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SOME PROBLEMS RELATED TO UNIVERSALITY OF MAPS IN INFINITE-DIMENSIONAL TOPOLOGY

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Some problems concerning universality, softness and invertibility of maps in infinite-dimensional topology are formulated and discussed.

1. Denotation and terminology

As usual, l^2 denotes the separable Hilbert space of square summable sequences,

$$\sigma = \{(x_i)_{i=1}^{\infty} \in l^2 | x_i = 0 \text{ for all but finitely } i\},$$
$$\Sigma = \{(x_i)_{i=1}^{\infty} \in l^2 | \sum_{i=1}^{\infty} (ix_i)^2 < \infty\}.$$

Let $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ and

$$Q^\infty = \varinjlim\{Q \to Q \times \{0\} \hookrightarrow Q \times Q \to Q \times Q \times \{0\} \hookrightarrow Q \times Q \times Q \to \dots\}$$

where $Q = \prod_{i=1}^{\infty} [-1,1]_i$ is the Hilbert cube.

For a class $\mathcal{C}$ of spaces let $\mathcal{C}^\infty = \{X \cong \varinjlim X_i | X_1 \hookrightarrow X_2 \hookrightarrow \dots$ is a sequence of elements of $\mathcal{C}$ and closed embeddings $\}$. We put $\sigma\mathcal{C} = \{X \cong \bigcup_{i=1}^{\infty} X_i | X_i \in \mathcal{C}$ are closed subsets of $X\}$.

Let $\mathcal{K}$ denote the class of all metrizable compacta and $\mathcal{K}_\omega$ (respectively $\mathcal{K}_n$, $\mathcal{K}_n^m$, $\mathcal{K}_\beta$) its subclass consisting of finite-dimensional compacta (respectively of compacta of dimension $\leq n$, of compacta of dimension $\leq n$ and embeddable in $\mathbb{R}^m$, of compacta of small inductive dimension $\leq \beta$).

For classes $\mathcal{C}$, $\mathcal{D}$ of spaces we put

$$M(\mathcal{C}, \mathcal{D}) = \{f : X \to Y | f \text{ is continuous, } X \in \mathcal{C},\ Y \in \mathcal{D}\}.$$

Finally, let $\mathcal{M}$ denote the class of separable metrizable spaces. For undefined notions see, e.g., [1,2].

The following definition is due to E.V.Shchepin [3]. Further, all maps are assumed to be continuous.

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1.1. Definition. A map $f : X \to Y$ is called $\mathcal{C}$-soft, $\mathcal{C}$ being a class of spaces which is topological and closed hereditary, if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow[\psi]{} & Y \end{array}$$

where A is closed subset of a space $Z \in \mathcal{C}$ there exists a map $\Phi : Z \to X$ with $\Phi|A = \varphi$ and $f\Phi = \psi$.

A map $f : X \to Y$ is called $\mathcal{C}$-invertible if for every map $\psi : Z \to Y$, $Z \in \mathcal{C}$ there exists a map $\varphi : Z \to X$ such that $f\varphi = \psi$.

Let $\mathcal{A}$ be a class of maps. A map $f : X \to Y$ is called $\mathcal{A}$-universal if for every map $f' : X' \to Y'$, $f' \in \mathcal{A}$ there exist embeddings $i : X' \to X$, $j : Y' \to Y$ such that $fi = jf'$.

2. Open problems.

2.1. *Let X be a separable linear metric space and $\mathcal{C}(X)$ a class of compacta embeddable into X. Is there a $\mathcal{C}(X)$-soft map of X onto Q?*

In particular:

2.2. *Is there a map $f : E_\beta \to Q$ where E_β is the absorbing set for the class $\mathcal{K}(\beta)$ (it is realized as a metric linear space; see* [4]*) which is $\mathcal{K}_\beta$-soft?*

2.3. *Is there a $\mathcal{K}_n^m$-invertible map $f : X \to Q$ where $X \in \mathcal{K}_n^m$?*

The problems 2.3 and 2.4 have the affirmative answers for $n \geq 2m+1$ [5].

