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ON HOMEOMORPHISMS OF HYPERSPACES OF CONVEX SUBSETS

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The famous Shchepin's results on spectral analisys of homeomorphisms of normal functor-powers in tne category of compacta are exstended to "normal" functors from the category of convex compacta to the category of compacta. It is stated that for a metrizable convex compacta K, L, M, which are distinct from a point, and $\tau \geq \omega_2$ the spaces K^τ, the hyperspace of convex closed subsets of L^τ and the hyperspace of convex closed subsets of M^τ, which are convex hulls of $\leq n$ points, $n \geq 2$, are mutually non-homeomorphic. For any homeomorphism between the hyperspaces of convex closed subsets of K^τ and L^τ the preservation of finite-dimensional elements and finite-dimensional elements with n extreme points for $n \in A$, A necessarily contains 1 and 2 and coincides with $\mathbb{N}$ or its initial segment, is proved.

Let $\mathcal{C}onv$ be the category with convex compacta as objects and affine continuous maps as arrows. For a convex compactum X denote by $cc\,X$ the subset in $exp\,X$ consisting of closed convex sets in X. It is easy to see that $cc\,X$ is closed in $exp\,X$. Let $f : X \to Y$ be an arrow in $\mathcal{C}onv$. Then define $cc\,f$ as the restriction of $exp\,f$ to $cc\,X$. Since $exp\,(cc\,X) \subset cc\,Y$, the functor $cc : \mathcal{C}onv \to \mathcal{C}omp$ is correctly defined.

Lemma 1. *The functor cc is mono- and epimorphic, continuous and preserves empty set, singleton, intersections, preimages and weight of infinite compacta.*

Let $cc_n\,X \subset cc\,X$ consist of convex closed sets in X which are closed hulls of $\leq n$ points (affine images of $(n-1)$-dimensional simplex). Obviously, the subfunctor cc_n of cc is obtained.

Lemma 2. *The functor cc_n has all the properties mentioned in Lemma 1.*

In the sequel for any subset F of a convex compactum X we denote by $Cv(F)$ the convex hull of F (i.e., the intersection of convex sets in X which contain F), by $\overline{Cv}(F)$ the closed convex hull of F (i.e., the intersection of closed convex sets).

Evidently, for any F we have $\overline{Cv}(F) = \overline{Cv(F)}$ (bar denotes the closure operator). If $F_1, \dots, F_k \in cc\,X,\ k \in \mathbb{N},$ then $\overline{Cv}(F_1 \cup \dots \cup F_k) = Cv\,(F_1 \cup \dots \cup F_k)$.

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Lemma 3. *Let B be a closed convex set in a convex compactum X and for any element of a family $\{F_\alpha \mid \alpha \in \mathcal{A}\}$ of closed sets in X there is $B = \overline{Cv}(F_\alpha)$. Then $B = \overline{Cv}(\bigcap\limits_{\alpha\in\mathcal{A}} F_\alpha)$.*

Corollary. *For any $B \in cc\,X$ there exists (necessarily unique) the least closed set in X such that the closed convex hull of it coincides with B.*

Denote it by $ex\,B \in exp\,X$. By Krein-Milman theorem [1], $ex\,B$ coincides with the closure of extreme points of B.

In the sequel all the spaces are convex compacta and maps are continuous if otherwise is not specified.

Lemma 4. *Let $A, B \in cc_n\,X$ with $ex\,A \subset ex\,B$. Then A and B can be connected by an ordered arc in $cc_n\,X$.*

§1. On reduction and induced homeomorphism of diagrams.

