\circ, U(R)\rangle$. Suppose to the contrary that $R/\mathcal{J}(R) = \langle e_1\rangle \oplus \langle e_2\rangle \oplus S$ for some idempotents e_1, e_2 of $R/\mathcal{J}(R)$ such that $2e_1 = 2e_2 = 0$ and for some semisimple ring S.

Now let e be an identity of ring $R/\mathcal{J}(R)$. Then $e = e_1 + e_2 + e_3$, where e_3 is the identity of ring S and any unit u of ring $R/\mathcal{J}(R)$ can be written as $u = e_1+e_2+v$, for some $v \in U(S)$. Since, by assumption, $R = \langle R_\circ, U(R)\rangle$ we have in particular $e_1 \in \langle \bar{R}_\circ, U(R/\mathcal{J}(R))\rangle$ with $\bar{R}_\circ = \langle R_\circ, \mathcal{J}(R)\rangle/\mathcal{J}(R)$, and so $e_1 = n_1u_1 + \cdots + n_ku_k$ for some $k \in \mathbb{N}$, $n_i \in \bar{R}_\circ$, $u_i \in U(R/\mathcal{J}(R))$ $(i = 1, \dots, k)$. From the last equality it follows that $e_1 = \sum_{i=1}^k n_ie_1$ and $0 = \sum_{i=1}^k n_ie_2$. This contradicts our choice of e_1, e_2 so that the necessity of the lemma is proved.

($\Leftarrow$) Let at most one simple component of the quotient ring $R/\mathcal{J}(R)$ be isomorphic to the field $GF(2)$. Show that $\langle R_\circ, U(R)\rangle = R$. Since $1+\mathcal{J}(R)$ is a normal subgroup of $U(R)$, we have $\langle \bar{R}_\circ, U(R/\mathcal{J}(R))\rangle = \langle \bar{R}_\circ, U(R)/(1 + \mathcal{J}(R))\rangle$ and so it is enough to prove that $R/\mathcal{J}(R) = \langle \bar{R}_\circ, U(R/\mathcal{J}(R))\rangle$.

If $R/\mathcal{J}(R)$ is a simple ring, then without loss of generality $R/\mathcal{J}(R) = \mathcal{M}_n(P)$ for some skew field P and some positive integer n and therefore $U(R/\mathcal{J}(R)) = GL_n(P)$. Consequently $R/\mathcal{J}(R) = \langle \bar{R}_\circ, U(R/\mathcal{J}(R))\rangle$.

Now suppose that $R/\mathcal{J}(R) = S_1 \oplus \cdots \oplus S_m$ $(m \geq 2)$, $S_i = \mathcal{M}_{n_i}(P_i)$ for some skew field P_i and some positive integer n_i $(i = 1, \dots, m)$. Then as it is well known, $U(R/\mathcal{J}(R)) \cong U(S_1) \otimes \cdots \otimes U(S_n)$ is a direct product of groups. It is sufficient to consider only the case $m = 2$. Let e_i be an identity of S_i. It is enough to show that $e_i \in \langle \bar{R}_\circ, U(R/\mathcal{J}(R))\rangle$, $i = 1, 2$.

Suppose the characteristic of S_1 or S_2 is different 2. Say, let $\operatorname{char} S_1 \neq 2$. Then from $2e_1 + e_2 \in U(R/\mathcal{J}(R))$ it follows that $e_1 = (2e_1 + e_2) - (e_1 + e_2) \in \langle \bar{R}_\circ, U(R/\mathcal{J}(R))\rangle$. Now let $\operatorname{char} S_1 = \operatorname{char} S_2 = 2$. By the assumptions S_1 and S_2 are not simultaneously isomorphic to $GF(2)$. So suppose without loss of generality that $S_1 \neq GF(2)$. If S_1 is a skew field then there is $u \in U(S_1)$ such that $u \neq e_1$. Therefore $e_1 - u \in U(S_1)$ and so $e_1 = (e_1 - u + e_2) + (u + e_2) \in \langle \bar{R}_\circ, U(R/\mathcal{J}(R))\rangle$.

Suppose that $S_1 = \mathcal{M}_n(P)$, with $n > 1$, $\operatorname{char} P = 2$. Then there exist $u, v \in U(S_1)$ such that $u \neq e_1$, $v \neq e_1$ and $u + v = e_1$, for example,

$$u = \begin{pmatrix} f & f \\ f & 0 \end{pmatrix} \quad \text{та} \quad v = \begin{pmatrix} 0 & f \\ f & f \end{pmatrix},$$

where f is the identity of P. Consequently, $e_1 = (u+e_2)+(v+e_2) \in \langle \bar{R}_\circ, U(R/\mathcal{J}(R))\rangle$. Since $e_1, e_2 \in \langle \bar{R}_\circ, U(R/\mathcal{J}(R))\rangle$, we have $R = \langle R_\circ, U(R)\rangle$, proving the lemma.

