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ON NORMING MARKUSHEVICH BASES

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A subclass of Markushevich bases in separable Banach spaces such that there exists a way to reconstruct vectors from their generalized Fourier series is introduced. It is proved that this subclass coincides with the class of norming Markushevich bases.

Introduction of a sort of "coordiante system" in a Banach space is an important tool in investigation of various problems. For separable Banach spaces the most common coordinate systems are introduced using biorthogonal sequences.

Let us recall the corresponding definitions. A sequence $\{x_i\}_{i=1}^\infty$ in a Banach space is called *minimal* if for every $i \in \mathbb{N}$ the vector x_i is not contained in the closed linear span of $\{x_j: \ j \in \mathbb{N},\ j \neq i\}$. A sequence in a Banach space X is called *fundamental* if its closed linear span coincides with X.

It is easy to verify that for a fundamental minimal sequence $\{x_i\}_{i=1}^\infty$ there exists a unique sequence $\{x_i^*\}_{i=1}^\infty$ of continuous linear functionals satisfying the condition

$$x_i^*(x_j) = \delta_{ij} \ (i, j \in \mathbb{N}).$$

This sequence is called the *biorthogonal sequence* of $\{x_i\}_{i=1}^\infty$.

So if $\{x_i\}_{i=1}^\infty$ is a fundamental minimal sequence then to each $x \in X$ there corresponds a generalized Fourier series

$$\sum_{i=1}^\infty x_i^*(x)x_i.$$

The vector x is uniquely determined by this series if and only if the biorthogonal sequence $\{x_i^*\}_{i=1}^\infty$ is total (that is: $(\forall x \neq 0)(\exists i \in \mathbb{N})(x_i^*(x) \neq 0)$).

Fundamental minimal sequences with total biorthogonal sequences were studied by A.I.Markushevich [M]. In particular he proved that every separable Banach space has such a sequence. In this connection fundamental minimal sequences with total biorthogonal are usually called *Markushevich bases*.

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Many authors have introduced different classes of biorthogonal systems for which there exist ways of reconstruction of a vector from its generalized Fourier series. The survey of results of this kind can be found in [S].

Now we introduce a natural notion of a way of reconstruction which we consider in this paper.

Definition. A Markushevich basis $\{x_i\}_{i=1}^{\infty}$ with the biorthogonal sequence $\{x_i^*\}_{i=1}^{\infty}$ is called a *reconstructing basis* if there exist mappings

$$T_n : \text{lin}\{x_i\}_{i=1}^{n} \to \text{lin}\{x_i\}_{i=1}^{n},$$

such that

$$(\forall x \in X)(\lim_{n\to\infty} ||x - T_n(\sum_{i=1}^{n} x_i^*(x)x_i)|| = 0). \tag{1}$$

Recall that a subspace $M \subset X^*$ is called *norming* if there exists $c > 0$ such that

$$(\forall x \in X)(\exists f \in M)(||f|| = 1)(|f(x)| \geq c||x||).$$

A Markushevich basis $\{x_i\}_{i=1}^{\infty}$ is called *norming* if its biorthogonal sequence $\{x_i^*\}_{i=1}^{\infty}$ spans a norming subspace in X^*.

The purpose of the present paper is to prove the following result.

Theorem. *Every reconstructing Markushevich basis is a norming one.*

Let us make some remarks before proof of this result.

1. V.P.Fonf [F] proved analogous result with the restriction that the mappings $\{T_n\}_{n=1}^{\infty}$ are continuous. His proof is less direct, it uses results of V.A.Vinokurov, Yu.I.Petunin and A.N.Plichko [VPP] and it seems impossible to use Fonf's argument in non-continuous case. So, in particular, we give a more direct proof of Fonf's result.

2. M.I.Kadets [K] proved that if $\{x_i\}_{i=1}^{\infty}$ is a norming Markushevich basis then there exist continuous mappings

$$T_n : \text{lin}\{x_i\}_{i=1}^{n} \to \text{lin}\{x_i\}_{i=1}^{n},$$

satisfying (1). By this result and the theorem we get.

Corollary. *A Markushevich basis is reconstructing if and only if it is norming.*

Proof of the theorem. Suppose that reconstructing Markushevich basis $\{x_i\}_{i=1}^{\infty}$ is not norming and let $\{T_n\}_{n=1}^{\infty}$ be the corresponding mappings.

Recall that if U and V are subspaces of a Banach space X then the number

$$\delta(U, V) = \inf\{||u - v|| : u \in S(U), v \in V\}$$

is called the *inclination* of U to V.

