OPERATOR CALCULUS FOR A CONVOLUTION ALGEBRA OF SCHWARTZ DISTRIBUTIONS ON SEMIAxis

O.V. Lopushansky, S.V. Sharin

Dedicated to Professor V. Lyantse on his 75th birthday


In this article we construct for a special algebra of Schwartz distributions a counterpart of the functional calculus for unbounded linear operators which are generators of one-parametric groups of the class $(C_0)$ over Banach spaces.

A counterpart of the functional calculus for a special algebra of distributions on the axis of unbounded linear operators which are generators for the one-parametric semigroups of $(C_0)$-class over Banach spaces is constructed.

Considered algebra of distributions is defined as the image of Fourier transformation $\mathcal{F} : f \rightarrow \hat{f}$ of algebra $\mathcal{D}'_+$ of Schwartz distributions on the semiaxis with the convolution operation $\ast$ instead of multiplication and is denoted by $\mathcal{D}'_+$. The multiplication in algebra $\mathcal{D}'_+$ is determined by the following relations

$$\hat{\delta} = 1 \quad \text{and} \quad \hat{f} \cdot \hat{h} = \hat{f \ast h} \quad \text{for all} \quad f, h \in \mathcal{D}'_+, \quad (1)$$

where $\delta$ is the Dirac function. Let us notice that operation defined in such way is different from the multiplication of distributions which was considered in the known article of L. Schwartz [1].

Functional calculus $\mathcal{G} : \mathcal{D}'_+ \ni \hat{f} \rightarrow \hat{f}(A)$ is determined by the formula

$$\hat{f}(A) \hat{x} = \int_0^\infty e^{-itA} (f \ast x)(t) \, dt, \quad \text{where} \quad \hat{x} = \int_0^\infty e^{-itA} x(t) \, dt, \quad (2)$$

and $e^{-itA}$ is the semigroup in Banach space $E$ with generator $-iA$, and $f \ast x$ is a cross-correlation of the distribution $f \in \mathcal{D}'_+$ with an arbitrary $E$-valued smooth finite function $x(t)$ on the semiaxis. This calculus is realizing an isomorphism of algebras, in particular,

$$\mathcal{G}(1) = I \quad \text{and} \quad \mathcal{G} (\hat{f} \cdot \hat{h}) = \hat{f}(A) \hat{h}(A). \quad (3)$$

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1Another approaches for extending the multiplication on the classes of generalized functions were considered in the articles [2–4].
The domain of definition of operators \( \hat{f}(A) \) which is constructed from the elements \( \hat{x} \) is dense in \( E \).

Let us remark that \( \mathcal{D}_+ \) coincides with the space of generalized boundary values on the real axis of some space \( \mathcal{H}_- \) of analytic functions in the lower complex half-plane and contains a respective Hardy class \( H^2_+ \). That is why the isomorphisms \( \mathcal{D}_+ \cong \mathcal{D}_+ \) and \( \mathcal{D}_+ \cong \mathcal{H}_- \) determined by Fourier transformation may be treated as a generalization of Paley-Wiener theorem for the Hardy class.

1. Let us consider the complex space of Schwartz distributions \( \mathcal{D}' \equiv \mathcal{D}'(\mathbb{R}) \) on the axis \( \mathbb{R} \) and its subspace \( \mathcal{D}'_+ \) of distributions \( f \) with the supports \( \text{supp} \, f \) on the semiaxis \( (0, +\infty) \). We shall describe the space of test functions for \( \mathcal{D}'_+ \).

Let \( \mathbf{D} \equiv \mathbf{D}(\mathbb{R}) \) be the complex space of finite functions \( \varphi = \varphi(t) \) on \( \mathbb{R} \) with the continuous derivatives \( \varphi^{(n)}(t) \) of all orders. We shall denote by \( \mathbf{D}' \) the subspace of the functions \( \varphi \) with supports \( \text{supp} \, \varphi \) on segment \( [-\nu, \nu] \) for an arbitrary number \( \nu > 0 \) and topology will be determined by the collection of norms

\[
\|\varphi\|_{n, \nu} = \sup_{t \in [-\nu, \nu]} |\varphi^{(n)}(t)|. \]

It is easy to see that \( \mathbf{D} = \bigcup_{\nu > 0} \mathbf{D}' \). Let us determine the locally convex topology of the inductive limit \( \lim_{\nu \to +\infty} \mathbf{D}' \) on \( \mathbf{D} \).

Canonical bilinear form which gives a duality of spaces \( \mathbf{D}' \) and \( \mathbf{D} \) will be denoted as \( \langle f, \varphi \rangle \), and the dual pair itself as \( \langle \mathbf{D}', \mathbf{D} \rangle \).

Taking into account the duality \( \langle \mathbf{D}', \mathbf{D} \rangle \), a dual space to \( \mathbf{D}'_+ \) will be described as the quotient space \( \mathbf{D}/(\mathbf{D}'_+)^{\circ} \), where \( (\mathbf{D}'_+)^{\circ} = \{ \varphi \in \mathbf{D} : \text{supp} \, \varphi \subset (-\infty, 0) \} \) is a polar of \( \mathbf{D}'_+ \). The quotient space \( \mathbf{D}/(\mathbf{D}'_+)^{\circ} \) may be identified with the space of functions \( \mathbf{D}_+ = \{ \varphi_+(t) \setminus \varphi \in \mathbf{D} \} \), where \( \varphi_+ = \theta \varphi \) is the product of Heaviside function \( \theta(t) = \begin{cases} 
1, & t \geq 0 \\
0, & t < 0 
\end{cases} \) multiplied by \( \varphi \in \mathbf{D} \).

Let us determine on \( \mathbf{D}_+ \) the topology of inductive limit \( \lim_{\nu \to +\infty} \mathbf{D}'_+ \). Norms

\[
\|\varphi\|_{n, \nu} = \sup_{t \in [0, \nu]} |\varphi^{(n)}(t)|
\]

are considered on the spaces \( \mathbf{D}'_+ = \{ \varphi_+ \in \mathbf{D}_+ : \text{supp} \, \varphi_+ \subset [0, \nu] \} \). Note that in zero point the right derivatives of functions \( \varphi_+ \) are defined only.