Lemma 5. *Let $\mathcal{A}$ be a set, $|\mathcal{A}| \geq \omega_2$ and $\mathcal{B}$ be a directed upward family of countable subsets of $\mathcal{A}$ with $\cup\mathcal{B} = \mathcal{A}$. Then for any $n \in \mathbb{N}$ there exist $B_1, \ldots, B_n \in \mathcal{B}$ such that for any $j \in \{1, \ldots, n\}$ the inequality $B_j \setminus \bigcup_{i\neq j} B_i \neq \emptyset$ holds.*

Lemma 6. *Let the family $\mathcal{B}$ from Lemma 1 be closed with respect to countable unions and finite intersections. Then for any $n \in \mathbb{N}$ there exist $B_0, \ldots, B_n \in \mathcal{B}$ such that for any $i, j \in \{1, \ldots, n\}$, $i \neq j$ we have $B_0 \neq B_i$, $B_0 = B_i \cap B_j$.*

Lemma 7. *Let F_1, F_2 be bicommutative continuous epimorphic functors from the category of (convex) compacta to $\mathcal{C}omp$, K and L be (convex) metrizable compacta, τ be an infinite index set, $H : F_1(K^\tau) \to F_2(L^\tau)$ be a homeomorphism. Then the set of those $B \subset \tau, |B| \leq \omega$ for which we can supplement the diagram :*

$$\begin{array}{ccc} F_1(K^\tau) & \xrightarrow{H} & F_2(L^\tau) \\ {\scriptstyle F_1\pi^\tau_B}\big\downarrow & & {\scriptstyle F_2\pi^\tau_B}\big\downarrow \\ F_1(K^B) & \xrightarrow{h} & F_2(L^B) \end{array}$$

with a morphism h, is closed with respect to countable unions and finite intersections.

Following [2] denote by $\pi^2 X$ and $\pi^3 X$ the commutative diagrams :

$$\begin{array}{ccc} X\times X\times X\times X & \xrightarrow{\pi^{24}_{13}} & X\times X \\ {\scriptstyle \pi^{34}_{12}}\big\downarrow & & \big\downarrow{\scriptstyle \pi^3_1} \\ X\times X & \xrightarrow{\pi^2_1} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X\times X\times X & \xrightarrow{\pi^2_{13}} & X\times X \\ {\scriptstyle \pi^3_{12}}\big\downarrow & & \big\downarrow{\scriptstyle \pi^3_1} \\ X\times X & \xrightarrow{\pi^2_1} & X \end{array}$$

respectively.

Indices below specify factors onto which the projection is taken and indices above correspond to factors which are omitted.

Let X, $Y_1, \ldots, Y_n$ be topological spaces, $n \in \mathbb{N}$. Let us denote by $\mathcal{D}(X, Y_1, \ldots, Y_n)$ the diagram indexed by all subsets of the set $\{1, \ldots, n\}$. The respective object equals $X \times Y_{i_1} \times \cdots \times Y_{i_k}$ for $\{i_1, \ldots, i_k\}$ and arrows are all projections going from objects with indices more as sets to objects with "less" index. If $X = Y_1 = \cdots = Y_n$, the respective diagram $D(X, Y_1, \ldots, Y_n) = D(X, \ldots, X)$ is denoted by $\nu^n X$. Obviously $\nu^2 X$ coincides with $\pi^3 X$.

In the sequel FD, where F is a functor and D is a diagram, denotes the diagram which is obtained by application of the functor to the spaces and the maps of the diagram D.

The following is an extension of the results of [2,3] to the case of $\mathcal{C}onv$.

Proposition 1. *Let K and L be convex metrizable compacta, F and L be functors from $\mathcal{C}onv$ to $\mathcal{C}omp$ for which all conditions of "normality" hold (e.g., cc, cc_n or the lifting of a normal functor in $\mathcal{C}omp$ to $\mathcal{C}onv$). Then any homeomorphism of diagrams $F_1(\pi^2 K^\omega) \to\to F_2(\pi^2 L^\omega)$ induces a homeomorphism of reduced diagrams $F_1(\pi^3 K^\omega) \to F_2(\pi^3 L^\omega)$.*

The proof can be obtained by simple replacement of points with finite supports in the proof of the similar theorem from [2], for points with supports containing in affine images of finite-dimensional simplices. (Here the notion of support is considered as in usual compact case - as the least convex compactum, which, when a functor is applied, contains the point.) The fact is that for a "normal" functor $\mathcal{C}onv \to \mathcal{C}omp$ and a convex compactum X such points are dense in X.

