Математичні студії
випуск 5

УДК 515.12

# ON MANIFOLDS MODELED ON DIRECT LIMITS OF $\mathcal{C}$-UNIVERSAL ANR'S

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ABSTRACT. Eugen Pentsak. *On manifolds modeled on direct limits of $\mathcal{C}$-universal ANR's* // Matematychni Studii, **5** (1995) P.107–116.

Countable dirict limits of strongly $\mathcal{C}$-universal ANR-spaces are considered. Characterization theorems for these spaces and corresponding manifold are proved. Besides, open (closed) embedding theorem and classification theorems are proved for manifolds.

The subject of this paper, namely, the investigation of spaces that are countable direct limits of ANRs and corresponding manifolds, continues the works of R. Heisey [1,2,3], K. Sakai [4, 5], V. T. Liem [6] devoted to the cases $R^\infty = \varinjlim R^n$ and $Q^\infty = \varinjlim Q^n$. In these papers the characterization theorems for $R^\infty$- and $Q^\infty$-manifolds, open (closed) embedding theorems, triangulation and classification theorems for these manifolds are proved.

It is natural to consider the countable direct limits of metrizable ANRs which are strongly universal for some classes of separable metrizable spaces. The examples of such ANRs are those of M. Bestvina and J. Mogilski [7] which are strongly universal for the absolute Borelian classes. In this paper we give characterizations of such spaces and corresponding manifolds. A good illustration of our situation is the space $s^\infty = \varinjlim s^n$ which appeared in [8] ($s$ denotes the pseudointeriour of the Hilbert cube).

Note also that our results can also be applied to the spaces $\Lambda_\alpha^\infty = \varinjlim \Lambda_\alpha^i$, $\Omega_\alpha^\infty = \varinjlim \Omega_\alpha^i$, $1 \le \alpha \le \omega_1$, $\Pi_n^\infty = \varinjlim \Pi_n^i$, $n \ge 1$, where $\Lambda_\alpha$ (respectively $\Omega_\alpha$, $\Pi_n$ ) are absorbing sets for the absolute Borelian additive class $\mathcal{A}_\alpha$ (respectively absolute Borelian multiplicative class $\mathcal{M}_\alpha$, absolute projective class $\mathcal{P}_n$); see [9].

The author wishes to express his gratitude to T. Banakh and M. Zarichnyi for valuable comments and to R.Cauty for the careful reading of the preliminary version of the paper and useful remarks.

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1991 *Mathematics Subject Classification.* 57N20.

## 1. Preliminaries

Let $\mathcal{C}$ be a class of separable metric spaces. We say that $\mathcal{C}$ is a topological class if for every $C \in \mathcal{C}$ and every homeomorphism $h : C \to D$ it follows that $D \in \mathcal{C}$. A topological class $\mathcal{C}$ is closed-hereditary if every closed subset of any $C \in \mathcal{C}$ belongs to $\mathcal{C}$. A topological class $\mathcal{C}$ is closed-additive if $C \in \mathcal{C}$ whenever $C$ can be expressed as the union of two of its closed subsets that belong to $\mathcal{C}$. We say that the class $\mathcal{C}$ is $[0,1]$-stable if for every $C \in \mathcal{C}$ it follows that $C \times [0,1] \in \mathcal{C}$ .

All classes of spaces considered in this paper are topological closed-additive closed-hereditary [0,1]-stable, and are subclasses of the class of separable metric spaces; all maps are continuous.

The classes of metrizable compact spaces, finite dimensional metrizable compact spaces, complete-metrizable separable spaces are examples of such classes.

By $cov(X)$ we denote the set of all open covers of $X$. For maps $f,g : X \to Y$ and for $\mathcal{U} \in cov(Y)$ the symbol $(f,g) \prec \mathcal{U}$ means that for each $x \in X$ there is $U \in \mathcal{U}$ such that $\{f(x), g(x)\} \subset U$. A map $f : X \to Y$ is a near-homemorphism if for any $\mathcal{U} \in cov(Y)$ there is a homeomorphism $h : X \to Y$ such that $(h,f) \prec \mathcal{U}$. A homotopy $h : X \times [0,1] \to Y$ is called $\mathcal{U}$-homotopy provided for each $x \in X$ the set $h(\{x\} \times [0,1])$ is contained in some $U \in \mathcal{U}$. $X \in \mathbf{ANR}(\mathbf{AR})$ means that $X$ is an $\mathbf{ANR}(\mathbf{AR})$ for the class of separable metric spaces. For $X, Y \in \mathbf{ANR}$ a map $f : X \to Y$ is said to be a fine homotopy equivalence if for any $\mathcal{U} \in cov(Y)$ there is a map $g : Y \to X$ such that $f \circ g$ is $\mathcal{U}$-homotopic to $id_Y$ and $g \circ f$ is $f^{-1}(\mathcal{U})$-homotopic to $id_X$. Let $A \subset X$. By $\mathcal{U} \cap A$ we denote the cover of $A$ formed by the sets $U \cap A$, where $U \in \mathcal{U}$. For $A \subset X, \mathcal{U} \in cov(X)$ let $st(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$ and for a collection $\mathcal{B}$ of subsets of $X$ let $st(\mathcal{B},\mathcal{U}) = \{st(B,\mathcal{U}) : B \in \mathcal{B}\}$. We write $st(\mathcal{U}) = st(\mathcal{U},\mathcal{U})$. $\mathcal{V} \in cov(X)$ is a refinement of $\mathcal{U}$, denoted by $\mathcal{V} \prec \mathcal{U}$, if each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$. If $st(\mathcal{V}) \prec \mathcal{U}$, then we write $\mathcal{U} \succ^* \mathcal{V}$. Denote by $\mathcal{U} \bigwedge \mathcal{V}$ the open cover of $X$ formed by the sets $U \cap V$ where $U \in \mathcal{U}$, $V \in \mathcal{V}$. Let $\overline{A}$, $intA$, $\partial A$ denote the closure, the interiour and the boundary of $A$ respectively.