2.5. *A metrizable compactum X has property C if for every sequence $\{\mathcal{C}_n\}_{n=1}^\infty$ of open covers of X there is an open cover $\mathcal{U} = \bigcup_{n=1}^{\infty}$ of X such that a family $\mathcal{U}_n$ is disjoint and refines $\mathcal{C}_n$ for each $n \geq 1$ (see, e.g.,* [6]*). Let $\mathcal{K}_C$ be a class of metrizable compacta with property C. Is there a $\mathcal{K}_C$-soft map $f : X \to Q$ where $X \in \sigma\mathcal{K}_C$?*

2.6. *Let $\varphi_\omega : \mathbb{R}^\infty \to Q^\infty$ be a $(\mathcal{K}_\omega^\infty, \mathcal{K}^\infty)$-universal $\mathcal{K}_\omega$-soft map* [7]*. Are there weaker topologies τ_1, τ_2 on $\mathbb{R}^\infty$ and Q^∞ respectively such that the map $\varphi_\omega : (\mathbb{R}^\infty, \tau_1) \to (Q^\infty, \tau_2)$ is $\mathcal{K}_\omega$-soft and $(\mathbb{R}^\infty, \tau_1) \cong \sigma$, $(Q^\infty, \tau_2) \cong \Sigma$?*

2.7. *Let φ_ω be as in 2.6. Does the functor of free (abelian) topological group preserve $(\mathcal{K}_o^\infty, \mathcal{K}^\infty)$-universality and $\mathcal{K}_\omega$-softness of φ_ω?*

2.8. *Let G be a topological group which is a Σ-manifold. Are there a topological group H which is a σ-manifold and a homeomorphism $\varphi : H \to G$ which is $\mathcal{K}_\omega$-soft map?*

Let $\mathcal{K}_n(\mathbb{Z})$ denote the class of metrizable compacta X with $\dim_{\mathbb{Z}} X \leq n$. J. Dydak and J. Mogilski [8] proved that there is no $\mathcal{K}_n(\mathbb{Z})$-invertible map of X onto Q for any $X \in \mathcal{K}_n(\mathbb{Z})$.

2.9. *Let $\mathcal{K}_\omega(\mathbb{Z}) = \bigcup_{n=1}^{\infty} \mathcal{K}_n(\mathbb{Z})$. Is there a $\mathcal{K}_\omega(\mathbb{Z})$-invertible map of $X \in \mathcal{K}_\omega(\mathbb{Z})$ onto Q?*

2.10. *Let $\varphi_\omega : \mathbb{R}^\infty \to Q^\infty$ be a map mentioned in 2.6. Is φ_ω locally homeomorphic to itself?*

3. Discussion

We expect affirmative answers to the questions of Section 2. It turns out that existence of soft (universal, unvertible etc.) map often implies existence of absorbing set and/or absorbing pair in the sence of [1,2]. Here is a rather typical example of such implication.

3.1. Theorem. *Let $\mathcal{C}$ be a class of Polish spaces X of cohomological dimension* $\dim_{\mathbb{Z}} X < \infty$. *There exists an $(\mathcal{M}_1, \mathcal{C})$-absorbing pair $(l^2, \Omega(\mathcal{C}))$.*

Proof. Dydak and Mogilski [8] showed that for any $n \in \mathbb{N}$ there exists a universal space in the class of complete spaces X with $\dim_{\mathbb{Z}} X \leq n$. In fact, the following fact is implicitly proved in [8].

Let $C \subset [0,1]$ denote the standard middle-third Cantor set. For each subset $A \subset C \times Q$ and each $c \in C$ let $A(c) = \{x \in Q | (c,x) \in A\}$.

Dydak and Mogilski [8] showed that there exists a universal space in the class of complete spaces X with $\dim_{\mathbb{Z}} X \leq n$. The following fact is implicitely proved in [8].

There exists a complete subspace $Y_n \subset C \times Q$ with $\dim_{\mathbb{Z}} Y_n = n$ *satisfying the property: for every complete subspace $Y' \subset Q$ with* $\dim_{\mathbb{Z}} Y' \leq n$ *there exists $c \in C$ with $Y_n(c) = Y'$.*

This is aslightly stronger property that invertibility of $pr_2|Y_n$.