Corollary 2.3. *If R is a semilocal ring and $2 \in U(R)$ then $R = \langle R_\circ, U(R)\rangle$.*

Lemma 2.4. [6, Lemma 2.4a)]. *Let A is a nil ring. Then A^+ is a p-group iff an adjoint group A° is a p-group.*

Lemma 2.5. [7, Lemma 2.3]. *The adjoint group R° of a local ring R is periodic if and only if $\operatorname{char} R = p^k$ $(k \in \mathbb{N})$ for some prime p, $\mathcal{J}(R)$ is a nil ideal and the quotient ring $R/\mathcal{J}(R)$ is an absolute field.*

Lemma 2.6. *Let C be a commutative local ring of prime power characteristic p^k, $\mathcal{J}(C)$ a nil ideal. If P is locally nilpotent p-group then the group ring $C[P]$ is local, the Jacobson radical $\mathcal{J}(C[P])$ is a nil ideal and $C/\mathcal{J}(C) \cong C[P]/\mathcal{J}(C[P])$.*

Proof. Let $D \cong C/\mathcal{J}(C)$ and suppose I is the kernel of the canonical epimorphism $C[P] \to D[P]$. Then $I = \mathcal{J}(C)C[P]$ is a nil ideal. Moreover, $D[P]$ is a local algebra by Theorem 16.2 [2, p.73]. It follows that $C[P]$ is also local. By Proposition 1.11 [2] $D[P]/\omega_{D[P]}(P) \cong D$ and therefore $\mathcal{J}(D[P]) = \omega_{D[P]}(P)$ is a locally nilpotent ideal by Proposition 5.7 [2]. Therefore $\mathcal{J}(C[P])$ is a nil ideal. We conclude that $\mathcal{J}(C[P]) = I + \sum_{g \in P}(1-g)C[P]$. Since

$$C[P]/\mathcal{J}(C[P]) \cong (C[P]/I)/(\mathcal{J}(C[P])/I) \cong D[P]/\mathcal{J}(D[P]) \cong D,$$

we have $C/\mathcal{J}(C) \cong C[P]/\mathcal{J}(C[P])$. This proves the lemma.

Theorem 2.7. *Let R be a local ring. Then the following statements are equivalent:*
1) $U(R)$ is a periodic locally nilpotent group.
2) R is a homomorphic image of a group ring $C[P]$ with P a locally nilpotent p-group, C is a homomorphic local image of the group ring $\mathbb{Z}_{p^k}[G]$ of an abelian locally cyclic p'-group G.

Proof. 1) $\Rightarrow$ 2). Since $F = R/\mathcal{J}(R)$ is a skew field, F is an absolute field and $\operatorname{char} F = p$ for some prime p by Lemma 2.1 [7]. Moreover, $\operatorname{char} R = p^k$ for some positive integer k by Lemma 2.4 and F^* is an abelian locally cyclic p'-group by Lemma 2.1 [7]. Since $U(R)$ is periodic and locally nilpotent, we have

$$U(R) = (1 + \mathcal{J}(R)) \times G \quad \text{with} \quad G \cong F^*$$

and $1+\mathcal{J}(R)$ is a p-group by Lemma 2.4. But $R = \langle R_\circ, U(R)\rangle$ by Lemma 2.2 and so $R = \langle R_\circ, 1+\mathcal{J}(R), G\rangle$ with $R_\circ \cong \mathbb{Z}_{p^k}$. Put $C = \langle R_\circ, G\rangle$. Then $R = \langle C, 1+\mathcal{J}(R)\rangle$ is a homomorphic image of the group ring $C[1+\mathcal{J}(R)]$. Clearly, C is a local ring which is a homomorphic image of the group ring $R_\circ[G]$.
2) $\Rightarrow$ 1). Obviously $\operatorname{char} C \leq p^k$. If H is a finitely generated subgroup of P, then H is a nilpotent p-group and $U(C[H])$ is a nilpotent group by Theorem 1 [8]. Let $s_1, \dots, s_n$ be any elements of $U(C[P])$. Then obviously $s_1, \dots, s_n \in U(C[H_\circ])$ for some finitely generated subgroup $H_\circ$ of G and therefore the subgroup $\langle s_1 \dots, s_n\rangle$ is nilpotent. We conclude that $U(C[P])$ is a locally nilpotent group. Hence $U(R)$ is also locally nilpotent.

If x is an arbitrary element of $U(R)$ then $x \in U(C_1[H])$ for some finitely generated p-subgroup H of G and some subring $C_1 = \mathbb{Z}_{p^k}[G_1]$, where G_1 is a cyclic p'-group. Since the subring C_1 is finite, the subgroup $U(C_1[H])$ is also finite and consequently the group $U(R)$ is periodic. This finishes the proof of the theorem.

Corollary 2.8. *Let R be a semiperfect ring, $2 \in U(R)$. Then $U(R)$ is periodic locally nilpotent group iff $R = R_1 \oplus \cdots \oplus R_s$ is a ring direct sum of local rings $R_1, \dots, R_s$ which satisfies the conditions of Theorem 2.7.*

The proof of the corollary can be obtained by combining the results of Theorem 2.7 and Corollary 2.3.

Corollary 2.9. *Let R be a local ring with commutative Jacobson radical $\mathcal{J}(R)$ (in particular, $\mathcal{J}(R)^2 = 0$). If the group $U(R)$ is periodic and locally nilpotent then R is a commutative ring.*

The proof follows from Theorem 2.7.

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Received 10.06.96