By an absolute (neighbourhood) extensor for the class $\mathcal{C}$ (abbreviated $\mathbf{AE}(\mathcal{C})$ $\big(\mathbf{ANE}(\mathcal{C})\big)$ we mean a space $X$ such that every map $f : B \to X$ where $B$ is a closed subset of $A$, $A \in \mathcal{C}$ has an extension $F : A \to X$ ($F : U \to X$ where $U$ is an open neighbourhood $B$ in $A$).

A closed subset $A \subset X$ is called a $Z$-set, if for any $\mathcal{U} \in cov(X)$, any map $f : Q \to X$ of the Hilbert cube there exists a map $\overline{f} : Q \to X \backslash A$ such that $(\overline{f}, f) \prec \mathcal{U}$.

A space $X$ is called strongly $\mathcal{C}$-universal if for any $A \in \mathcal{C}$, a closed subset $B$ of $A$, $f : A \to X$ such that the restriction $f|_B : B \to X$ is a $Z$-embedding , $\mathcal{U} \in cov(X)$ there exists a $Z$-embedding $h : A \to X$ such that $h|_B = f|_B$ and $(f,h) \prec \mathcal{U}$.

We say that a space $X$ is a direct limit of $X_n$, $X_1 \subset X_2 \subset X_3 \subset \dots$ (briefly $X = \varinjlim X_n$) if $X = \cup_{n=1}^{\infty} X_n$ and $U$ is an open set in $X$ if and only if $U \cap X_n$ is an open set in $X_n$ for every $n \in \mathbb{N}$. We assume that each $X_n$ is closed in $X_{n+1}$ for $n \in \mathbb{N}$.

## 2. Characterization theorem.

We begin the proof of the main theorem with a few elementary Lemmas.

**Lemma 1.** *Let $X = \varinjlim X_n$ and let $A \subset X$ be a closed metrizable subset. Then for every $a \in A$ there exist an open neighbourhood $V$ of $a$ in $X$ and $n \in \mathbb{N}$ such that $V \cap A \subset X_n$.*

Lemma 1 is a simple consequence of Lemma 2.

**Lemma 2.** *Let $X = \varinjlim X_n$ and let $f : A \to X$ be a map of a metrizable space $A$ into $X$. Then for every $a \in A$ there exist an open neighbourhood $V$ of $a$ and $n \in \mathbb{N}$ such that $f(V) \subset X_n$.*

*Proof.* Let $d$ be an arbitrary metric on $A$ compatible with its topology, $f(a) \in X_{n_0}$ for some $n_0 \in \mathbb{N}$ and let $U_n = \{y \in A : d(a, y) < \frac{1}{n}\}$. Suppose that the statement of Lemma is not true. Then we can choose $a_n \in U_n$ so that $f(a_n) \in X_{k_n} \backslash X_{k_{n-1}}$ for some $k_n > k_{n-1}$ and $k_1 > n_0$, $n \in \mathbb{N}$. Since $f(a) \notin B = \{f(a_n) : n \in \mathbb{N}\}$ and $B$ is closed, we obtain a contradiction with the assumption.

**Lemma 3.** *If $X = \varinjlim X_n$ and $X_n$ are metrizable, then $X$ is paracompact.*

*Proof.* See [10].

**Lemma 4.** *Let $X \in$ **ANE**$(\mathcal{C})$, $\mathcal{U} \in cov(X), A \in \mathcal{C}$, $f : A \to X$. Then for every closed subset $B$ of $A$ and for every $\mathcal{U}$-homotopy $h : B \times [0; 1] \to X$ such that $h(x, 0) = f(x)$ for $x \in B$ and for every open neigbourhood $V$ of $B$ in $A$ there exists an $\mathcal{U}$-homotopy $H : A \times [0; 1] \to X$ such that $H|_{B\times[0;1]} = h$ and $H(x, t) = f(x)$ for every $x \in A \backslash V$, $t \in [0; 1]$.*

*Proof.* This Lemma can be proved by standard methods of ANR theory; see a "controlled" version of the Borsuk Homotopy Extension Theorem [11].