**Lemma 1.** Spaces \( \mathbf{D}_+ \) and \( \mathbf{D}/(\mathbf{D}'_+)^{\circ} \) are topologically isomorphic and bilinear form of \( \langle \mathbf{D}', \mathbf{D} \rangle \) induces a duality \( \langle \mathbf{D}'_+, \mathbf{D}_+ \rangle \).

**Proof.** Representatives of the same class of equivalence \( [\varphi] \equiv \{ \varphi \in \mathbf{D} : \theta \varphi = \varphi_+ \} \) coincide on \( [0, +\infty) \). Thus, inequality \( \|\varphi\|_{n, \nu} = \inf_{\varphi \in [\varphi]} \sup_{t \in [-\nu, \nu]} |\varphi^{(n)}(t)| \geq \sup_{t \in [0, \nu]} |\varphi^{(n)}(t)| = \|\varphi_+\|_{n, \nu} \) holds for the quotient-norm whenever \( \nu > 0 \), and the mapping \( [\varphi] \to \varphi_+ \) from \( \mathbf{D}/(\mathbf{D}'_+)^{\circ} \) to \( \mathbf{D}_+ \) is continuous. Evidently, it is a surjective mapping and one can use the theorem about the open mapping for denumerable inductive limits of Fréchet spaces (see [5]). \( \bullet \)

An immediate corollary of lemma and reflexivity of spaces is that the strong topology on \( \mathbf{D}_+ \) relative to the duality \( \langle \mathbf{D}'_+, \mathbf{D}_+ \rangle \) is induced by the strong topology relative to the duality \( \langle \mathbf{D}', \mathbf{D} \rangle \).

Let us consider the complex Banach space \( (E, \|\cdot\|) \) and the space \( \mathcal{D}_+(E) \) of finite \( E \)-valued functions \( x = x(t) \) on the semiaxis \( [0, +\infty) \) with infinite quantity of continuous (two-sided) derivatives \( x^{(n)}(t) \) on the interval \( (0, +\infty) \) and derivatives on the right in zero. Similarly to the scalar case, the topology of inductive limit
will be determined on $\mathcal{D}_+(E)$,

$$
\mathcal{D}_+(E) = \lim_{\nu \to +\infty} \mathcal{D}_+^\nu(E), \quad \text{where} \quad \mathcal{D}_+^\nu(E) = \{ x \in \mathcal{D}_+(E) : \text{supp} \ x \subset [0, \nu] \}
$$

are Fréchet spaces with norms $\| x \|_n = \sup_{t \in [0, n]} \| x^{(n)}(t) \|$.

**Lemma 2.** (a) The following topological isomorphism

$$
\mathcal{D}_+(E) \simeq E \otimes \mathcal{D}_+
$$

is valid, and in the right side there is a completion of projective tensor product.

(b) For an arbitrary function $x(t) \in \mathcal{D}_+(E)$ there exists a number $\nu > 0$ such that $x(t) \in \mathcal{D}_+^\nu(E)$, and $x(t)$ can be written as absolutely convergent series in the space $\mathcal{D}_+^\nu(E)$ in the following way

$$
x(t) = \sum_{n=1}^{\infty} \lambda_n x_n \otimes \varphi_n(t),
$$

where $\sum_n |\lambda_n| < \infty$, the sequences of functions $\{\varphi_n(t)\}$ in $\mathcal{D}_+^\nu$ and elements $\{x_n\}$ in $E$ converge to zero.

(c) Linear mapping

$$
\mathcal{D}_+ \ni f \to \langle f, \ x(t) \rangle \in E, \quad \text{where} \quad \langle f, \ x(t) \rangle = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n(t) \rangle x_n,
$$

is nuclear.

**Proof.** Proposition (a) is based on the nuclear property of space $\mathcal{D}_+$, which follows from nuclearity of $\mathcal{D}$ ([6], III, 8) and from the theorem on “heredity” ([6], III, 7.4) of this property under quotient-maps. Let $E \tilde{\otimes} \mathcal{D}_+$ be the completion of tensor product in the so called topology of uniform convergence on equicontinuous subsets of the dual spaces $E'$ and $\mathcal{D}_+'$ ([6], III, 6). It is known [7] that an isomorphism $\mathcal{D}_+(E) \simeq E \tilde{\otimes} \mathcal{D}_+$ holds. An isomorphism $E \otimes \mathcal{D}_+ \simeq E \tilde{\otimes} \mathcal{D}_+$ is realized due to nuclearity of $\mathcal{D}_+$ ([6], IV, 9.4), i.e., $\mathcal{D}_+(E) \simeq E \otimes \mathcal{D}_+$.

It is well known (see [7]) that $E \tilde{\otimes} \lim_{\nu \to +\infty} \mathcal{D}_+^\nu \simeq \lim_{\nu \to +\infty} E \tilde{\otimes} \mathcal{D}_+^\nu$. Hence, $\mathcal{D}_+(E) \simeq \lim_{\nu \to +\infty} E \otimes \mathcal{D}_+^\nu$. For each $x \in \mathcal{D}_+(E)$ there exists $\nu > 0$ such that $x \in E \otimes \mathcal{D}_+^\nu$. The element $x$ can be developed into a series of the form (4) via the theorem ([6], III, 6.4) about representation of the elements of the completion of projective tensor product of Fréchet spaces, applied to $E \otimes \mathcal{D}_+^\nu$. Proposition (c) is a direct corollary of proposition (b). ●

2. Now we define an operation $\ast$ from formula (1) which is a generalization of the cross-correlation. For an arbitrary $f \equiv \langle f, \cdot \rangle \in \mathcal{D}_+'$ and $x(t) \in \mathcal{D}_+$ the cross-correlation may be determined by the formula

$$
(f \ast x)(t) = \begin{cases} 
(f, \ x(\cdot + t)), & t \geq 0 \\
0, & t < 0
\end{cases},
$$

where $\langle f, \ x(\cdot + t) \rangle$ is defined by relation (5). In the case, if distribution $f$ is regular, $\langle f, \ x(\cdot + t) \rangle = \int_0^\infty f(s) \ x(s + t) \ ds$. 