Proposition 2. *Let K, L, F_1, F_2 satisfy the conditions of Proposition 1, $\tau \geq \omega_2$ be a cardinal, $H : F_1(K^\tau) \to F_2(L^\tau)$ be a homeomorphism and $F_1(K^\tau)$ and $F_2(L^\tau)$ be represented as the limits of canonical σ-spectra for $F_1((K^\omega)^\tau)$ and $F_2((L^\omega)^\tau)$. Then H induces a homeomorphism of cofinal σ-closed and σ-complete subspectra such that:*

1. *for any element of their index set the subspectra contain homeomorphic (through the restriction of the subspectra homeomorphism) diagrams $F_1(\nu^2 K^\omega)$ and $F_2(\nu^2 L^\omega)$ with all objects having indices more than above mentioned element;*
2. *if the diagram $F_1(\nu^2 K^\omega)$ (or, which is equivalent, $F_2(\nu^2 L^\omega)$) is bicommutative, then the same statement holds for $F_1(\nu^n K^\omega)$ and $F_2(\nu^n L^\omega)$ for any $n \in \mathbb{N}$.*

Remark. It is easy to see that (b) holds for functors *exp*, *cc*, the forgetful functor $\mathcal{C}onv \to \mathcal{C}omp$ (but not for cc_n, $n \geq 2$).

Proposition 3. *Let K be convex metrizable compactum, $\tau > \omega$, and $cc(K^\tau)$ be represented as a limit of the canonical σ-spectrum (or its closed cofinal subspectrum). Then $z \in cc(K^\tau)$ is a singleton, a segment, a finite-dimensional element or an finite-dimensional element with finite number of extreme points iff there exists such index element B_0 that for any $B \geq B_0$ (i.e., $B_0 \subset B$ as sets) the respective condition holds for $cc\,\pi_B(z)$.*

§2. On homeomorphisms of cc_n's of powers of metrizable convex compacta.

Now restrict our attention to the spaces of the form $cc_n(K^\tau)$ where τ is an infinite cardinal number, K is a metrizable convex compactum distinct from a point.

The diagram $cc_n\,\nu^2(K^\omega)$ will be considered. We use the following notation :

$$\chi(x_2,x_3)=(cc_n\,\pi^3_{12})^{-1}(x_2)\cap(cc_n\,\pi^2_{13})^{-1}(x_3)$$

for $x_2, x_3 \in cc_n\,(K^\omega \times K^\omega)$. Evidently $\chi(x_2,x_3)\neq\emptyset$ only if $cc_n\,\pi^2_1(x_2)=cc_n\,\pi^3_1(x_3)$.

Lemma 8. *Let X be an arcwise connected space, $n=\{0,1,\dots,n-1\}$, X^n is identified with $C(n,X)$. If $B\subset X$, $|B|<n$ and $\varphi,\psi\in X^n$, $\varphi(n)\supset B$, $\psi(n)\supset B$, then there exists a path $\lambda: I\to X^n$ such that $\lambda(0)=\varphi$, $\lambda(1)=\psi$ and for any $t\in I$ there is $\lambda(t)(n)\supset B$.*

Lemma 9. *Let X, Y be arcwise connected spaces, π_1 and π_2 are projections of the product $X\times Y$ to the factors, $F\subset exp_{n-1}X$, $G\subset exp_n Y$. Then the set of $V\in exp_n(X\times Y)$ with $\pi_1(V)\supset F$ and $\pi_2(V)\supset G$ is arcwise connected.*

Proposition 4. *The most number of arcwise connected components of $\chi(x_2,x_3)$ for fixed $x_1\in cc_n\,(K^\omega)$ and different $x_2,x_3\in cc_n\,(K^\omega\times K^\omega)$ such that $cc\,\pi^2_1(x_2)=cc\,\pi^3_1(x_3)=x_1$ equals $(n-|ex\,x_1|+1)!$.*

Proof. Let $x_1\in cc_n\,(K^\omega)$, $x_2,x_3\in cc_n\,(K^\omega\times K^\omega)$, $x_1=cc_n\,\pi^2_1(x_2)=cc_n\,\pi^3_1(x_3)$ and $A(x_2,x_3)=\pi^2_1(ex\,x_2)\cup\pi^3_1(ex\,x_3)$.

