Математичні студії
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FIVE OPEN PROBLEMS IN THE THEORY OF ENTIRE FUNCTIONS

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Let l be a positive continuous function defined on $[0,+\infty)$. An entire function f is said to be a function of *bounded l-index* if there exists a number $\nu \in \mathbb{Z}_+$ such that

$$\frac{|f^{(j)}(z)|}{j!l^j(|z|)} \le \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(|z|)} \ : 0 \le k \le \nu\right\} \tag{1}$$

for every $z \in \mathbb{C}$ and $j \in \mathbb{Z}_+$ [1]. If $l(|z|) \equiv 1$, we obtain the definition of entire function of bounded index. Many papers are devoted to investigations of properties of entire functions of bounded index and their applications (see [2] for references). Counterparts of the main theorems from [2] for entire functions of bounded l-index are obtained in [3–5]. The fact that the function l is involved in the definition 1 leads naturally to some new problems, in particular, to the problem of existence, for a pregiven function l, of an entire function of bounded l-index which satisfies some growth conditions.

It is easy to see that every polynomial is a function of bounded l-index, for every function l. This is not the case for entire transcendental functions. Actually [6], for a function l there exists an entire transcendental function of bounded l-index if and only if $rl(r) \to \infty$ $(r \to \infty)$. In [7], entire functions with growth of logarithm of maximum modulus close to $\int_0^r l(t)dt$ are indicated for functions l satisfying some additional conditions. On the other hand, if A_k are zeros of an entire function f, p_k their multiplicity, with $\overline{\lim\limits_{k\to+\infty}}\, p_k = \pm\infty$, then f cannot be a function of bounded l-index, for every positive continuous on $[0,+\infty)$ function l.

The following natural problem arizes.

Problem 1. *Let $p_k = O(1)$, $k \to \infty$. Is there a positive continuous on $[0,+\infty)$ function l such that f is a function of bounded l-index?*

An entire function of exponential type $\sigma > 0$ is said to be of sine type [8] if there exist $m \ge 0$, $M > 0$, and $H > 0$ such that $m \le |f(x+iy)|\exp\{-\sigma|y|\} \le M$ for every $x \in \mathbb{R}$ and $y \in \mathbb{R}\backslash(-H,H)$. A necessary and sufficient conditions are given in [8] for a bounded sequence (ψ_k) in order that every entire function with zeros $a_k + \psi_k$ is of sine type, a_k being zeros of a sine type function. On the other hand [9], every function of sine type is a function of bounded l-index. Thus the following important problem arizes.

Problem 2. *Let f be an entire function of bounded index with zeros a_k. For what bounded sequences (ψ_k) the function with zeros $a_k+\psi_k$ is also of bounded index?*

The following problem is also connected with functions of bounded l-index.

Problem 3. *Let l be a positive continuous on $[0,+\infty)$ function such that $xl(x)\to +\infty$, $l\big(x+O\big(\frac{1}{l(x)}\big)\big)=O\big(l(x)\big)$, $x\to+\infty$ and f an entire function with zeros a_k. Put*

$$G_r(f)=\mathbb{C}\backslash\bigcup_k\left\{z:|z-a_k|\le\frac{r}{l(|a_k|)}\right\},\quad r>0,$$

$$n\Big(r,z_0,\frac{1}{f}\Big)=\sum_{|a_k-z_0|\le r}1$$

and assume that for every $r>0$ there exists $M(r)>0$ such that

$$\left|\frac{f'(z)}{f(z)}\right|\le M(r)l(|z|)$$

for every $z\in G_r(f)$ and

$$\overline{\lim_{r\to\infty}}\frac{n(r)}{rl(r)}<1.$$

Is there $k\in\mathbb{N}$ such that $n\big(\frac{1}{l(|z_0|)},z_0,\frac{1}{f}\big)\le k$ for every $z_0\in\mathbb{C}$?

Having in mind the representation of the sine as the canonical product, we may regard the entire function

$$\pi_0(z)=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^2}\right),$$

where (λ_n) is an increasing to $+\infty$ sequence of positive numbers, as its generalization. Since D. Polya's paper [10], the function π_0 is often used in investigations of properties of entire functions determined by gap power series, in the interpolation theory, and another aspects of analysis. The function π_0 has the same maximum modulus as the function

$$\pi(z)=\prod_{n=1}^{\infty}\Big(1+\frac{z^2}{\pi^2}\Big),$$

and the latter is positive onto a positive ray. Thus, it is more convenient to formulate the following problem for the function π.

Problem 4. *Let $\sum_{n=1}^{\infty}\frac{1}{\lambda_n}<\infty$ and $\pi(z)=\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n$. Then π is an entire function not exceeding the minimal type of the first order and by the Hadamard formula $\sqrt[n]{a_n}\to 0$ $(n\to\infty)$. Then $f(z)=\sum_{n=0}^{\infty}a_nz^n$ is an entire function. Does the relation*

$$\int_1^{\infty}\frac{\ln\ln f(r)}{r^2}dr<\infty$$

holds?

The following problem is also related to gap power series and interpolation.

Problem 5. *Let* (λ_n) *be an increasing sequence of natural numbers such that* $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} < \infty$ *and*

$$\Lambda_n = \prod_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{\lambda_m^2}{|\lambda_m^2 - \lambda_n^2|} .$$

Is there an increasing sequence of natural numbers (p_n) *such that*

$$\int_1^{\infty} \frac{\ln \ln \Psi(r)}{r^2} dr < \infty,$$

where

$$\Psi(r) = \sum_{n=1}^{\infty} \Lambda_n p_n! \left(\frac{r}{\lambda_n} \right)^{p_n} ?$$

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