Lemma 3. For an arbitrary $f \in \mathcal{D}'_+$ and $x(t) \in \mathcal{D}_+(E)$ we have $(f \ast x)(t) \in \mathcal{D}_+(E)$, moreover, \( \text{supp}(f \ast x) \subset [0, \mu] \), where \( \mu = \max\{t \geq 0 : x(t) \neq 0\} \). In addition,

\[ (f \ast x)^{(n)} = f \ast x^{(n)}. \]

Proof. For nonzero $f$ and $x$, the function $(f \ast x)(t)$ is different from identically zero-valued one for $t \geq 0$ such that there exists $s \in (\text{supp } f) \cap (\text{supp } x \cdot + t)$. However,

\[ s \in \text{supp } x(\cdot + t) \iff s + t \in \text{supp } x \iff t \in \text{supp } x - s. \]

Since $s$ runs over $\text{supp } f$, we have $t \in \text{supp } x - \text{supp } f$. Hence and from definition (6) one can obtain $\text{supp}(f \ast x) \subset (\text{supp } x - \text{supp } f) \cap [0, +\infty)$.

Using continuity of functional $f$ we can write

\[ (f \ast x)'(t) = \lim_{s \to 0} s^{-1} \langle f, x(\cdot + t + s) - x(\cdot + t) \rangle = \langle f, x'(\cdot + t) \rangle = (f \ast x')(t). \]

Counterparts can be obtained for the derivatives of higher orders. \( \bullet \)

Let $L(\mathcal{D}_+)$ and $L(\mathcal{D}_+(E))$ be algebras of the linear continuous operators on the spaces $\mathcal{D}_+$ and $\mathcal{D}_+(E)$ respectively. The algebra $L(\mathcal{D}_+)$ contains an operator of left translation

\[ T_s : \mathcal{D}_+ \ni \varphi(t) \to \theta(t)\varphi(t + s), \quad s \in [0, +\infty), \]

and the scalar operation of cross-correlation,

\[ K_f : \mathcal{D}_+ \ni \varphi(t) \to (f \ast \varphi)(t) \]

determined by some distribution $f \in \mathcal{D}'_+$.

Evidently, operators $I \otimes T_s$ and $I \otimes K_f$, where $I$ is the unity operator on $E$, belong to the algebra $L(\mathcal{D}_+(E))$. Using formulae (4) and (5) one can obtain

\[ I \otimes T_s : \mathcal{D}_+(E) \ni x(t) \to \theta(t)x(t + s), \quad I \otimes K_f : \mathcal{D}_+(E) \ni x(t) \to (f \ast x)(t). \]

Lemma 4. For an arbitrary $f \in \mathcal{D}'_+$ the operator $I \otimes K$, where $K = K_f$, is nuclear and satisfies the following condition

\[ I \otimes KT_s = I \otimes T_sK. \]  \( \tag{7} \)

Vice versa, for an arbitrary operator $K \in L(\mathcal{D}_+)$ which satisfies condition (7) there exists a unique distribution $f \in \mathcal{D}'_+$ such that $K = K_f$ and $(I \otimes K)x = f \ast x$ for all $x \in \mathcal{D}_+(E)$.

Proof. The following equalities $(I \otimes K)x(s) = (f \ast x)(s) = \langle f, x(\cdot + s) \rangle = \langle f, T_s x \rangle$ are valid. Using series (4) from Lemma 2 and continuity $T_s$ on $\mathcal{D}_+$, we can get the series development $(I \otimes K)x(s) = \sum_n \lambda_n \langle f, T_s \varphi_n \rangle x_n$ in which the sequence \( \{T_s \varphi_n\}_n \) converges to zero in $\mathcal{D}_+$. Applying the criterium of nuclearity from (6), III, 7.1 ensures nuclearity of $K_f$. Further,

\[ (I \otimes KT_s)x(t) = (f \ast T_s x)(t) = \langle f, T_s x(\cdot + t) \rangle = \langle f, \theta(\cdot + s)x(\cdot + t + s) \rangle = (f \ast x)(t + s) = (I \otimes T_s)(f \ast x)(t) = (I \otimes T_sK)x(t) \]
and condition (7) holds true.

Conversely. The linear continuous functional \( f : \varphi \to (K\varphi)(0) \) determines a distribution \( f \in \mathcal{D}'_+ \) for an arbitrary \( \varphi \in \mathcal{D}_+ \). We obtain \( (K\varphi)(0) = \langle f, \varphi \rangle = (f * \varphi)(0) \) via the evident formula \( \langle f, \varphi \rangle = (f \ast \varphi)(0) \). Putting \( T_s \varphi(t) \) instead of \( \varphi(t) \) and using condition (7), one can obtain the relation \( (K\varphi)(s) = (f \ast \varphi)(s) \), and thus \( (I \otimes K)x = f \ast x \).

3. Consider the Fourier transformation on the space \( \mathcal{D}_+ \)

\[
F : \mathcal{D}_+ \ni \varphi \rightarrow \widehat{\varphi}, \quad \text{where} \quad \widehat{\varphi}(\zeta) = \int_0^\infty e^{-it\zeta}\varphi(t) dt, \quad \zeta \in \mathbb{R}.
\]

Introduce the topology of inductive limit in the Fourier transformation image \( \widehat{\mathcal{D}}_+ \equiv \{ \widehat{\varphi} : \varphi \in \mathcal{D}_+ \} \)

\[\widehat{\mathcal{D}}_+ = \lim_{\nu \rightarrow +\infty} \widehat{\mathcal{D}}^\nu_+ ,\]

where the spaces \( \widehat{\mathcal{D}}^\nu_+ \equiv \{ \widehat{\varphi} : \varphi \in \mathcal{D}_+^\nu \} \) are topologized by norms \( \| \widehat{\varphi} \|_n = \| \varphi \|_n \).

Let the space \( \widehat{\mathcal{D}}^\nu_+ \) of the linear continuous functionals be dual to the space \( \widehat{\mathcal{D}}_+ = \lim_{\nu \rightarrow +\infty} \widehat{\mathcal{D}}^\nu_+ \). Let us denote by

\[
\mathcal{F} = 2\pi(F^{-1})' : \mathcal{D}_+ \ni \varphi \rightarrow \widehat{\varphi} \in \widehat{\mathcal{D}}_+ ,
\]

the mapping adjoint to the inverse Fourier transformation \( F^{-1} \) relatively to \( \langle \mathcal{D}_+^\nu, \mathcal{D}_+^\nu \rangle \), where \( \widehat{\varphi} = \mathcal{F}(\varphi) \). Further we shall call it the Fourier transformation of distributions of the space \( \mathcal{D}_+^\nu \). The restriction of \( \mathcal{F} \) onto \( \mathcal{D}_+ \), evidently, coincides with \( F \).