For any affine $f: X\to Y$ and $V\in cc\,X$, $W\in cc\,Y$ one can see that $cc\,f(v)=W$ iff $ex\,W\subset f(ex\,V)\subset W$. As a consequence we obtain

$$ex\,x_1\subset\pi^2_1(ex\,x_2)\cap\pi^3_1(ex\,x_3)\subset A(x_2,x_3)\subset x_1.$$

(In the last inclusion we consider x_1 as a subset in K^ω.)

Assume $x_4\in\chi(x_2,x_3)$. Then :

$$\overline{Cv}((ex\,x_4\cap(\pi^3_{12})^{-1}(ex\,x_2))\cup(ex\,x_4\cap(\pi^2_{13})^{-1}(ex\,x_3)))\in\chi(x_2,x_3).$$

Let B be a minimal subset of the finite set in the parentheses above such that $Cv(B)\in\chi(x_2,x_3)$. Then $B=ex\,Cv(B)\subset ex\,x_4$, $\pi^3_{12}(B)\supset ex\,x_2$, $\pi^2_{13}(B)\supset$ $\supset ex\,x_3$.

By Lemma 4, x_4 can be connected with $Cv(B)$ by decreasing arc entirely laying in $\chi(x_2,x_3)$. For any $z\in A(x_2,x_3)$ we have ;

$$\pi^3_{12}(B\cap(\pi^{23}_1)^{-1}(z))\supset ex\,x_2\cap(\pi^2_1)^{-1}(z)\ ;\ \pi^2_{13}(B\cap(\pi^{23}_1)^{-1}(z))\supset ex\,x_3\cap(\pi^3_1)^{-1}(z).$$

Suppose $|supp\,x_2\cap(\pi^2_1)^{-1}(z)|\neq|supp\,x_3\cap(\pi^3_1)^{-1}(z)|$ or both the inclusions above are strict. Then the conditions of Lemma 6 are satisfied and there is a path $\lambda: I\to$ $\to exp_{|B\cap(\pi^{23}_1)^{-1}(z)|}\,((\pi^3_{12})^{-1}(x_2)\cap(\pi^2_{13})^{-1}(x_3)\cap(\pi^{23}_1)^{-1}(z))$ connecting $B\cap(\pi^{23}_1)^{-1}(z)$ with a set B_z which projects bijectively to those of the sets $ex\,x_2\cap(\pi^2_1)^{-1}(z)$ and $ex\,x_3\cap(\pi^3_1)^{-1}(z)$ which have the number of elements not less than the other one.

Then it is easy to construct a path $\mu: I\to exp_{|B|}\,((\pi^3_{12})^{-1}(x_2)\cap(\pi^2_{13})^{-1}(x_3)\cap$ $\cap(\pi^{23}_1)^{-1}(A(x_2,x_3)))$ such that $\mu(0)=B$, $Cv(\mu(t))\subset\chi(x_2,x_3)$ for any $t\in I$, and for $C=\mu(1)$ the following holds :

* *$Cv(C)\in\chi(x_2,x_3)$, $\pi^{23}(C)=A(x_2,x_3)$ and for any $z\in A(x_2,x_3)$ the set $C\cap(\pi^{23}_1)^{-1}(z)$ is projected bijectively to those of the sets $ex\,x_2\cap(\pi^2_1)^{-1}(z)$ and $ex\,x_3\cap(\pi^3_1)^{-1}(z)$ which have the number of elements not less than the other one (including the case of equal numbers).*

Thus the number of arcwise connected connected components of $\chi(x_2, x_3)$ is equal to the maximal number of pairs which mutually can not be connected by a path in $\chi(x_2, x_3)$ with each point satisfying $(*)$.

For a set C of the form $(*)$ the number of elements always equals:

$$|C| = \sum_{z \in A(x_2,x_3)} \max\left(|ex\, x_2 \cap {\pi_1^2}^{-1}(z)|, |ex\, x_3 \cap {\pi_1^3}^{-1}(z)|\right) = S(x_2, s_3)$$

For $S(x_2, x_3) > n$ evidently $\chi(x_2, x_3) = \emptyset$.