**Lemma 5.** *Assume that $X \in$ **ANR**, $X \in \mathcal{C}$ and $X$ is strongly $\mathcal{C}$-universal, $A$ is a $Z$-set in $X$, $C \in \mathcal{C}$ and $D$ is a closed subset of $C$, $f : C \to X$ is a map such that the restriction $f|D : D \to X$ is a $Z$-embedding. Then for every $\mathcal{U} \in cov(X)$ there exists a closed embedding $g : C \to X$ such that $g|D = f|D$, $g(C\backslash D) \subset X\backslash A$ and $g$ is $\mathcal{U}$-homotopic to $f$.*

*Proof.* Since $X \in$ **ANR**, by [10], there exists a refinement $\mathcal{U}_0 \in cov(X)$ of $\mathcal{U}$ such that every two $\mathcal{U}_0$-close maps are $\mathcal{U}$-homotopic. Let $Y$ be an adjunction space $C \cup_i A$ where $i : D \cap f^{-1}(A) \to X$ is the restriction of $f$. Considering the natural quotient map $p : C \oplus A \to C \cup_i A$ we can obtain that $p|C$ and $p|A$ are closed embeddings. Since the class $\mathcal{C}$ is closed-additive, we see that $Y \in \mathcal{C}$. Applying the property of strong $\mathcal{C}$-universality of $X$ to the map $h : Y \to X$ defined as

$$h(y) = \begin{cases} p^{-1}(y), & \text{for } y \in p(A) \\ f(p^{-1}(y)), & \text{for } y \in p(C) \end{cases}$$

we can find a closed embedding $k : Y \to X$ such that $k|D \cup_i A = h|D \cup_i A$ and $(k, h) \prec \mathcal{U}_0$. Clearly, $g = h \circ p|C$ satisfies all the conditions of Lemma.

**Lemma 6.** *Let $B \subset A$ be a closed subset of a metric space $A$ and let $B = \cup_{n=1}^{\infty} B_n$ where $B_n$ are closed subset of $A$, $B_n \subset intB_{n+1}$ for each $n \in \mathbb{N}$. Then there exists a collection $\{A_i\}_{i=1}^{\infty}$ such that $A_i \subset A$ is a closed in $A$, $A_i \subset intA_{i+1}$, $A = \cup_{i=1}^{\infty} A_i$ and $A_i \cap B = B_i$ for each $i \in \mathbb{N}$.*

*Proof.* The collection $\{A_i\}_{i=1}^{\infty}$ will be constructed inductively. Let $A_1 = B_1$, $A_2 = \{x \in A : d(x, A_1 \cup B_1) \le d(x, B\backslash B_1)$ or $d(x, B) \ge 1\}$ where $d$ is the metric on $A$ ; $A_3 = \{x \in A : d(x, A_2 \cup B_2) \le d(x, B\backslash B_2)$ or $d(x, B) \ge \frac{1}{2}\}$ and so on. It is easy to see that the collection $\{A_i\}_{i=1}^{\infty}$ has the required properties.

Let $\mathcal{M}^{\infty}$ is the class of separable metric spaces that are direct limits of sequences consisting of metrizable spaces and closed embeddings.

**Lemma 7.** *If $X = \varinjlim X_n$ and $X_n \in$* **ANR** *for $n \in \mathbb{N}$, then $X \in$* **ANE**$(\mathcal{M}^{\infty})$.

*Proof.* See [12].

**Lemma 8.** *If $X = \varinjlim X_n$ and $X_n \in$* **AR** *for $n \in \mathbb{N}$, then $X \in$* **AE**$(\mathcal{M}^{\infty})$.

*Proof.* Moreover, any direct limit of $ANR$s is an $ANE$ for the class of all stratifiable spaces (see [13]).

The following theorem is the main result in this section.

**Theorem 1.** *Let $X = \varinjlim X_n$ and let $X_n \in \mathcal{C}$ be a strongly $\mathcal{C}$-universal* **ANR** *which is a $Z$-set in $X_{n+1}$ for $n \in \mathbb{N}$. Then $X$ is strongly $\mathcal{C}$-universal.*

*Proof.* Let us note, that every closed metrizable set is a $Z$-set in $X$ and every closed embedding is a $Z$-embedding in $X$ respectively.