The following bilinear form

\[
\langle \hat{f}, \hat{\varphi} \rangle \equiv \langle \mathcal{F}f, \mathcal{F}\varphi \rangle = 2\pi \langle f, \varphi \rangle ,
\]

where \( f \in \mathcal{D}_+^\nu \) and \( \varphi \in \mathcal{D}_+ \), defines a new duality \( \langle \widehat{\mathcal{D}}_+^\nu, \mathcal{D}_+ \rangle \).

Now consider the algebra \( L(\widehat{\mathcal{D}}_+) \) of linear continuous operators on the space \( \widehat{\mathcal{D}}_+ \). Denote by \( \circ \) the composition of operators in \( L(\widehat{\mathcal{D}}_+) \). Further we assume that topology of the uniform convergence on compacta is determined on algebra \( L(\widehat{\mathcal{D}}_+) \).

Note that the operator \( \mathcal{T}^*_\delta = F \cdot T_s \cdot F^{-1} \), where \( T_s \) is the left translation on the space \( \mathcal{D}_+ \), belongs to the algebra \( L(\widehat{\mathcal{D}}_+) \). The scalar operation of the cross-correlation \( K_f = f* \) with a distribution \( f \in \mathcal{D}_+^\nu \) defines uniquely the operator \( F \cdot K_f \cdot F^{-1} \), which also belongs to algebra \( L(\widehat{\mathcal{D}}_+) \).

Define the strong topology relative to the duality \( \langle \mathcal{D}_+^\nu, \mathcal{D}_+ \rangle \) on the convolution algebra \( \mathcal{D}_+^\nu \).

The following theorem is a fundamental theorem of the scalar part of the functional calculus.

**Theorem 1.** Transformation

\[
\Lambda : \mathcal{D}_+^\nu \ni \varphi \rightarrow f_\Lambda \in L(\widehat{\mathcal{D}}_+) ,
\]

where

\[
(f_\Lambda \hat{\varphi})(\zeta) = \int_0^\infty e^{-it\zeta}(f \ast \varphi)(t) dt \quad \text{and} \quad F : \mathcal{D}_+ \ni \varphi \rightarrow \hat{\varphi} \in \widehat{\mathcal{D}}_+ ,
\]
realizes a topological isomorphism of the convolution algebra $\mathcal{D}'_+$ on the subalgebra of algebra $L(\mathcal{D}_+)$ of those operators which commute with the operator $\hat{T}_s$. Besides,
\[ \Lambda(f \ast h) = f_\Lambda \circ h_\Lambda \quad \text{for all } f, h \in \mathcal{D}'_+ \]  
and the operator $\circ$ in the subalgebra $\Lambda(\mathcal{D}'_+)$ is continuous.

Proof. Since $f \ast \phi = (f, T_s \phi)$ and $T_s \in L(\mathcal{D}_+)$, from the properties of integral (9) it follows that the bilinear mapping $\mathcal{D}'_+ \times \mathcal{D}_+ \ni (f, \phi) \mapsto f_\Lambda \hat{\phi} \in \mathcal{D}_+$ is separately continuous. The Fourier transformation $F : \mathcal{D}_+ \rightarrow \mathcal{D}_+$ realizes the topological isomorphism, and hence the mapping $\mathcal{D}'_+ \times \mathcal{D}_+ \ni (f, \hat{\phi}) \mapsto f_\Lambda \hat{\phi} \in \mathcal{D}_+$ is also separately continuous. One can obtain from the Banach-Steinhaus theorem, which can be applied to $f_\Lambda \hat{\phi}$ due to barrelledness of the spaces $\mathcal{D}'_+$ and $\mathcal{D}_+$, that the bilinear mapping $\mathcal{D}'_+ \times \mathcal{D}_+ \ni (f, \hat{\phi}) \mapsto f_\Lambda \hat{\phi} \in \mathcal{D}_+$ is hypocontinuous. Otherwise, $\Lambda$ is a continuous mapping from $\mathcal{D}'_+$ to the algebra $L(\mathcal{D}_+)$ with the topology of the uniform convergence on compacta.

It follows from Lemma 4 that the image $\Lambda(\mathcal{D}'_+)$ is a maximal commutative subalgebra of the element $\hat{T}_s$. The maximality property ensures "closedness" of $\Lambda(\mathcal{D}'_+)$ in the algebra $L(\mathcal{D}_+)$. Since $\mathcal{D}_+$ is a Montel space, the topologies of uniform convergence on compacta and on boundary sets coincide. Finally, $\mathcal{D}_+$ is bornological, and hence the space $L(\mathcal{D}_+)$ coincides with the space of all boundary linear operators. And the latter is complete in the topology of uniform convergence on bounded sets (9, 1.10.9).

By Lemma 3 $\text{supp}(f \ast \phi) \subset [0, \nu]$, where $\mu = \max\{t \geq 0 : x(t) \neq 0\}$, i.e., for arbitrary $\nu > 0$, subspaces $\mathcal{D}'_+$ are invariant with respect to operators of subalgebra $\Lambda(\mathcal{D}'_+)$. Hence $\Lambda(\mathcal{D}'_+)$ belongs to the projective limit of the form $\text{lim}_{\nu \rightarrow +\infty} L(\mathcal{D}'_+)$, where topologies of uniform convergence on compacta are also determined in the algebras $L(\mathcal{D}'_+)$ of Fréchet algebras and $\text{lim}_{\nu \rightarrow +\infty} L(\mathcal{D}'_+)$ is embedded isomorphically into $L(\mathcal{D}_+)$. Thus, it is possible to apply the open mapping theorem in the form [5] for $\Lambda$, i.e., $\Lambda$ is a topological linear isomorphism. To show that $\Lambda$ is a homomorphism of algebras, one can use the following relations
\[ ((f \ast h)_\Lambda(\hat{\phi})(\xi) = \int_0^{\infty} e^{-it\xi} \langle (f \ast h) \hat{\phi}, t \rangle dt = \int_0^{\infty} e^{-it\xi} \langle f(h)(\cdot), \varphi(\cdot + t) \rangle dt \]  
\[ = \int_0^{\infty} e^{-it\xi} \langle f(\cdot), \langle h(\cdot), \varphi(\cdot + t) \rangle \rangle dt = \int_0^{\infty} e^{-it\xi} \langle f(\cdot), \langle h(\varphi)(\cdot + t) \rangle \rangle dt \]  
\[ = \int_0^{\infty} e^{-it\xi} \langle (f \ast (h \ast \varphi))(t) \rangle dt = (f_\Lambda(h \ast \varphi))(\xi) = ((f_\Lambda \circ h_\Lambda) \hat{\phi})(\xi). \]