One can easily see that for $S(x_2, x_3) < n$ the "free point" enables us to use Lemma 6 in order to connect by a path in $\chi(x_2, x_3)$ any two sets of the form $(*)$.

Let $S(x_2, x_3) = n$. Then any element of $\chi(x_2, x_3)$ is a set of the form $(*)$ and any path in $\chi(x_2, x_3)$ is a "convex hull" of $|A(x_2, x_3)|$ paths connecting for any $z \in A(x_2, x_3)$ in $cc_n\,({\pi_{12}^3}^{-1}(x_2) \cap {\pi_{13}^2}^{-1}(x_3) \cap {\pi_1^{23}}^{-1}(z))$ the sets $Cv\,(ex\, x_4' \cap {\pi_1^3}^{-1}(z)$ and $Cv\,(ex\, x_4'' \cap \cap {\pi_1^3}^{-1}(z)$.

Thus the number of of arcwise connected components of $\chi(x_2, x_3)$ is equal to the product of the numbers of arcwise connected components of the sets

$$\begin{gathered}\chi_z = \{Cv(C) \,|\, |C| \le \max\left(|ex\, x_2 \cap {\pi_1^2}^{-1}(z)|, |ex\, x_3 \cap {\pi_1^3}^{-1}(z)|\right), C \text{ is projected} \\ \text{bijectively to those of } ex\, x_2 \cap {\pi_1^2}^{-1}(z) \text{ and } ex\, x_3 \cap {\pi_1^3}^{-1}(z) \\ \text{which have not less number of elements than the other one}\}\end{gathered}$$

One can deduce from Lemma 9 that the set χ_z is arcwise connected if $|ex\, x_2 \cap \cap {\pi_1^2}^{-1}(z)| \ne |ex\, x_3 \cap {\pi_1^3}^{-1}(z)|$ and discrete with $|\chi_z| = k!$ if $|ex\, x_2 \cap {\pi_1^2}^{-1}(z)| = = |ex\, x_3 \cap {\pi_1^3}^{-1}(z)| = k$.

Thus the number of arcwise connected components of $\chi(x_2, x_3)$ can not exceed $(n - |ex\, x_1| + 1)!$. This number is reached when both x_2 and x_3 are $(n-1)$-dimensional simplices with sets of vertices projecting to $ex\, x_1$ in such way that preimages of all points of $ex\, x_1$ but one through both projections are single point sets.

The proposition is proved.

Theorem 1. *Let K, L, M be convex metrizable compacta distinct from a point, $\tau \ge \omega_2$. Then:*

1. *if $m, n \in \mathbb{N}$, $m \ne n$, then $cc\,(K^\tau) \not\cong cc_m\,(L^\tau) \not\cong cc_n\,(M^\tau)$ (including the case $m = 1$, $cc_m\,(L^\tau) \cong L^\tau$);*
2. *if $n \in \mathbb{N}$ and $H : cc_n\,(K^\tau) \to cc_n\,(L^\tau)$ is a homeomorphism, then H preserves powers of the sets of extreme points in $cc_n\,(K^\tau)$.*

The *Proof* relies on the previous Lemmae and the following:

The diagrams $\nu^2 K^\omega$ and $cc\,\nu^2 K^\omega$ are bicommutative, the first one (but not the second) has bijective characteristic map, the diagram $cc_n\,\nu^2 K^\omega$, $n \ge 2$ is not bicommutative and has different maximal numbers of arcwise connected components of characteristics of pairs for different n. It follows from the previous Proposition that the component of any homeomorphism of diagrams $H : cc_n\,(\nu^2 K^\omega) \to cc_n\,(\nu^2 L^\omega)$ which corresponds to the space $cc_n\,(K^\omega)$ preserves the number of extreme points.

§3. On homeomorphisms of cc's of powers of metrizable convex compacta.