We shall show that for any $\mathcal{U} \in cov(X)$, $A \in \mathcal{C}$, a closed subset $B \subset A$, $f : A \to X$ such that $f|B : B \to X$ is a closed embedding, there exists a closed embedding $g : A \to X$ such that $g|B = f|B$ and $(f, g) \prec \mathcal{U}$. We find family $\{\mathcal{U}_i\}_{i=1}^{\infty}$, $\mathcal{U}_i \in \mathrm{cov}(X)$, $i \in \mathbb{N}$ such that $\mathcal{U} = \mathcal{U}_0 \succ^* \mathcal{U}_1 \succ^* \mathcal{U}_2 \succ^* \dots$. Since $f(B)$ is a closed metrizable set, by Lemma 1, we can choose an open cover $\mathcal{V} = \{V_i\}_{i=1}^{\infty}$ of $X$ such that for every $i \in \mathbb{N}$ we have $V_i \cap f(B) \subset X_{n_i}$ for some $n_i \in \mathbb{N}$. Let $\{W_i\}_{i=1}^{\infty}$ be an open refinement of $\mathcal{V}$ such that $\overline{W}_i \subset V_i$ for $i \in \mathbb{N}$. By Lemma 2, we can choose an open cover $\mathcal{U}' = \{U_i'\}_{i=1}^{\infty}$ of $A$ such that for every $i \in \mathbb{N}$ we have $f(U_i') \subset X_{k_i}$ for some $k_i \in \mathbb{N}$. Having the cover $\mathcal{U}' \bigwedge \{f^{-1}(W_i)\}_{i=1}^{\infty}$ of $A$ we can construct a collection of closed sets $\mathcal{T} = \{T_i\}_{i=1}^{\infty}$ such that $A = \cup_{i=1}^{\infty} T_i$ and for every $i \in \mathbb{N}$ $T_i \subset int\, T_{i+1}$ and $f(T_i) \subset (\cup_{k=1}^{i} W_{n_k}) \cap X_{p_i}$ for some $n_i, p_i \in \mathbb{N}$ such that $p_i < p_{i+1}$.

For every $n \in \mathbb{N}$ we construct a collection $\mathcal{V}_n = \{V_n^i\}_{i=1}^{\infty}$ such that $W_n \subset \overline{W}_n \subset V_n^1 \subset \overline{V}_n^1 \subset V_n^2 \subset \ldots \subset V_n$. We shall prove the theorem by induction.

First, we consider $f|T_1 : T_1 \to W_{n_1}$. Using strongly $\mathcal{C}$-universality of $V_{n_1} \cap X_{p_2}$, the fact that $f(T_1) \subset W_{n_1} \subset \overline{W}_{n_1} \subset V_n'$ and Lemma 5 find a closed embedding $\tilde{g}_1 : T_1 \to X_{p_2}$ such that $\tilde{g}_1|T_1 \cap B = f|T_1 \cap B$, $\tilde{g}_1(T_1\backslash B) \cap X_{p_1} = \emptyset$ and $\tilde{g}_1$ is $(\mathcal{U}_1 \cap (V_{n_1}^1 \cap X_{p_2})) \cup \{Xp_2 \backslash \overline{W}_{n_1}\}$-homotopic to $f|T_1$ and this homotopy is stationary on $T_1 \cap B$. By Lemma 4, we have a map $g_1' : T_2 \to (\cup_{k=1}^{2} V_{n_k}^1) \cap X_{p_2}$ such that $g_1'|\partial T_2 \cup (T_2 \cap B) = f|\partial T_2 \cup (T_2 \cap B)$, $g_1'|T_1 = \tilde{g}_1$ and $g_1'$ is $\mathcal{U}_1$-homotopic to $f|T_2$. Let us put for $x \in A$

$$g_1(x) = \begin{cases} g_1'(x), & \text{for } x \in T_2 \\ f(x), & \text{for } x \in A \backslash int T_{k+1}. \end{cases}$$

Suppose that for $n = k$ we have a map $g_k : A \to X$ such that $g_k|(A\backslash int\, T_{k+1}) \cup B = f|(A\backslash int\, T_{k+1}) \cup B$, $g_k|T_k$ is a closed embedding, $g_k(T_k\backslash(B \cup T_{k-1})) \cap X_{p_k} = \emptyset$,

$g_k|T_{k-1} = g_{k-1}|T_{k-1}$, $g_k(T_{k+1}) \subset (\cup_{l=1}^{k+1} V_{n_l}^{k+1}) \cap X_{p_{k+1}}$ and $g'_k$ is $\mathcal{U}_k$-homotopic to $g_{k-1}$. By Lemma 5 we can find a closed embedding $\tilde{g}_{k+1} : T_{k+1} \to X_{p_{k+2}}$ such that $\tilde{g}_{k+1}|T_{k+1} \cap (B \cup T_k) = g_k|T_{k+1} \cap (B \cup T_k)$, $\tilde{g}_{k+1}(T_{k+1} \backslash (B \cup T_k)) \cap X_{p_{k+1}} = \emptyset$ and $\tilde{g}_{k+1}$ is $\{X_{p_{k+2}} \backslash \cup_{l=1}^{k} \overline{V}_{n_l}^{k}\} \cup (\mathcal{U}_{k+1} \cap (\cup_{l=1}^{k+1} V_{n_l}^{k+1} \cap X_{p_{k+2}}))$-homotopic to $g_k|T_{k+1}$, and this homotopy is stationary on $T_{k+1} \cap B$, and, by Lemma 4, a map $g'_{k+1} : T_{k+2} \to \cup_{l=1}^{k+1} V_{n_l}^{k+1} \cap X_{p_{k+2}}$ such that $g'_{k+1}|\partial T_{k+2} \cup (T_{k+2} \cap B) = f|\partial T_{k+2} \cup (T_{k+2} \cap B)$, $g'_{k+1}|T_{k+1} = \tilde{g}_{k+1}$ and $g'_{k+1}$ is $\mathcal{U}_{k+1}$-homotopic to $g_k|T_{k+2}$. Then we can define a map $g_{k+1} : A \to X$ as

$$g_{k+1}(x) = \begin{cases} g'_{k+1}(x), & \text{for } x \in T_{k+1} \\ g_k(x), & \text{for } x \in A \backslash intT_{k+1}. \end{cases}$$

One can easily see that $g = lim_{i\to\infty} g_i : A \to X$ is the desired closed embedding.