The continuity of the multiplication follows from the Arens Theorem [10], because $\Lambda(\mathcal{D}'_+)$ is a Fréchet algebra.

Theorem 1 states a topological isomorphism of the algebras $\mathcal{D}'_+ \xrightarrow{\Lambda} \Lambda(\mathcal{D}'_+)$. On the other hand, the Fourier transformation realizes topological isomorphism of the spaces $\mathcal{D}'_+ \xrightarrow{F} \mathcal{D}_+$. Thus, the following topological isomorphism of the spaces is valid
\[ \mathcal{D}_+ \xrightarrow{\Lambda \cdot F^{-1}} \Lambda(\mathcal{D}'_+). \]

This allows us to formulate the following proposition.
Theorem 2. The space \( \widehat{\mathcal{D}}' \) is a Fréchet algebra in the strong topology with respect to multiplication (1), which is topologically isomorphic to convolution algebra \( \mathcal{D}' \) and to the algebra of operators \( \Lambda(\mathcal{D}'_+) \).

Remark. The Fourier transformation possesses the property
\[
\widehat{f \ast h} = \hat{f} \cdot \hat{h} \quad \text{for all } f, h \in \mathcal{D}_+,
\]
where in the right side of the equation there is the usual multiplication of the continuous functions. Thus the subalgebra \( \Lambda(\mathcal{D}_+) = \{ f_\Lambda : f \in \mathcal{D}_+ \} \) in \( \Lambda(\mathcal{D}'_+) \) is topologically isomorphic to the function algebra \( \widehat{\mathcal{D}}_+ \). The subspace \( \mathcal{D}_+ \) is dense in the space \( \mathcal{D}'_+ \) relatively to weak topology, due to Lemma 1. Since \( \mathcal{D}'_+ \) is a Montel space, \( \overline{\mathcal{D}}_+ = \mathcal{D}'_+ \) in the strong topology. Therefore \( \overline{\mathcal{D}}_+ = \mathcal{D}'_+ \). This means that the space \( \overline{\mathcal{D}}'_+ \) with multiplication (1) is an algebra which has a dense subalgebra of functions \( \mathcal{D}_+ \). In particular, the product of arbitrary distributions \( \widehat{f}, \widehat{h} \) in \( \overline{\mathcal{D}}'_+ \) can be defined uniquely by passing to the limit
\[
\widehat{f} \cdot \widehat{h} = \lim_{n \to \infty} (\widehat{f}_n \cdot \widehat{h}_n), \text{ where } \widehat{f} = \lim_{n \to \infty} \widehat{f}_n, \quad \widehat{h} = \lim_{n \to \infty} \widehat{h}_n \text{ and } \widehat{f}_n, \widehat{h}_n \in \overline{\mathcal{D}}_+.
\]

Further, to calculate the Fourier transformation of some concrete examples we shall use relation (8) and Theorem 2.

Example 1. The equality \( \widehat{\delta} = 1 \) is valid, where 1 is the unit function in \( \widehat{\mathcal{D}}'_+ \). One can use the inverse Fourier transformation to obtain
\[
\langle \delta, \varphi \rangle = 2\pi \langle \delta, \varphi \rangle = 2\pi \varphi(0) = \int_{-\infty}^{+\infty} e^{it\xi} \varphi(\xi) d\xi = \langle 1, \varphi \rangle.
\]
Since \( \delta \)-function is the unit in the convolution algebra \( \mathcal{D}'_+ \), \( \widehat{f} \cdot 1 = 1 \cdot \widehat{f} = \widehat{f} \) for all \( \widehat{f} \in \overline{\mathcal{D}}'_+ \).

Example 2. Degrees of \( \delta \)-function in algebra \( \widehat{\mathcal{D}}'_+ \) are calculated by formula
\[
\delta^n = \frac{1}{(2\pi)^n} \theta^\ast \ldots \ast \theta, \quad \text{in particular, } \delta = \frac{1}{2\pi} \hat{\theta}.
\]
This follows from formula (1) and equalities
\[
\langle \delta, \varphi \rangle = \varphi(0) = \int_0^\infty e^{-it\xi} \varphi(t) dt = \langle \theta, \varphi \rangle = \frac{1}{2\pi} \langle \hat{\theta}, \varphi \rangle.
\]

Example 3. The following formula is valid for generalized derivatives of the Dirac function
\[
\delta^{(n)} = \frac{1}{2\pi} (it)^n \theta(t).
\]
Indeed,
\[
\langle \delta^{(n)}, \varphi \rangle = (-1)^n \langle \delta, \varphi^{(n)} \rangle = \langle \delta, (it)^n \varphi(t) \rangle = \frac{1}{2\pi} \langle \hat{\theta}, (it)^n \varphi(t) \rangle = \langle \theta, (it)^n \varphi(t) \rangle = \frac{1}{2\pi} \langle (it)^n \theta(t), \varphi \rangle.
\]
4. Denote by $\mathcal{H}_-$ the class of functions $\hat{f}(\xi)$ which are analytic in lower complex half-plane $\{\xi = \zeta + i\eta: \eta < 0, \zeta \in \mathbb{R}\}$ and satisfy the conditions:
(a) for each fixed $\eta < 0$ the function $\zeta \to \hat{f}(\zeta + i\eta)$ belongs to the space $\widehat{\mathcal{D}}_+$;
(b) the family $\{\hat{f}(\cdot + i\eta): \eta < 0\}$ is bounded in the space $\widehat{\mathcal{D}}_+$ endowed with the strong topology relatively to the duality $\langle \widehat{\mathcal{D}}_+, \mathcal{D}_+ \rangle$.