Lemma 10. *Let X, Y be convex compacta, $|Y| \neq 1$. Then the subspace $\mathcal{A}(X,Y) \subset \mathcal{C}(X,Y)$ which consists of affine continuous maps from X to Y is compact iff X is finite-dimensional.*

Corollary. *Let X, Y, U be convex compacta, $U \subset X$, $|Y| \neq 1$. Then the set $B_0(U,Y) \subset exp(X \times Y)$ of convex closed sets in $X \times Y$ which are projected bijectively onto U is compact iff U is finite-dimensional.*

The *Proof* uses the fact that the graph mapping $\mathcal{C}(U,Y) \to exp\,(X \times Y)$ is an imbedding. Obviously the topology on $B_0(U,Y)$ does not depend on X into which U is affinely imbedded.

Lemma 11. *Let X, Y_1, Y_2, Z, Z_0 be convex compacta, $Z \subset X \times Y_1$, $\pi_1^2(Z) = Z_0 \subset X$, $|Y_2| \neq 1$. Then the set $\{V \subset X \times Y_1 \times Y_2, \mid\ V$ is convex and closed, $\pi_{12}^3(V) = Z$, $\pi_{13}^2(V) = Z_1\}$ consists of a single element for any such $Z_1 \in cc\,(X \times Y_2)$ that $\pi_1^3(Z_1) = Z_0$ iff Z projects bijectively onto Z_0.*

Lemma 12. *Let X, X', Y_1, Y_1', Y_2, Y_2' be convex compacta, $|Y_2| \neq 1$, $|Y_2'| \neq 1$, and let $H : cc\,D(X, Y_1, Y_2) \to cc\,D(X', Y_1', Y_2')$ be a homeomorphism of diagrams (see §1). Then the restriction of H to the diagram $cc\,(X \times Y_1) \xrightarrow{cc\,\pi_1^2} cc\,X$ preserves the following property of pairs (U,V), $U \in cc\,X$, $V \in cc\,(X \times Y_1)$: V projects bijectively onto U.*

The *Proof* relies on Lemmae 10,11.

Lemma 13. *Let $X, X', Y_1, Y_1', Y_2, Y_2'$ be convex metrizable compacta, $|Y_1| \neq 1$, $|Y_1'| \neq 1$, $|Y_2| \neq 1$, $|Y_2'| \neq 1$, and let $H : cc\,D(X, Y_1, Y_2) \to cc\,D(X', Y_1', Y_2')$ be a homeomorphism of diagrams (see §1). Then the component of H which corresponds to $cc\,X$ preserves the property of elements of $cc\,X$ to be finite-dimensional.*

Let $V_i \in (X \times Y_i)$, $pr_1(V_i) = U \in cc\,X$, $\quad i = 1, 2$. Let us call V_1 and V_2 to be equivalent (over U) iff there exists such $W \in cc\,(X \times Y_1 \times Y_2)$ that $\pi_{12}^3(W) = V_1$, $\pi_{13}^2(W) = V_2$ and W is projected bijectively onto V_1 and V_2. It is equivalent to the existence of a homeomorphism $h : V_1 \to V_2$ such that $pr_1 \circ h = pr_1$ (it is called a fibrewise homeomorphism over U). The notation is $V_1 \sim V_2$.

If we weaken the requirements to the bijectivity of $\pi_{12}^3|_W$ only (i.e., h is only an epimorphism), then h is called a fibrewise epimorphism and V_2 is a fibrewise image of V_1. We denote it $V_2 \prec V_1$.

One can see easily that "$\sim$" really is an equivalent relation and "$\prec$" is a preorder on the class or preimages of the point $U \in cc\,X$ through all possible projections of the form $cc\,pr_1 : cc\,(X \times Y) \to cc\,X$.