Remark. From the proof of Theorem 1 we see that a map $g : A \to X$ can be choose $\mathcal{U}$-homotopic to $f$ by the homotopy which is stationary on $B$.

**Corollary 1.** *If $X = \varinjlim X_n$ where $X_n$ is a strongly $\mathcal{C}$-universal* **AR** *and a $Z$-set in $X_{n+1}$ for $n \in \mathbb{N}$, then $X$ is strongly $\mathcal{C}$-universal.*

**Theorem 2.** *If $X, Y \in \mathbf{AE}(\mathcal{C})$ are strongly $\mathcal{C}$-universal spaces that can be expressed as $X = \varinjlim X_n, Y = \varinjlim Y_n$ where $X_n, Y_n \in \mathcal{C}$ are $Z$-sets in $X$ and $Y$ respectively then $X \cong Y$.*

*Proof.* Let $f'_1 : X_1 \to Y$ be a constant map, i.e. $f'_1(x) = y_0 \in Y$ for all $x \in X_1$. Using strongly $\mathcal{C}$-universality of $Y$ we can find a $Z$-embedding $f_1 : X_1 \to Y$. Since $X \in AE(\mathcal{C})$ we can find a map $g'_1 : f_1(X_1) \cup Y_1 \to X$ such that $g'_1(y) = f_1^{-1}(y)$ for all $y \in f_1(X_1)$. Using strongly $\mathcal{C}$-universality of $X$ there exist a $Z$-embedding $g_1 : f_1(X_1) \cup Y_1 \to X$ such that $g_1|f_1(X_1) = g'_1|f_1(X_1)$. Similarly, we can find maps $f_2, g_2, f_3, g_3, \ldots,$ such that the diagram

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_2 \cup g_1(Y_1) & \longrightarrow & \ldots \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \downarrow \\ f(X_1) \cup Y_1 & \longrightarrow & f_2(X_2) \cup Y_2 & \longrightarrow & \ldots \end{array}$$

is commutative. Then $f = \varinjlim f_i : X \to Y$ is the required homeomorphism .

## 3.Manifolds

In the sequel we fix a space $X(\mathcal{C}) \in \mathbf{AE}(\mathcal{C})$ satisfying the following conditions : (i) $X(\mathcal{C}) = \varinjlim X_i$ where $X_i \in \mathcal{C}$; (ii) $X_i$ is strongly $\mathcal{C}$-universal $ANR$; (iii) $X_i$ is a $Z$-set in $X_{i+1}$ for all $i \in \mathbb{N}$.

We can regard manifolds modeled on $X(\mathcal{C})$ ($X(\mathcal{C})$-manifolds) i.e. paracompact spaces locally homeomorphic to an open set of $X(\mathcal{C})$. We shall assume that each $X(\mathcal{C})$-manifold admits a countable atlas.

If for a class $\mathcal{C}$ exists a strong universal $AR$-space $X \in \mathcal{C}$ we can find a $Z$-embedding $i : X \to X$. Note that this condition is satisfied by every class $\mathcal{C} \in \{\mathcal{M}_\alpha : \alpha < \omega_1\} \cup \{\mathcal{A}_\alpha : \alpha < \omega_1\} \cup \{\mathcal{P}_n : n \in \mathbb{N}\}$ and $X \in \{\Omega_\alpha : \alpha < \omega_1\} \cup \{\Lambda_\alpha :$

$\alpha < \omega_1\} \cup \{\Pi_n : n \in \mathbb{N}\}$ which is the respective $\mathcal{C}$-absorbing set (see [7,9] ).Then the direct limit

$$X \xrightarrow{i} X \xrightarrow{i} X \xrightarrow{i} \dots$$

will be homeomorphic to the $X(\mathcal{C})$.

**Lemma 9.** *Every $X(\mathcal{C})$-manifold $M$ can be expressed as $\varinjlim M_i$ where $M_i \in \mathcal{C}$.*

*Proof.* Let $\mathcal{V} = \{V_i\}_{i=1}^{\infty}$ be an open cover of $M$ such that there exists a homeomorphism $h_i : V_i \to U_i$ where $U_i$ is an open set in $X(\mathcal{C})$ for $i \in \mathbb{N}$. Using paracompactness of $M$, we can choose a closed locally finite refinement $\mathcal{W} = \{W_i\}_{i=1}^{\infty}$ of $\mathcal{V}$ suth that $W_i \subset V_i$ for all $i \in \mathbb{N}$. Let us put $M_1 = h_1^{-1}(h_1(W_1) \cap X_1)$, $M_2 = h_1^{-1}(h_1(W_1) \cap X_2) \cup h_2^{-1}(h_2(W_2) \cap X_2)$ and so on. It is easy to check that $M = \varinjlim M_i$ and $M_i \in \mathcal{C}$.