**Theorem 3.** Every distribution $\hat{f} \in \widehat{\mathcal{D}}_+$ admits a unique extension to a function $\hat{f}(\cdot + i\eta)$ of the class $\mathcal{H}_-$ and this extension is determined by relation

$$\langle \hat{f}(\cdot + i\eta), \varphi \rangle = \langle \hat{f}, \varphi_\eta \rangle, \quad \text{where } \varphi_\eta(t) = \varphi(t)e^{i\eta}, \quad \varphi \in \mathcal{D}_+. $$

(11)

Vice versa, for each function $\hat{f}(\cdot + i\eta) \in \mathcal{H}_-$ there exists a distribution $\hat{f} \in \widehat{\mathcal{D}}_+$ such that

$$\lim_{\eta \to 0} \hat{f}(\cdot + i\eta) = \hat{f}$$

(12)
in the strong topology of $\widehat{\mathcal{D}}_+$.

**Proof.** Formula (11) follows from relation (8) and a simple fact that $\varphi_\eta \in \mathcal{D}_+$ if $\varphi \in \mathcal{D}_+$. The family $\{\varphi_\eta: \eta < 0\}$ is bounded in $\mathcal{D}_+$ for each function $\varphi \in \mathcal{D}_+$. In virtue of (11), for each distribution $\hat{f} \in \widehat{\mathcal{D}}_+$ the family $\{\hat{f}(\cdot + i\eta): \eta < 0\}$ is bounded in the weak topology and therefore in the strong topology $\widehat{\mathcal{D}}_+$, i.e., $\hat{f}(\xi) \in \mathcal{H}_-$.

Converse proposition is a corollary of the Montel property of the space $\widehat{\mathcal{D}}_+$ in the strong topology. This means that the bounded set $\{\hat{f}(\cdot + i\eta): \eta < 0\}$ is compact if $\hat{f}(\cdot + i\eta) \in \mathcal{H}_-$ and limit (12) exists. ♦

5. Now consider the main theorem on operator calculus in the algebra $\widehat{\mathcal{D}}_+$. Further, the operator $-iA$ will be a generator of the semigroup of the linear boundary operators $\{e^{-itA}: t \geq 0\}$ that act in the Banach space $E$. Define a subspace in $E$,

$$\widehat{\mathcal{D}}_+(E) \equiv \{\hat{x}: x = x(t) \in \mathcal{D}_+(E)\},$$

where the elements $\hat{x}$ are determined by formulae (2).

**Lemma 5.** If $\{e^{-itA}: t \geq 0\}$ is a semigroup of class $C_0$, i.e., $e^{-itA}|_{t=0} = I$ and the function $0 \leq t \to e^{-itA}y$ is continuous for each $y \in E$, then the subspace $\widehat{\mathcal{D}}_+(E)$ is dense in $E$ and the map

$$\mathcal{F}_A: \mathcal{D}_+(E) \ni x \to \hat{x} \in \widehat{\mathcal{D}}_+(E)$$

(13)
is bijective.

**Proof.** Prove density of $\widehat{\mathcal{D}}_+(E)$ in $E$. Let $x' \in E'$, where $E'$ is the dual space to $E$ of the linear continuous functionals, then the following relation is valid

$$\langle x', \hat{x} \rangle = \int_0^\infty \langle x', e^{-itA}x(t) \rangle \, dt.$$

(14)
For the following elements $x(t) = y \otimes \varphi \in \mathcal{D}_\nu^+(E)$, where $y \in E$ and $\varphi(t) \in \mathcal{D}_+^\nu$, the integrand in (14) is equal to $\langle x', e^{-itA} y \rangle \varphi(t)$. Suppose that for some $x' \in E'$, $\langle x', \widehat{x} \rangle = 0$ for all $x \in \mathcal{D}_\nu(E)$, in particular, for all $y \in E$ and $\varphi \in \mathcal{D}_+$. From (14) it follows that $\langle x', e^{-itA} y \rangle = 0$ for all $y \in E$. Since $e^{-itA}$ is a semigroup of class $(C_0)$, we obtain $x' = 0$. The subspace $\mathcal{D}_\nu(E)$ is dense in $E$ according to Hahn-Banach theorem.

Prove injectivity of $\mathcal{F}_A$. Let $x(t) \in \mathcal{D}_\nu(E)$ be a nonzero function such that $\langle x', \widehat{x} \rangle = 0$ for all $x' \in E'$. Suppose that $x(t) = y \otimes \varphi(t)$, where $y \in E$, $\varphi(t) \in \mathcal{D}_+$. As $e^{-itA}$ is a semigroup of class $(C_0)$, $\langle x', e^{-itA} y \rangle$ is not identically zero-valued for each $x' \in E'$. According to the Hill-Yosida theorem about the representation of the semigroups of this class ([11], IX, 7), the limit in the following formula

$$\langle x', e^{-itA} y \rangle = \lim_{n \to \infty} \sum_{m=0}^{\infty} \frac{(-it)^m}{m!} \langle x', A^m \left( I + \frac{i}{n} A \right)^{-m} y \rangle$$

uniformly converges on each segment $[0, \nu]$ for all $x' \in E'$. According to the Weierstrass theorem, polynomials uniformly approximate an arbitrary continuous function on segment $[0, \nu]$. Thus, there is no nonzero function $\varphi(t) \in \mathcal{D}_\nu^+$ such that

$$\int_0^\infty \langle x', e^{-itA} y \rangle \varphi(t) \, dt = 0.$$

Hence, $y \otimes \varphi(t) = 0$ for each number $\nu > 0$.