Lemma 14. *Let X, Y, Y' be convex compacta, $|Y_1| > 1$, $U \subset X$, $V \subset X \times Y$ be closed convex sets, $pr_1(V) = U$. then the following are equivalent:*

1. *for any $V' \in cc(X \times Y')$, $\quad pr_1(V') = U, V' \prec V$ we have: either $V' \sim V$ or $pr_1 \mid_{V'} : V' \to U$ is a bijection ($\Leftrightarrow V' \in B_0(U,Y)$) (but not simultaneously);*
2. *the maximal dimension of the preimages of points of U through the projection $pr_1 \mid_V : V \to U$ equals 1 (we denote it $V \in B_1(U,Y)$).*

Analogously we write $V \in B_n(U,Y)$ if the respective maximal dimension equals n. If the condition on Y is strengthened: $\dim Y' \geq n$ for some $n \in \mathbb{N}$, then this lemma permits a generalization:

The following is equivalent :

(1) *for any* $V' \in cc\,(X \times Y'), pr_1(V') = U, V' \prec V$ *exactly one of the following holds:* $V' \in B_0(U,Y')$, $V' \in B_1(U,Y')$, $\dots$, $V' \in B_{n-1}(U,Y')$, $V' \sim V$;

(2) $V \in B_n(U,Y)$.

Lemma 15. *Let* X, Y *be convex compacta,* $U \in cc\,X$.

(1) $B_n(U,Y) \neq \emptyset \iff \dim Y \geq n \iff$ *there exists an affine imbeddind* $I^n \hookrightarrow Y$, *and, consequently,* $X \times I^n \hookrightarrow X \times Y$;

(2) *for any* $V \in B_n(U,Y)$ *there exists such* $V' \in B_n(U,I^n)$ *that* $V \sim V'$.

Lemma 16. *Let* $U \subset X$, Y *be convex compacta,* $|Y| \neq 1$. *Then the following are equivalent:*

(1) *for any* $V_1, V_2 \in B_1(U,Y)$ *we have* $V_1 \sim V_2$;

(2) $U = \{x\}, \quad x \in X$.

Lemma 17. *Let* $U \subset X$, $\dim U < \omega$ *and* $i : I^n \hookrightarrow Y$ *be an affine imbedding. Then the imbedding* $cc\,(1_U \times i)\,\big|_{B_n(U,I^n)}: B_n(U,I^n) \to B_n(U,Y)$ *induces the map of the quotient spaces* $B_n(U,I^n)/_\sim \to B_n(U,Y)/_\sim$ *which is a homeomorphism.*

In the sequel we denote $B_n(U,I^n) = B_n(U)$.

Lemma 18. *The quotient space* $C_n(U) = B_n(U)/_\sim$ *is a compactum and the quotient map* $\varphi_n(U,I^n) = \varphi_n(U) : B_n(U) \to C_n(U)$ *is open if* U, X, Y *are as in Lemma 17.*

Lemma 19. *Let* U, Y, Z *be convex compacta,* $\dim U < \omega$, $\dim Y \geq n$, $\dim Z \geq M$, $n, m \in \omega$, $V \in B_n(U,V)$. *Then the natural imbedding* $B_m(V,Z) \hookrightarrow B_{m+n}(U,Y \times Z)$ *induces a continuous map* $\theta^m_n(U, Y \times Z)$ *of the quotient spaces:*

$$\begin{array}{ccc} B_m(V,Z) & \xrightarrow{\varphi_m(V,Z)} & C_m(V,Z) \\ \downarrow & & \downarrow{\scriptstyle \theta^m_n(U,Y\times Z)} \\ B_{m+n}(U,Y\times Z) & \xrightarrow{\varphi_{m+n}(U,Y\times Z)} & C_{m+n}(U,Y\times Z) \end{array}$$

Lemma 20. *Let* U, Y_1, Y_2 *be convex compacta,* $\dim U < \omega$, $\dim Y_i \geq 1$, $V_i \in B_1(U,Y_i)$, $i = 1,2$. *Let us consider the diagram* $cc\,D(U,Y_1,Y_2)$. *The set* $\{W \in cc\,(U \times Y_1 \times Y_2) \mid \pi^3_{12}(W) = V_1,\ \pi^2_{13}(W) = V_2,\ \pi^3_{12}\,\big|_W$ *and* $\pi^2_{13}\,\big|_W$ *are "monos"*$\}$ *can be:*

(1) *empty iff* $V_1 \nsim V_2$ *(i.e.,* V_1 *and* V_2 *are not equivalent);*

(2) *consisting of two elements iff there is such* $V \in B_1(U)$, $V \sim V_1 \sim V_2$, *that* V *is symmetric (i.e.,* $(1_U \times s)(V) = V, s : I \to I, s(t) = 1 - t$*);*

(3) *consisting of one element iff* $V_1 \sim V_2$ *but they are equivalent to no symmetric* $V \in B_1(U)$.

