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SEQUENTIALLY COMPACT HAUSDORFF CANCELLATIVE SEMIGROUP IS A TOPOLOGICAL GROUP

B. Bokalo, I. Guran

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We prove the statement announced in the title

Recall [1] that a group G with a topology is a paratopological group if the multiplication in G is continuous.

Reznichenko [5] has proved that a Tychonoff pseudocompact group with continuous multiplication is a topological group. Pfister [4] proved that a countable compact regular paratopological group is a topological group. Yur'eva [7] has recently shown that a sequential countable compact Hausdorff semigroup with two-side cancellations is a topological group.

We give an answer to a question posed by Robbie and Svetlichny in [6], i.e., we prove that a sequentially compact Hausdorff paratopological group is a topological group.

In sake of completeness we include some definitions and proofs here.

A topological space X is called sequentially compact [3] if X is a Hausdorff space every sequence of points in X has a convergent subsequence.

Lemma 1 [2]. *If $V^2 \subset U$ where V and U are open neighborhoods of the identity of a paratopological group, then $(\overline{V^{-1}})^{-1} \subset U$.*

Proof. We show that $\overline{V^{-1}} \subset U^{-1}$. Let $x \in \overline{V^{-1}}$. Then $xV \cap V^{-1} \neq \varnothing$. Therefore there exist $v_1, v_2 \in V$ suct that $uv_1 = v_2^{-1}$ and $x = v_2^{-1} v_1^{-1} \in V^{-1}V^{-1} \subset (V^{-1})^2 \subset U^{-1}$.

Definition 2. A paratopological group G is called topologically periodic if for each $x \in G$ and every neighborhood U of the identity there exists an integer n such that $x^n \in U$.

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Theorem 3. *Let G be a countable compact Hausdorff topologically periodic paratopological group. Then G is a topological group.*

Proof. Let U be an arbitrary neighborhood of the identity and $\{V_i | i \in \omega\}$ a family of neighborhoods of the identity such that $V_0 = U$ and $V_{i+1}^2 \subset V_i$ for each $i \in \omega$. By lemma 1 we have $(\overline{V_{i+1}^{-1}})^{-1} \subset V_i$.

We show that $F = \{\overline{V_i^{-1}} | i \in \omega\} \subset U$. Let $x \in F$, $x \in \overline{V_i^{-1}}$ for each $i \in \omega$. Then $x^{-1} \in (\overline{V_{i+1}^{-1}})^{-1} \subset V_i$ for each $i \in \omega$. Choose $n \in \omega$ such that $x^n \in V_1$, then $(x^{-1})^{n-1} \in ((\overline{V_{i+1}^{-1}})^{-1})^{n-1} \subset V_i^{n-1}$ for each $i \in \omega$. Now choose $i_0 \in \omega$ such that $V_{i_0}^{n-1} \subset V_1$. Then $x = x^n (x^{-1})^{n-1} \in V_1 V_1 \subset U$. Therefore $F \subset U$. Since G is a countable compact group, there exist $i_1, \ldots, i_k$ such that $\cap\{\overline{V_{i_j}^{-1}} | j = 1, \ldots, k\} \subset U$. Hence G is a topological group.

Lemma 4. *Let G is a sequentially compact paratopological group. Then G is topologically periodic.*

Proof. Let x an arbitrary element from G. Then $S = \overline{\{x^n | n \in \omega \setminus \{0\}\}}$ is a sequentially compact abelian topological semigroup with two-side cancellations. Therefore, by Lemma B.1 [5] S is group. Then $e \in \overline{\{x^n | n \in \omega \setminus \{0\}\}}$.

Theorem 3 and Lemma 4 imply the following

Theorem 5. *A sequentially compact Hausdorff paratopological group is a topological group.*

The Robbie-Svetlichny Lemma B.1 from [5] and Theorem 5 imply the following:

Theorem 6. *A sequentially compact Hausdorff cancellative topological semigroup is a topological group.*

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Department of Mathematics, Lviv State University, Lviv, 290602, Ukraine

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