Taking instead of $y$ and $\varphi(t)$ elements of Hamel basises in the spaces $E$ and $\mathcal{D}_\nu^+$ respectively, we conclude that no nonzero function $x(t)$ such that $\langle x', \widehat{x} \rangle = 0$ for all $x' \in E'$ exists in the tensor product $E \otimes \mathcal{D}_\nu^+$. This means that the bilinear form $E' \times (E \otimes \mathcal{D}_\nu^+) \ni (x', x) \mapsto \langle x', \widehat{x} \rangle$ leads spaces $E'$ and $E \otimes \mathcal{D}_\nu^+$ into duality. The projective topology of the tensor product is determined on $E \otimes \mathcal{D}_\nu^+$. It follows from (14) that for arbitrary $x(t) = \sum_n x_n \otimes \varphi_n(t) \in E \otimes \mathcal{D}_\nu^+$ the following inequality holds

$$| \langle x', \widehat{x} \rangle | \leq \alpha e^{\beta \nu} \| x' \| \| x(t) \|_\infty,$$

where numbers $\alpha > 0$ and $\beta \in \mathbb{R}$ are such that $\| e^{-itA} \| \leq \alpha e^{\beta \nu}$, $\| x' \|$ denotes the norm of functional and $\| x(t) \|_\infty = \inf \{ \sum_i \| x_i \|_\infty \| \varphi_i \| : x(t) = \sum_{i=1}^m x_i \otimes \varphi_i(t), m \in \mathbb{N} \}$. Hence, if the strong topology on the dual space $E'$ is determined, then the bilinear form $\langle x', \widehat{x} \rangle$ is continuous on $E' \times (E \otimes \mathcal{D}_\nu^+)$. Continuous injection $E \otimes \mathcal{D}_\nu^+ \subset (E \otimes \mathcal{D}_\nu^+)^\nu$ exists, where $(E \otimes \mathcal{D}_\nu^+)^\nu$ is the second strong dual space relatively to the duality which is generated by the bilinear form $\langle x', \widehat{x} \rangle$. And the strong topology is determined on the space $(E \otimes \mathcal{D}_\nu^+)^\nu$. The space $(E \otimes \mathcal{D}_\nu^+)^\nu$ is complete and for the completion $\tilde{E} \otimes \mathcal{D}_\nu^+$ the continuous injection $\tilde{E} \otimes \mathcal{D}_\nu^+ \subset (E \otimes \mathcal{D}_\nu^+)^\nu$ is preserved. Since the space $E'$ separates points of $(E \otimes \mathcal{D}_\nu^+)^\nu$, it separates points of $\tilde{E} \otimes \mathcal{D}_\nu^+$, i.e., there is no nonzero function $x(t) \in \tilde{E} \otimes \mathcal{D}_\nu^+$ such that $\langle x', \widehat{x} \rangle = 0$ for all $x' \in E'$.

To complete the proof of injectivity of $F_A$ we shall use the isomorphism $\mathcal{D}_\nu^+(E) \simeq \lim \mathcal{D}_\nu^+(E)$ described above in the proof of Lemma 2.

In virtue of Lemma 5, in the space $\tilde{\mathcal{D}_\nu^+(E)}$ one can introduce a Hausdorff topology of the inductive limit

$$\tilde{\mathcal{D}_\nu^+(E)} = \lim \mathcal{D}_\nu^+(E), \quad \mathcal{D}_\nu^+(E) = \{ \widehat{x} : x = x(t) \in \mathcal{D}_\nu^+(E) \}.$$
and on the spaces $\hat{D}_+^*(E)$ the topology is determined by norms

$$\|\hat{x}\|_n = \|x\|_n = \sup_{t \in [0,\nu]} \|x^{(n)}(t)\|.$$ 

It is easy to see that $\mathcal{F}_A$ is a topological isomorphism.

Let us consider an algebra $L(\hat{D}_+(E))$ of the linear continuous operators over the space $\hat{D}_+(E)$. We define on the space $L(\hat{D}_+(E))$ the topology of uniformly convergence on the bounded sets.

**Theorem 4.** In conditions of Lemma 5, the mapping

$$\mathcal{G} : \hat{D}_+^* \ni \hat{f} \to \hat{f}(A) \in L(\hat{D}_+(E))$$

(15)

which is determined by formulae (2) realizes a topological isomorphism of the distributions algebra $\hat{D}_+^* = \{\hat{f} : f \in D_+\}$ onto a closed subalgebra of the algebra $L(\hat{D}_+(E))$ of those operators which commute with the operator $\mathcal{F}_A \cdot (I \otimes T_s) \cdot \mathcal{F}_A^{-1}$. Here, $T_s$ is the operator of left translation over the functions of the space $D_+$.

Proof is analogous to that of Theorem 1. We shall only emphasize some new arguments.

The Banach–Steinhaus theorem may be applied to the bilinear mapping $\hat{D}_+^* \times \hat{D}_+(E) \ni (\hat{f}, \hat{x}) \to \hat{f}(A) \hat{x} \in \hat{D}_+(E)$, because the space $\hat{D}_+(E)$ is barrelled as the inductive limit of Fréchet spaces.

Since the topology of uniform convergence on the bounded sets defined in the algebra $L(\hat{D}_+(E))$ is more strong than topology of uniform convergence on the compact sets, we don’t use the Montel property. Since the space $\hat{D}_+(E)$ is bornological, this is enough for conclusion that the algebra $L(\hat{D}_+(E))$ coincides with the algebra of the bounded operators.

**Example 4.** For an arbitrary operator $A$ from Theorem 4 and natural number $n$ the following relation

$$\hat{f}(A) \hat{x}^{(n)} = (iA)^n \hat{f}(A) \hat{x} - \sum_{k=0}^{n-1} (iA)^{n-k-1} \langle f, x^{(k)} \rangle$$

holds.

**Example 5.** Since $2\pi \delta = \hat{\theta}$, applying to the relation $\theta \ast \delta' = \delta' \ast \theta = \delta$ the Fourier transformation, we obtain

$$\delta \ast \hat{\delta}' = \hat{\delta}' \ast \delta = \frac{1}{2\pi},$$

i.e., the Dirac function $\delta$ has an inverse element $2\pi \hat{\delta}'$ in the algebra $\hat{D}_+^*$. Using Theorem 4 one can conclude that for an arbitrary $\hat{y} \in \hat{D}_+(E)$ the equation

$$2\pi \cdot \hat{\delta}'(A) \hat{x} = \hat{y}$$

has a unique solution

$$\hat{x} = \delta(A) \hat{y} = \frac{1}{2\pi} \int_0^\infty y(t + s) \, ds.$$
6. Let us consider the subspace $D_0(E) \equiv \{ x(t) \in D_+(E) : \text{supp } x(t) \subset (0, +\infty) \}$ and the subalgebra $D'_0 \subset D_+$ of distributions whose support is located in zero. We denote by $D_0(E) \equiv \{ \hat{x} : x \in D_0(E) \}$ and $D'_0 \equiv \{ \hat{f} : f \in D'_0 \}$ their images under the transformations $F_A$ and $\hat{\cdot}$, respectively. Clearly, the continuous projection from the space $D_+(E)$ on the subspace $D_0(E)$ exists and $D'_0$ is a subalgebra of $D'_+(E)$.