Lemma 21. *Let* U, Y_i, Y_i' *be convex compacta,* $\dim U < \omega$, $\dim Y_i \geq m$, $\dim Y_i' \geq n, i = 1,2$ *and* $V_1 \sim V_2$, $V_1' \sim V_2'$. *Then there exists a fibrewise homeomorphism between the sets* $\chi(V_i, V_i') = \{V \in cc(U \times Y_i \times Y_i') \mid cc\pi^3_{12}(V) = V_i, cc\pi^2_{13}(V) = V_i'\}$, $i = 1,2$, *induced by a homeomorphism of* $(V_i \times Y_i') \cap (V_i' \times Y_i) \subset U \times Y_i \times Y_i'$, $i = 1,2$, *being the "fibrewise product" of the respective fibrewise homeomorphisms* $V_1 \to V_2$ *and* $V_1' \to V_2'$.

Thus this homeomorphism takes elements to equivalent over U, preserves the maximal dimensions of preimages of points when elements are projected (i.e., the image of $B_k(U, Y_1 \times \times Y_1') \cap \chi(V_1, V_1')$ is $B_k(U, Y_2 \times Y_2') \cap \chi(V_2, V_2')$, the image of $B_k(V_1, Y_1') \cap \chi(V_1, V_1')$ is $B_k(V_2, Y_2') \cap \chi(V_2, V_2')$ and the image of $B_k(V_1', Y_1) \cap \chi(V_1', V_1)$ is $B_k(V_2', Y_2) \cap \chi(V_2, V_2')$). Consequently, such properties of a set of the form χ concerning with the "relative dimensions" of its elements and their projections to the faces of the product depend only on the respective classes of equivalence. The same holds for more number of factors (the set $\chi(V_1, \ldots, V_n)$ can be defined analogously).

Lemma 22. *Let* U *be convex compactum,* $0 < \dim U < \omega$. *The following are equivalent:*

(1) U *is a segment;*
(2) *Let* $V_1, V_2, V_1', V_2' \in B_1(U)$ *be symmetric,* $V_1 \nsim V_2$, $V_1' \nsim V_2'$ *and* $|\chi(V_1, V_2)| = |\chi(V_1', V_2')| = 1$; *then either* $V_1 \sim V_1'$, $V_2 \sim V_2'$ *or* $V_1 \sim V_2'$, $V_2 \sim V_1'$.

Lemma 23. *Let* X, X', Y_i, Y_i', $i = 1,2,3$ *be convex compacta, all* Y_i *and* Y_i' *be distinct from a point and let* $H : cc\,D(X, Y_1, Y_2, Y_3) \to cc\,D(X', Y_1', Y_2', Y_3')$ *be a homeomorphism of diagrams. Then the restriction of* H *to* $cc\,X$ *preserves points and segments (i.e., 0- and 1-dimensional elements).*

Taking into account Proposition 5 and previous Lemmae, we obtain a

Theorem 2. *Let* K, L *be convex metrizable compacta distinct from a point,* $\tau \geq \omega_2$ *be a cardinal,* $H : cc\,(K^\tau) \to cc\,(L^\tau)$ *be a homeomorphism. Then:*

(1) H *preserves the property of elements to be finite-dimensional;*
(2) *the set of those* $n \in \mathbb{N}$ *that all finite-dimensional elements with* n *extreme points are sent to finite-dimensional elements with the same number of extreme points, coincides with* $\mathbb{N}$ *or* $\{1, \ldots, m\}$ *for some* $m \geq 2$.

Remark. The author has some reasons to suppose that the above mentioned set necessarily coincides with $\mathbb{N}$ but can not prove it.

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