**Lemma 10.** *Every open subset of the space $X(\mathcal{C})$ is $\mathcal{C}$-universal.*

*Proof.* We can proof this Lemma by the same argument as in the proof of Proposition 2.1 (see [7]) replacing $\epsilon$-closed by $\mathcal{U}$-closed, where $\mathcal{U} \in cov(X(\mathcal{C}))$.

**Theorem 3.** *Every $X(\mathcal{C})$-manifold is strongly $\mathcal{C}$-universal.*

*Proof.* Let $M$ be an $X(\mathcal{C})$-manifold. We have to show that for every $\mathcal{U} \in cov(M)$, $A \in \mathcal{C}$, a closed subset $B$ of $A$, a map $f : A \to M$ such that $f|B : B \to M$ is a closed embedding, there exists a closed embedding $g : A \to M$ such that $g|B = f|B$ and $(f, g) \prec \mathcal{U}$.

We find a family $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of open covers of $M$ such that $\mathcal{U} = \mathcal{U}_0 \succ^* \mathcal{U}_1 \succ^* \mathcal{U}_2 \succ^* \dots$ and $\mathcal{V} = \{V_i\}_{i=1}^{\infty} \in cov(M)$ such that there exists a homeomorphism $h_i : R_i \to V_i$ where $R_i$ is an open set in $X(\mathcal{C})$ for every $i \in \mathbb{N}$. Let $W^1 = \{W_j^1\}_{j=1}^{\infty}$, $W^2 = \{W_j^2\}_{j=1}^{\infty}$ be star-refinements of $\mathcal{V}$ such that $W_j^2 \subset \overline{W}_j^2 \subset W_j^1 \subset \overline{W}_j^1 \subset V_{n_j}$ for every $j \in \mathbb{N}$ and for some $n_j \in \mathbb{N}$.

Since, by Lemma 9, $M = \varinjlim M_i$, for every $j \in \mathbb{N}$ we can find an open family $\{U_j^n\}_{j=1}^{\infty}$ such that $\overline{W}_j^2 \subset U_j^n \subset \overline{U}_j^n \subset U_j^{n+1}$ for every $n \in \mathbb{N}$ and $\cup_{n=1}^{\infty} U_j^n = W_j^1$.

We shall construct maps $g_n : A \to M$ by induction over $n = 1, 2, \dots$ such that the following conditions are satisfied :

1) $g_n|A_{n-1} \cup B$ is a closed embedding ;

2) $g_n|A_{n-2} \cup B \cup (A \backslash intA_n) = g_{n-1}|A_{n-2} \cup B \cup (A \backslash intA_n)$;

3) $g_n|A \backslash intA_n = f|A \backslash intA_n$;

4) $g_n$ is $\mathcal{U}_n$-close to $g_{n-1}$.

Now we shall describe the process of construction of the maps $g_n : A \to M$ and $A_n$ for every $n \in \mathbb{N}$. Let $A_1 = f^{-1}(\overline{W}_1^2)$,$A_0 = \emptyset$,$g_1 = f$. It is easy to see that the conditions 1)-4) are satisfied.

Let $A_2 = f^{-1}(\overline{U}_1^1)$, $A_1' = f^{-1}(\overline{U}_1^2)$. Since $f(A_1) \subset W_1^1 \subset V_{n_1}$ and $h_{n_1}^{-1} \circ f : A_1' \to R_{n_1}$, using Lemma 10 and remark following Theorem 1 we can find a closed embedding $\tilde{g}_1 : A_1' \to R_{n_1}$, $\mathcal{U}_1^1$-homotopic to $h_{n_1}^{-1} \circ f|A_1$ by a homotopy $H_1$, which is stationary on $A_1' \cap B$, and $\tilde{g}_1|A_1' \cap B = h_{n_1}^{-1} \circ f|A_1' \cap B$ where $\mathcal{U}_1^1 = h_{n_1}^{-1}(\mathcal{U}_1 \cap V_{n_1}) \wedge \{U_{s_1^1}^2, V_{n_1} \backslash \overline{U}_{s_1^1}^1\} \wedge \dots \wedge \{U_{s_k^1}^2, V_{n_k} \backslash \overline{U}_{s_k^1}^1\}$ and $s_1^1, s_2^1, \dots, s_k^1$ are such that $W_{s_j^1}^1 \cap W_1^1 \neq \emptyset$ for $j = 1, 2, \dots, k$.