Let’s define the topology of uniform convergence at the bounded sets in the algebra $L(D_0(E))$ of the linear continuous operators over the subspace $D_0(E)$ of $D_+(E)$.

**Theorem 5.** Let $0 \leq t \to e^{-itA}$ be a semigroup of class $(C_0)$. Then

(a) the subspace $D_0(E)$ is invariant with respect to the action of operators $\{ \hat{f}(A) : f \in D'_0 \}$ and is dense in $E$,

(b) the mapping

$$G : D'_0 \ni \hat{f} \to \hat{f}(A) \in L(D_+(E))$$

realizes a topological isomorphism of the algebra $D'_0$ onto the maximal commutative subalgebra of operator $F_A \cdot (I \otimes I_s) \cdot F_A^{-1}$ in $L(D_+(E))$ such that the subspace $D_0(E)$ is invariant,

(c) the mapping

$$G_0 : D'_0 \ni \hat{f} \to \hat{f}_0(A) \in L(D_0(E)),$$

where $\hat{f}_0(A) \equiv \hat{f}(A)|_{D_0(E)}$ is the restriction of the operator $\hat{f}(A)$ onto the subspace $D_0(E)$, is a topological injective isomorphism of algebras,

(d) the isomorphism $G_0$ satisfies the relation

$$\delta^{(n)}(A) \hat{x} = (-iA)^n \hat{x}, \quad f^{(n)}(A) \hat{x} = (-iA)^n \hat{f}(A) \hat{x} = \hat{f}(A) (-iA)^n \hat{x}, \quad (16)$$

for all $f \in D'_0$ and $\hat{x} \in D_0(E)$.

**Proof.** Invariance of the subspace $D_0(E) = \{ \hat{f} : f \in D'_0 \}$ relative to the action of the operators $\{ \hat{f}(A) : f \in D'_0 \}$ follows from injection $\text{supp}(f \ast x) \subset (\text{supp } x - \text{supp } f) \cap [0, +\infty)$, which was stated in the proof of Lemma 3. Indeed, in our case $\text{supp } f = \{0\}$, that is why $\text{supp}(f \ast x) \subset \text{supp } x$, and hence $\hat{f}(A) \hat{x} = F_A(f \ast x) \in D_0(E)$.

Prove density $D_0(E) = E$. It is known [8] that for any $n \in \mathbb{N}$ there exists a smooth function $\varphi_n = \varphi_n(t)$ with the properties: $\varphi_n \geq 0$; $\text{supp } \varphi_n \subset (0, n^{-1})$; $\int_0^\infty \varphi_n(t) \, dt = 1$. The following inequalities hold for an arbitrary $y \in E$

$$\|y_n - y\| \leq \int_0^\infty \|e^{-itA}y - y\| \varphi_n(t) \, dt \leq \sup_{t \in (0, n^{-1})} \|e^{-itA}y - y\|,$$

where $y_n = y \otimes \varphi_n$. Since $e^{-itA}$ is a $(C_0)$-semigroup, $\sup_{t \in (0, n^{-1})} \|e^{-itA}y - y\| \to 0$ for all $y \in E$.

Hence, $\lim_{n \to \infty} y_n = y$ for all $y \in E$.

It follows from Theorem 4 that the mapping $G$ realizes an isomorphism of the subalgebra $D'_0$ into the algebra $L(D_+(E)).$
Let us supplement Lemma 4 by the following way. The subspace $D_0(E)$ is invariant with respect to the action of the operator $I \otimes K_f$ for arbitrary $f \in D_0'$. Conversely, the unique distribution $f \in D_0'$ such that $K = K_f$ and $(I \otimes K)x = f \star x$ for all $x \in D_+(E)$ exists for an arbitrary operator $K \in L(D_+)$ which preserves invariance of the subspace $D_0$, and satisfies condition (7). The direct statement has been checked above. The linear continuous functional $f : D_+ \ni \varphi \to (K\varphi)(0)$ for which $(f, \varphi) = 0$ for all $\varphi \in D_0$ defines the distribution $f \in D_+^*$ with support in zero. Thus the inverse statement is valid.

From this fact it follows that $G$ realizes an isomorphism of the subalgebra $D_0'$ into the algebra $L(D_+(E))$ and its image is the maximal commutative subalgebra in $L(D_+(E))$ of the operator $F_A : (I \otimes T_s) \cdot F_A^{-1}$ and the subspace $D_0(E)$ is invariant with respect to action of the operators of the algebra $L(D_+(E))$. Thus, (b) is proved.

The mapping $D_0 \ni f \to f \star x$ has zero kernel for every non-zero $x \in D_0(E)$, and hence $G_0$ realizes an isomorphism of $D_0'$ on its image $G_0(D_0') \subset L(D_0(E))$.

To prove (d) we shall use the relation

$$\widehat{\delta(n)}(A) \widehat{x} = \int_0^\infty e^{-itA} \delta(n) \star x(t) dt = \int_0^\infty e^{-itA} x^{(n)}(t) dt.$$ 

Further, integrating by partials at the condition $x^{(n)}(0) = 0$, we obtain the first formula of (16). To complete the proof let us use the following relations

$$\widehat{f^{(n)}}(A) \widehat{x} = \int_0^\infty e^{-itA} \delta^{(n)}(f \star x)(t) dt = \delta^{(n)}(A) \widehat{f}(A) \widehat{x}.$$ 

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Institute of Applied Problems of Mechanics and Mathematics,
National Academy of Sciences, Naukova 3b, 290053 Lviv, Ukraine.

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