For natural numbers $1 \leq n_1 < m_1 < n_2 < m_2 \leq \infty$ let

$$a(n_1, m_1, n_2, m_2) := \delta(\text{lin}\{x_i\}_{i=n_1}^{m_1}, \text{lin}\{x_i\}_{i=n_2}^{m_2}).$$

We prove that since $\{x_i\}_{i=1}^{\infty}$ is non-norming then for every $n_1 \in \mathbb{N}$ we have

$$\lim_{m\to\infty}\lim_{r\to\infty} a(n_1, m, r, \infty) = 0. \tag{2}$$

Suppose this is not the case. Then for some $n_1 \in \mathbb{N}$ and $\tau > 0$ and arbitrary $m \geq n_1$ there exists $r(m) > m$ such that

$$\delta(\operatorname{lin}\{x_i\}_{i=n_1}^{m}, \operatorname{lin}\{x_i\}_{i=r(m)}^{\infty}) > \tau. \tag{3}$$

On the other hand since $\{x_i\}_{i=1}^{\infty}$ is non-norming Markushevich basis then for every $\varepsilon > 0$ there exists $x \in X$ such that $\sup\{|f(x)| : f \in \operatorname{lin}\{x_i^*\}_{i=1}^{\infty}, ||f|| \leq 1\} < \varepsilon||x||$.

It is clear that we may suppose that $x \in \operatorname{lin}\{x_i\}_{i=n_1}^{\infty}$. Let $\varepsilon = \tau/2$ and

$$x = \sum_{i=n_1}^{m_1} a_i x_i$$

be such that $\sup\{|f(x)| : f \in \operatorname{lin}\{x_i^*\}_{i=1}^{\infty}, ||f|| \leq 1\} < (\tau/2)||x||$.

On the other hand it follows from (3) that there exists $f \in S(X^*)$ such that

$$f(x_i) = 0, (i \geq r(m_1)) \tag{4}$$

and $f(x) > \tau||x||$. Equalities (4) implies that $f \in \operatorname{lin}\{x_i^*\}_{i=1}^{r(m_1)-1}$. We arrived at a contradiction.

Using (2) we find $n_1 \in \mathbb{N}$ such that $\lim_{r\to\infty} a(1, n_1, r, \infty) < 1/2$.

By compactness there exists $u_1 = \sum_{i=1}^{n_1} a_i x_i, ||u_1|| = 1$
such that

$$\lim_{r\to\infty} \operatorname{dist}(u_1, \operatorname{lin}\{x_i\}_{i=r}^{\infty}) \leq 1/2. \tag{5}$$

By definition of reconstructing basis it follows that for some $r_1 > n_1$ we have

$$||T_{r_1} u_1 - u_1|| < 1/2.$$

By (5) there exists $v_1 = \sum_{i=r_1+1}^{t_1} a_i x_i$
such that $||u_1 + v_1|| < 1$.

Let $n_2 > t_1$ be such that $\lim_{r\to\infty} a(t_1 + 1, n_2, r, \infty) < 1/4$. By compactness there exists $u_2 = \sum_{i=t_1+1}^{n_2} a_i x_i, ||u_2|| = 1$,
such that

$$\lim_{r\to\infty} \operatorname{dist}(u_2, \operatorname{lin}\{x_i\}_{i=r}^{\infty}) \leq 1/4. \tag{6}$$

Let $r_2 > n_2$ be such that

$$||T_{r_2}(u_1 + v_1 + u_2) - (u_1 + v_1 + u_2)|| < 1/4.$$

Using (6) we find

$$v_2 = \sum_{i=r_2+1}^{t_2} a_i x_i$$

such that $||u_2 + v_2|| < 1/2$.

We continue in an obvious way. Let $z=\sum_{i=1}^{\infty}(u_i+v_i)$. It is clear that this series converges. We have

$$\sum_{i=1}^{r_k} x_i^*(z)x_i = u_1+v_1+...+u_{k-1}+v_{k-1}+u_k.$$

Therefore

$$||T_{r_k}(\sum_{i=1}^{r_k} x_i^*(z)x_i) - \sum_{i=1}^{r_k} x_i^*(z)x_i|| < 2^{-k}.$$

On the other hand

$$||z-\sum_{i=1}^{r_k} x_i^*(z)x_i|| = ||v_k+\sum_{i=k+1}^{\infty}(u_i+v_i)|| \geq 1-2^{-k+1}-\sum_{i=k+1}^{\infty} 2^{-i+1}.$$

So

$$\lim_{n\to\infty} ||z-T_n(\sum_{i=1}^{n} x_i^*(z)x_i)|| \neq 0.$$

This contradicts the assumption that $\{x_i\}_{i=1}^{\infty}$ is a recontructing basis. The theorem is proved.

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