Let $\alpha_1 : A_1' \to [0,1]$ be a map such that $\alpha_1(A_2) = 1$ and $\alpha_1(\partial A_1') = 0$. Then we can take

$$g_1(x) = \begin{cases} h_{n_1}(H_1(x, \alpha_1(x))), & \text{for } x \in A_1' \\ f(x), & \text{for } x \in X \setminus intA_1'. \end{cases}$$

Thus the map $g_1 : A \to M$ satisfies the conditions 1)-4).

Assume that $g_{k-1} : A \to M$ is constructed.

Let $A_k = g_{k-1}^{-1}(\overline{U}_k^k)$. In the same way as for the case $n = 2$ we can construct the map $g_k : A \to M$ which satisfies the conditions 1)-4). Using the property of $g_k : A \to M$ we can easy verify that $g = lim_{i\to\infty} g_i : A \to M$ is a closed embedding and $g|B = f|B$.

Let $M = \varinjlim M_i$, $N = \varinjlim N_i$ where $M_i, N_i \in \mathcal{C}$ for all $i \in \mathbb{N}$. By $C(M,N)$ we denote the space of all maps $f : M \to N$ equiped with the limitation topology which neighbourhood base in an $f \in C(M,N)$ consist of the sets $\{g \in C(M,N) : (f,g) \prec \mathcal{U}\}$ where $\mathcal{U} \in cov(N)$.

**Theorem 4.** *Every $X(\mathcal{C})$-manifold can be embedded as an open set in $X(\mathcal{C})$.*

*Proof.* Let $M$ be an $X(\mathcal{C})$-manifold. By Lemma 9, $M = \varinjlim M_i$ where $M_i \in \mathcal{C}$. The proof is by induction on $i$. Let $f_1' : M_1 \to X_1$ be a constant map, i.e. $f_1'(x) = x_1 \in X_1$ for all $x \in M_1$. Since $X_1$ is a strongly $\mathcal{C}$-universal $ANR$, there exists a homotopy $H_1 : M_1 \times I \to X_1$ such that $H_1(x,0) = f_1'(x)$ and the map $H_1(x,1) = f_1(x)$ is a $Z$-embedding. Since $M \in \mathbf{ANE}(\mathcal{C})$, we can find an open neigbourhood $V_1'$ of $f_1(M_1)$ in $X_1$ and a map $g_1' : V_1' \to M$ such that $g_1'|f_1(M_1) = f_1^{-1}$. Let us denote

$$H_1' : (f_1(M_1) \times I) \cup (X_1 \times \{0\}) \to X_1$$

as

$$H_1'(x,t) = \begin{cases} H_1(f_1^{-1}(x), t), & (x,t) \in f_1(M_1) \times I \\ x_1, & (x,t) \in X_1 \times \{0\}. \end{cases}$$

It is easy to show that there exists an open set $V_1$ in $X_1$ and a homotopy $H_1^* : \overline{V}_1 \times I \to X_1$ such that $f_1(M_1) \subset V_1 \subset \overline{V}_1 \subset V_1'$ and $H_1^*(x,0) = x_1$,$H_1^*(x,1) = x$ for all $x \in \overline{V}_1$. Using strong $\mathcal{C}$-universality of $M$ we can find a closed embedding $g_1 : \overline{V}_1 \to M$ such that $g_1|f_1(M_1) = f_1^{-1}$.

For $i = k$ we can find a $Z$-embedding $f_k : g_{k-1}(\overline{V}_{k-1}) \cup M_k \to X_k$ such that $f_k|g_{k-1}(\overline{V}_{k-1}) = g_{k-1}^{-1}$ and a homotopy $H_k : (g_{k-1}(\overline{V}_{k-1}) \cup M_k) \times I \to X_k$ such that $H_k(x,0) = x_1$ and $H_k(x,1) = f_k$ for every $x \in g_{k-1}(\overline{V}_{k-1}) \cup M_k$. Similarly as for $n = 1$ we can find an open neighbourhood $V_k$ of $f_k(g_{k-1}(\overline{V}_{k-1}) \cup M_k)$ in $X_k$ and a closed embedding $g_k : \overline{V}_k \to M$ such that

$$g_k|f_k(g_{k-1}(\overline{V}_{k-1}) \cup M_k) = f_k^{-1}$$

and a homotopy $H_k^* : \overline{V}_k \times I \to X_k$ such that $H_k^*(x,0) = x_1$, $H_k^*(x,1) = x$ for every $x \in \overline{V}_k$.

Proceeding similarly we obtain the commutative diagram

$$\begin{array}{ccccccc}
M_1 & \longrightarrow & M_2 \cup g_1(\overline{V}_1) & \longrightarrow & M_3 \cup g_2(\overline{V}_2) & \longrightarrow & \dots \\
{\scriptstyle f_1}\big\downarrow & & {\scriptstyle f_2}\big\downarrow & & {\scriptstyle f_3}\big\downarrow & & \big\downarrow \\
\overline{V}_1 & \longrightarrow & \overline{V}_2 & \longrightarrow & \overline{V}_3 & \longrightarrow & \dots \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \dots
\end{array}$$

Then the map

$$f = \varinjlim f_i : M = \varinjlim M_i \to \varinjlim V_i = V \subset X$$

is the desired open embedding.