There exists a countable family $\{C_i | i \in \mathbb{N}\}$ of mutually disjoint homeomorphic copies of C in I such that each nondegenerate subinterval of I contains some C_i. Let $h_i : C \to C_i$ be a homeomorphism and $C^* = \cup\{C_i | i \in \mathbb{N}\}$.

Put $Y_n^* = \cup\{(h_i \times Q)(Y_n) | i \in \mathbb{N}\}$.

For every subset $A \subset Q$ put

$$K(A) = \oplus_{n=1}^{\infty} Y_n^* \cap (C^* \times A)_n,$$
$$R(A) = \oplus_{n=1}^{\infty} (C^* \times A)_n$$

(here $\oplus$ denotes the topological sum and $(C^* \times A)_n$ is a copy of $C^* \times Y$, for each $n \in \mathbb{N}$).

Note that for each dense subset $A \subset Q$ the set $K(A)$ is dense in $I \times Q$. Indeed, given $\varepsilon > 0$, find a finite ε-net $S_\varepsilon \subset Q$, then, for each nondegenerate subinterval $J \subset I$, there exists $c \in J$ with $\{c\} \times S_\varepsilon \subset K(A)$.

Further, we apply the techniques of [9]. There exists an $(\mathcal{M}, \mathcal{M}_1)$-absorbing pair (s, Ω) [9]. Let $h : R(s) \to l^2$ be an embedding satisfying the following condition:

(*) for every $x \in R(s)$ and every closed subset $A \subset R(s)$ with $x \notin A$ there exists a continuous linear functional $\varphi : l^2 \to \mathbb{R}$ such that $\varphi(h(A)) = \{0\}$ and $\varphi(h(x)) = 1$ (see [10]).

Denote by E the closure of the linear span of the set $h(R(s))$ and by $H_{\mathcal{C}}$ the linear span of $h(K(\Omega))$. It follows from the results of [9] that the pair $(\prod_{l^2} E, \sum_{l^2} H_{\mathcal{C}})$ is strongly $(\mathcal{M}_1, \mathcal{C})$-universal.

Note that analogous results can be proved in the absolute Borel classes $\mathcal{A}_\alpha$, $2 \leq \alpha < \omega_1$, $\mathcal{M}_\alpha$, $2 \leq \alpha < \omega_1$ and in absolute projective classes $\mathcal{P}_i$, $i \in \mathbb{N}$.

REFERENCES

1. Bestvina M., Mogilski J. *Characterizing certain incomplete infinite-dimensional absolute retracts* // Michigan Math. J. 33 (1986). P.291–313.
2. Cauty R. *Ensembles absorbants pour les classes projectives* // Fund. Math. 143 (1993). P.23–36.
3. Shchepin E.V. *Functors and uncountable powers of compacta* // Uspekhi Mat. Nauk. 36 (1981). P.3–62. (in Russian).
4. Dranishnikov A.N. *Universal Menger compacta and universal maps* // Matem. sbornik. 129 (1986). № 1. P.121–139. (in Russian).
5. Dobrowolski T., Mogilski J. it Absorbing sets in the Hilbert cube related to transfinite dimension // Bull. Acad. Pol. Sci. Math. 38 (1990). No 1–12. P.185–188.
6. Rohm D.M. *Products of infinite-dimensional spaces* // Proc. Amer. Math. Soc. 108 (1990). No 4. P.1019–1023.
7. Zarichnyi M.M. *Functors generated by universal maps of injective limits of sequences of Menger compacta* // Matematika. 562 (1991). Riga. P.95–102. (in Russian).
8. Dydak J., Mogilski J. *Universal cell-like maps* // Amer. Math. Soc. 122 (1994). No 3. P.943–948.
9. Cauty R., Dobrowolski T., *Applying coordinate products to the topological identification of normed spaces* // Trans. Amer. Math. Soc. 337 (1993). P.625–649.
10. Bestvina M., Mogilski J. *Linear maps do not preserve countable-dimensi-onality* // Proc. Amer. Math. Soc. 93 (1985). P.661–666.

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