**Corollary 2.** *If $X(\mathcal{C}) = \varinjlim X_i$ where each $X_i \in \mathcal{C}$ is a strongly $\mathcal{C}$-universal AR, then for any $X(\mathcal{C})$-manifold there exists an open embedding in $X(\mathcal{C})$.*

**Theorem 5 (Characterization).** *A space $Y$ is an $X(\mathcal{C})$-manifold if and only if $Y$ is a strongly $\mathcal{C}$-universal $ANE(\mathcal{C})$ which can be expressed as a direct limit $Y = \varinjlim Y_i$ where every $Y_i \in \mathcal{C}$ is a $Z$-set in $Y$.*

**Theorem 6.** *The set of all closed embeddings is dense in $C(M, N)$ where $M = \varinjlim M_i$, $M_i \in \mathcal{C}$ and $N$ is $X(\mathcal{C})$-manifold.*

*Proof.* Let $f \in C(M, N)$, $\mathcal{U} \in cov(X(\mathcal{C}))$, $\mathcal{U} = \mathcal{U}_0 \succ^* \mathcal{U}_1 \succ^* \mathcal{U}_2 \succ^* \dots$ and $N = \varinjlim N_i$ where $N_i$ are those from Theorem 5. By Theorem 3 and the remark following Theorem 1, there exists a closed embedding $g_1 : M_1 \to N$ $\mathcal{U}_1$-homotopic to $f|M_1$ such that $g_1(M_1) \cap N_1 = \emptyset$. By Lemma 4, the map $g_1 : M_1 \to N$ has an extension $g_1' : M_2 \to N$ such that

$$g_1'|M_2 \backslash V(M_1) = f|M_2 \backslash V(M_1)$$

where $V(M_1)$ is an open neighbourhood of $M_1$ in $M_2$. Proceeding similarly we can construct a closed embedding $g_n' : M_n \to N$ $\mathcal{U}_n$-homotopic to $g_{n-1}|M_n : M_n \to N$ such that

$$g_n|M_{n-1} = g_{n-1}|M_{n-1}$$

and

$$g_n(M_{n+1} \backslash M_n) \cap N_n = \emptyset.$$

One can easily check that $g = \varinjlim g_i : \varinjlim M_i \to N$ is a closed embedding $\mathcal{U}$-close to $f$.

Notice that the obtained characterization resemblus the characterization of manifolds modeled on absorbing spaces of Bestvina and Mogilski [7].

**Corollary 3.** *Every $X(\mathcal{C})$-manifold can be embedded as a closed set in $X(\mathcal{C})$.*

**Theorem 7 (Classification).** *Every fine homotopy equivalence $f : M \to N$ between $X(\mathcal{C})$-manifolds is a near-homeomorphism.*

**Theorem 8.** *Every homotopy equivalence $f : M \to N$ between $X(\mathcal{C})$-manifolds is homotopic to homeomorphism.*

The proof of these two theorems is similar to the scheme given in [4].

**Theorem 9 (Triangulation).** *Every $X(\mathcal{C})$-manifold is homeomorphic to $|K| \times X(\mathcal{C})$ where $K$ is a countable locally finite simplicial complex .*

*Proof.* Since every $X(\mathcal{C})$-manifold has the homotopy type of a countable locally finite simplicial complex (see Theorem 4 and [17]) it is sufficient to show that for every countable locally finite simplicial complex $K$ the space $|K| \times X(\mathcal{C})$ is an $X(\mathcal{C})$-manifold.

Since $K$ is countable locally finite, we have $|K| = \varinjlim K_i$ where $K_i \in K_{i+1}$ is a compact polyhedron (and therefore $ANR$) for $i \in \mathbb{N}$. Since $X(\mathcal{C}) = \varinjlim X_i$ and $|K| = \varinjlim K_i$ is locally compact,

$$|K| \times X(\mathcal{C}) = \varinjlim K_i \times \varinjlim X_i = \varinjlim K_i \times X_i.$$

Since $K_i \times X_i$ is a closed subset of $I_i^{n_i} \times X_i$ for some $n_i \in \mathbb{N}$, $K_i \times X_i \in \mathcal{C}$. Because $|K| \in ANR$ and $X(\mathcal{C}) \in \mathbf{AE}(\mathcal{C})$, we have $|K| \times X(\mathcal{C}) \in \mathbf{ANE}(\mathcal{C})$.

Obviously $|K| \times X(\mathcal{C})$ is strongly $\mathcal{C}$-universal. Now the proof follows from Theorems 5,8, Lemma 7 and the fact that the natural projection $p : |K| \times X(\mathcal{C}) \to |K|$ is a homotopy equivalence.

However, we cannot speak about stability of such manifolds in the customary sense ($M \times X \cong M$), because it can be that $X \times X \ncong X$ (e.g.,$s^\infty \times s^\infty \ncong s^\infty$).

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*Received 15.03.94;*
*Revised 24.01.95.*